A Study Of The Fundamentals Of Soft Set Theory

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Abstract: In this paper, a systematic and critical study of the fundamentals of soft set theory, which include operations on soft sets and their properties, soft set relation and function, matrix representation of soft set among others, is carried out.

Index Terms - Soft set, soft subset, soft set operations, soft set relation and function, soft matrix.

1 INTRODUCTION

The concept of soft sets was first formulated by Molodtsov [1999] as a completely new mathematical tool for solving problems dealing with uncertainties. Molodtsov [1999] defines a soft set as a parameterized family of subsets of universe set where each element is considered as a set of approximate elements of the soft set. In the past few years, the fundamentals of soft set theory have been studied by various researchers. Maji et al. [2003] presented a detailed theoretical study of soft sets which includes subset and super set of a soft set, equality of soft sets, operations on soft sets such as union, intersection, AND and OR-operations among others. They also studied and discussed the basic properties of these operations. Pei and Miao [2005] redefined subset and intersection of soft sets and discussed the relationship between soft sets and information systems. Ali et al. [2009] introduced some new operations such as the restricted union, the restricted intersection, the restricted difference and the extended intersection of two soft sets and discussed their basic properties. Cagman and Enginoglu [2010] developed soft matrix theory and successfully applied it to a decision making problem. Babitha and Sunil [2010] introduced the concept of soft set relation and function and discussed many related concepts such as equivalence soft set relation, partition of soft sets, ordering on soft sets. In continuation of their work, Babitha and Sunil [2011] further worked on soft set relation and ordering by introducing the concept of anti-symmetric relation and transitive closure of a soft set relation. Yang and Guo [2011] introduced the notions of anti-symmetric closure of a soft set relation and obtained with proofs some results involving them. Sezgin and Atagun [2011], Ge and Yang [2011], Fuli [2011] etc., gave some modifications in the work of Maji et al. [2003] and also established some new results. Sezgin and Atagun [2011], also introduced the restricted symmetric difference of soft sets and investigated its properties with examples. Singh and Onyeozili [2012] obtained some results on distributive and absorption properties with respect to various operations on soft sets. Singh and Onyeozili [2012] proved that the operations defined on soft sets are equivalent to the corresponding operations defined on their soft matrices.

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The rest of this paper is organized as follows: Section 2 gives some basic definitions and results on soft sets. Section 3 discusses in detail, various operations of soft sets. Section 4 states without proofs many properties of soft set operations. Section 5 focuses on soft set relations and functions .Finally section 6 which comprises of two subsections, first discusses soft matrices and their basic operations while the second subsection concentrates on their properties.

2 PRELIMINARIES

In this section, we give some basic definitions and results on soft sets and suitably exemplify them.

Definition 2.1. [10] (Soft Set)

Let *U* be an initial universe set and *E* a set of parameters or attributes with respect to *U*. Let *P*(*U*) denote the power set of *U* and $A \subseteq E$. A pair (*F*, *A*) is called a soft set over *U*, where *F* is a mapping given by $F: A \rightarrow P(U)$. In other words, a soft set *(F, A)* over *U* is a parameterized family of subsets of *U*. For $e \in A$, $F(e)$ may be considered as the set of e-elements or e-approximate elements of the soft sets (*F*, *A*). Thus (*F*, *A*) is defined as

$$(F, A) = \{F(e) \in P(U) : e \in E,\ F(e) = \varnothing \text{ if } e \notin A\}.$$

Example 2.1

Assume that $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a universal set consisting of a set of six houses under consideration, $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters with respect to *U*, where each parameter $e_i, i = 1, 2, \ldots, 5$ stands for 'expensive', 'beautiful', 'cheap', 'modern', 'wooden', respectively and $A = \{e_1, e_2, e_3\} \subset E$. Suppose a soft set (*F*,*A*) describes the attractions of the houses, such that $F(e_1) = \{h_2, h_4\}, F(e_2) = \{h_1, h_3, h_5\}$ and $F(e_3) = \{h_3, h_4, h_5\}$. Then the soft set (*F*, *A*) is a parameterized family $\{F(e_i) : i = 1, 2, 3\}$ of subset of *U* defined as $(F, A) = \{F(e_1), F(e_2), F(e_3)\}$, i.e., $(F, A) = \{\{h_2, h_4\}, \{h_1, h_3, h_5\}, \{h_3, h_4, h_5\}\}$. The soft set (*F*,*A*) can also be represented as a set of ordered pairs as follows:

$(F,A)=\{(e_1,F(e_1)),(e_2,F(e_2)),(e_3,F(e_3))\}$ i.e., $(F,A)=\{(e_1,\{h_2,h_4\}),(e_2,\{h_1,h_3,h_5\}),(e_3,\{h_3,h_4,h_5\})\}$ Other notations for (*F*, *A*) are F_A or (F_A,E).

Definition 2.2 [9] (Soft subset/soft equal)
Let (*F*,*A*) and (*G*,*B*) be two soft sets over a common universe *U*, we say that

(a) (*F*,*A*) is a soft subset of (*G*,*B*) denoted $(F,A)\tilde{\subseteq}(G,B)$ if
  (i) $A\subseteq B$, and
  (ii) $\forall\ e\in A, F(e)$ and $G(e)$ are identical approximations.

(b) (*F*, *A*) is soft equal set to (*G*,*B*) denoted by (*F*,*A*) = (*G*,*B*) if $(F,A)\tilde{\subseteq}(G,B)$ and $(G,B)\tilde{\subseteq}(F,A)$.

Pei and Miao [11] pointed out that generally in (a) (ii) *F*(*e*) and *G*(*e*) may not be identical and so modified the definition of soft subset in the following way

Definition 2.3 [11] (Soft subset redefined)
For two soft sets (*F*, *A*) and (*G*, *B*) over a universe *U*, we say that (*F*, *A*) is a soft subset of (*G*,*B*) if

(i) $A\subseteq B$, and
(ii) $\forall\ e\in A,\ F(e)\subseteq G(e)$.

Example 2.2
Let $U=\{u_1,u_2,u_3,u_4,u_5,u_6\}$ be a universe set and $E=\{e_1,e_2,e_3,e_4,e_5\}$ be a set of parameters. Let $A=\{e_1,e_3,e_5\}\subset E$ and $B=\{e_1,e_2,e_3,e_5\}\subset E$.
Suppose (*F*,*A*) and (*G*,*B*) are two soft sets over *U* where
$F(e_1)=\{u_2,u_4\},F(e_2)=\{u_3,u_4,u_5\},F(e_5)=\{u_1\}$
and
$G(e_1)=\{u_2,u_4\},G(e_2)=\{u_1,u_3,u_4,u_5\},G(e_3)=\{u_3,u_4,u_5\},G(e_5)=\{u_1,u_4\}$
. Then $(F,A)\tilde{\subset}(G,B)$ since $A\subset B$ and $F(e)\subset G(e)\ \forall\ e\in A$. But $(G,B)\tilde{\not\subset}(F,A)$. Hence $(F,A)\neq(G,B)$.

Remark 2.1
Let (*F*, *A*) and (*G*, *B*) be soft sets over a common universe *U*. $(F,A)\tilde{\subset}(G,B)$ does not imply that every element of (*F*,*A*) is an element of (*G*,*B*). Therefore, the definition of classical subset does not hold for soft subset. For example, let $U=\{u_1,u_2,u_3,u_4\}$ be a universe and $E=\{e_1,e_2,e_3\}$ be a set of parameters such that if $A=\{e_1\},B=\{e_1,e_3\}$ and
$(F,A)=\{(e_1,\{u_2,u_4\})\},(G,B)=\{(e_1,\{u_2,u_3,u_4\}),(e_3,\{u_1,u_5\})\}$
, then $\forall\ e\in A,\ F(e)\subset G(e)$ and $A\subset B$. Hence $(F,A)\tilde{\subset}(G,B)$. Clearly $(e_1,F(e_1))\in(F,A)$ but $(e_1,F(e_1))\notin(G,B)$.

Definition 2.4 [9] (Not Set)
Let $E=\{e_1,e_2,e_3,\ldots,e_n\}$ be a set of parameters. The 'Not set of *E*', denoted by $\neg E$ is defined by
$\neg E=\{\neg e_1,\neg e_2,\neg e_3,\ldots,\neg e_n\}$, where $\neg e_i$ means not $e_i\ \forall\ i=1,2,3,\ldots,n$

Proposition 2.1[9]
Let *E* be a universal parameter set, *A* , *B* ⊂ *E,* then

i) $\neg(\neg A)=A$
ii) $\neg(A\cup B)=\neg A\cup\neg B$
iii) $\neg(A\cap B)=\neg A\cap\neg B$

Remark 2.2
It has been proved in [14] that $\neg A\neq A^c$ and that $\neg A\not\subset E$ and so proposition 2.1 above hold. But Ge and Yang[8] made the assumption that $\neg A\subset E$ and came up with the following proposition.

Proposition 2.2[8]
i) $\neg(A\cup B)=\neg A\cap\neg B$ (De Morgan's Law)
i) $\neg(A\cap B)=\neg A\cup\neg B$ (De Morgan's law)

Definition 2.5 [2]
Let *U* be a universe, *E* be a set of parameters and $A\subseteq E$.

a) *(F, A)* is called a **relative null soft set** with respect to *A*, denoted $\tilde{\Phi}_A$, if $F(e)=\emptyset$, $\forall\ e\in A$.
b) *(F , A)* is called a **relative whole soft set** or *A*-universal with respect to *A*, denoted $\tilde{U}_A$, if $F(e)=U$, $\forall\ e\in A$.
c) The relative whole soft set with respect to *E* denoted $\tilde{U}_E$ is called the **absolute soft set** over *U*.

Example 2.3
Let E={e_1,e_2 e_3, e_4 }. If $A=\{e_2,e_3,e_4\}$ such that $F(e_2)=\{u_2,u_4\}$, $F(e_3)=\emptyset$, , $F(e_4)=U$, then the soft set $(F,A)=\{(e_2,\{u_2,u_4\}),(e_4,U)\}$.

If $B=\{e_1,e_3\}$ such that the soft set $(G,B)=\{(e_1,\emptyset),(e_3,\emptyset)\}$, then the soft set *(G,B)* is a relative null soft set , ie $(G,B)=\tilde{\Phi}_B$.

If $C=\{e_1,e_2\}$ such that $H(e_1)=U$, $H(e_2)=U$, then the soft set *(H, C)* is a relative whole soft set $\tilde{U}_C$.

If $D=E$ such that $F(e_1)=U$, $\forall\ e_i\in E$, $i=1,2,3,4,$ Then the soft set $(F,D)=\tilde{U}_E$ is an absolute soft set.

Proposition 2.3 [13]
Let U *be* a universe, *E* a set of parameters, *A, B , C* $\subset E$. If *(F,A) , (G,B)* and *(H,C)* are soft sets over *U* , Then

- **i)** $(F,A) \,\tilde{\subseteq}\, \tilde{U}_A$.
- **ii)** $\tilde{\Phi}_A \,\tilde{\subseteq}\, (F,A)$
- **iii)** $(F,A) \,\tilde{\subseteq}\, (F,A)$
- **iv)** $(F,A) \,\tilde{\subseteq}\, (G,B)$ and $(G,B) \,\tilde{\subseteq}\, (H,C)$ *implies* $(F,A) \,\tilde{\subseteq}\, (H,C)$
- **v)** *(F,A) = (G,B)* and *(G,B) = (H,C) implies* *(F,A) = (H,C)*

Definition 2.6 [9] (Complement)
The complement of a soft set (*F*, *A*) denoted by $(F,A)^C$ is defined as $(F,A)^C = \left(F^C, \neg A\right)$ where

$F^C : \neg A \to P(U)$ is a mapping given by

$F^C(\alpha) = U - F(\neg\alpha)\ \forall \alpha \in \neg A$.

Later Ali et al.[2] introduced a new notion of complement called relative complement which is defined in the next definition.

Definition 2.7 [2] (Relative Complement)
The relative complement of a soft set (*F*,*A*) denoted by $(F,A)^r$ is defined by $(F,A)^r = \left(F^r, A\right)$ where F^r $: A \to P(U)$ is a mapping given by

$F^r(\alpha) = U - F(\alpha),\ \forall\ \alpha \in A$.

In view of the above discussion, we present the following example:

Example 2.4
Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a universe set and $E = \{e_1, e_2, e_3, e_4\}$ be a set of parameters. Suppose *A* = $\{e_2, e_3, e_4\} \subset E$ such that the soft set *(F,A)*=$\{e_2, \{u_2, u_4\}), (e_4, U)\}$, then

- ***i.*** $(F,A)^c = \{(\neg e_2, \{u_1, u_3, u_5\}), (\neg e_3, U)\}$
- ***ii.*** $(F,A)^r = \{(e_2, \{u_1, u_3, u_5\}), (e_3, U)\}$

Proposition 2.4
Let *(F,A)* be a soft set over a universe *U.* Then

- **i.** $(F,A)^c)^c = (F,A)$
- **ii.** $((F,A)^r)^r = (F,A)$
- **iii.** $\tilde{U}_A^C = \tilde{\Phi}_A = \tilde{U}_A^r$
- **iv.** $\tilde{\Phi}_A^C = \tilde{U}_A = \tilde{\Phi}_A^r$

3. SOFT SET OPERATIONS

Definition 3.1 [9]
Let (*F*, *A*) and (*G*, *B*) be two soft sets over a common universe *U*. Then:

- **(i)** the **union** of (*F*,*A*) and (*G*,*B*), denoted $(F,A)\,\tilde{\cup}\,(G,B)$ is a soft set (*H*,*C*), where $C = A \cup B$ and $\forall\ e \in C$

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & e \in A \cap B \end{cases}$$

- **(ii)** the **intersection** of (*F*,*A*) and (*G*,*B*) denoted $(F,A)\,\tilde{\cap}\,(G,B)$ is a soft set (H,C) where C=A∩B and $\forall$ e$\in$ C, *H*(*e*) = *F*(*e*) or *G*(*e*) (as both are same set).

- **(iii)** the **AND-operation** of (*F*,*A*) and (*G*,*B*) denoted (*F*,*A*) AND (*G*,*B*) or $(F,A) \wedge (G,B)$ is a soft set defined by $(F,A) \wedge (G,B) = (H, A \times B)$ where $H(a,b) = F(a) \cap G(b),\ \ \forall\ (a,b) \in A \times B$.

- **(iv)** the **OR-operation** of (*F*,*A*) and (*G*,*B*) denoted (*F*,*A*) OR (*G*,*B*) or $(F,A) \vee (G,B)$ is a soft set defined by $(F,A) \vee (G,B) = (H, A \times B)$ where $H(a,b) = F(a) \cup G(b),\ \ \forall\ (a,b) \in A \times B$.

Pei and Miao [11] pointed out that in Definition 3.1 (ii), *F*(*e*) and *G*(*e*) may not be the same set and thus revised the definition as follows:

Definition 3.2 [11] (Intersection redefined)
Let (*F*,*A*) and (*G*,*B*) be two soft sets over *U*. The intersection (also called bi-intersection by Feng et al. [6]) of (*F*,*A*) and (*G*,*B*) denoted $(F,A)\,\tilde{\cap}\,(G,B)$ is a soft set (*H*,*C*) where $C = A \cap B$ and $\forall\ e \in C,\ H(e) = F(e) \cap G(e)$. Moreover, Ahmad and Kharal [1] pointed out that in the above Definition 3.2, $A \cap B$ must be non-empty to avoid the degenerate case and hence improved the definition as follows:

Definition 3.3 [1] (Intersection redefined)
Let (*F*,*A*) and (*G*,*B*) be two soft sets over *U* with $A \cap B \neq \varnothing$. The intersection of (*F*,*A*) and (*G*,*B*) denoted $(F,A)\,\tilde{\cap}\,(G,B)$ is a soft set (*H*,*C*), where $C = A \cap B$ and $\forall\ e \in C,\ H(e) = F(e) \cap G(e)$. Ali et al. [2] later introduced the following operations.

Definition 3.4 [2]
Let (*F*,*A*) and (*G*,*B*) be two soft sets over *U*. Then

- **(i)** the **extended intersection** of (*F*,*A*) and (*G*,*B*) denoted $(F,A) \cap_\varepsilon (G,B)$ is a soft set (*H*,*C*), where $C = A \cup B$ and $\forall\ e \in C$,

$$H(e)=\begin{cases}F(e), \text{ if } e\in A-B\\ G(e), \text{ if } e\in B-A\\ F(e)\cap G(e), \text{ if } e\in A\cap B.\end{cases}$$

(ii) the **restricted intersection** (also called intersection by Pei and Miao [10] and bi(intersection by Feng et al. [6]) of (*F*,*A*) and (*G*,*B*), denoted $(F,A)\tilde{\cap}(G,B)$ is a soft set (*H*,*C*), where $C=A\cap B$ and $\forall\ e\in C,\ H(e)=F(e)\cap G(e)$.

(iii) the **restricted union** of (*F*,*A*) and (*G*,*B*), denoted $(F,A)\cup_R(G,B)$ is a soft set (*H*,*C*), where $C=A\cap B$ and $\forall\ e\in C$, $H(e)=F(e)\cup G(e)$.

(iv) the **restricted difference** of (*F*,*A*) and (*G*,*B*) denoted $(F,A)-_R(G,B)$ is a soft set (*H*,*C*) where $C=A\cap B$ and $\forall\ e\in C,\ H(e)=F(e)-G(e)$.

Sezgin and Atagun [13] in 2011, defined the following operation;

Definition 3.5 [13] (Restricted symmetric difference)

The restricted symmetric difference of (*F*,*A*) and (*G*,*B*) denoted $(F,A)\tilde{\Delta}(G,B)$ is a soft set defined by $(F,A)\tilde{\Delta}(G,B)=\left((F,A)\cup_R(G,B)\right)-_R((F,A)\tilde{\cap}(G,B))$ *or*

$(F,A)\tilde{\Delta}(G,B)=\left((F,A)-_R(G,B)\right)\cup_R\left((G,B)-_R(F,A)\right)$.

The above definition (3.5) can also be defined as follows:

Definition 3.6

The restricted symmetric difference of (*F*,*A*) and (*G*,*B*), denoted $(F,A)\tilde{\Delta}(G,B)$ is a soft set (*H*,*C*), where $C=A\cap B$ and $\forall\ e\in C,\ H(e)=F(e)\Delta G(e)$ (the symmetric difference of *F*(*e*) and *G*(*e*)).

Example 3.1

Let $U=\{h_1,h_2,h_3,h_4,h_5,h_6\}$ be a universe, $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set with respect to *U*, and $A=\{e_1,e_2,e_3\}\subset E$.

Let a soft set (*F*,*A*) over *U* be given by $(F,A)=\{(e_1,\{h_2,h_4\}),(e_2,\{h_1,h_3,h_5\}),(e_3,\{h_3,h_4,h_5\})\}$. Suppose $B=\{e_3,e_4,e_5\}$ and (*G*,*B*) is a soft set over U given by

$(G,B)=\{(e_3,\{h_1,h_2,h_3\}),(e_4,\{h_2,h_3,h_6\}),(e_5,\{h_2,h_3,h_4\})\}$. Then

(i) $(F,A)\tilde{\cup}(G,B)=\{(e_1,\{h_2,h_4\}),(e_2,\{h_1,h_3,h_5\}),(e_3,\{h_1,h_2,h_3,h_4,h_5\})$
$(e_4,\{h_2,h_3,h_6\}),(e_5,\{h_2,h_3,h_4\})\}$.

(ii) $(F,A)\cup_R(G,B)=\{(e_3,\{h_1,h_2,h_3,h_4,h_5\})\}$

(iii) $(F,A)\tilde{\cap}(G,B)=\{(e_3,\{h_3\})\}$

(iv) $(F,A)\cap_\varepsilon(G,B)=\{(e_1,\{h_2,h_4\}),(e_2,\{h_1,h_3,h_5\}),(e_3,\{h_3\})$
$(e_4,\{h_2,h_3,h_6\}),(e_5,\{h_2,h_3,h_4\})\}$

(v) $(F,A)-_R(G,B)=\{(e_3,\{h_3,h_5\})\}$

(vi) $(F,A)\tilde{\Delta}(G,B)=\{(e_3,\{h_1,h_2,h_4,h_5\})\}$

(vii) $(F,A)\wedge(G,B)=\{((e_1,e_3),\{h_2\}),((e_1,e_4),\{h_2\}),((e_1,e_5),\{h_4\})$
$((e_2,e_3),\{h_1,h_3\}),((e_2,e_4),\{h_3\}),((e_2,e_5),\{h_5\})$
$((e_3,e_3),\{h_3\}),((e_3,e_4),\{h_3\}),((e_3,e_5),\{h_3,h_4\})\}$

(viii) $(F,A)\vee(G,B)=\{((e_1,e_3),\{h_1,h_2,h_3,h_4\}),$
$((e_1,e_4),\{h_2,h_3,h_4,h_5\}),$
$((e_1,e_5),\{h_2,h_3,h_4\}),$
$((e_2,e_3),\{h_1,h_2,h_3,h_5\}),$
$((e_2,e_4),\{h_1,h_2,h_3,h_5,h_6\}),$
$((e_2,e_5),\{h_1,h_2,h_3,h_4,h_5\}),$
$((e_3,e_3),\{h_1,h_2,h_3,h_4,h_5\}),$
$((e_3,e_4),\{h_2,h_3,h_4,h_5,h_6\}),$
$((e_3,e_5),\{h_2,h_3,h_4,h_5\})\}$.

4. PROPERTIES OF SOFT SET OPERATIONS

Various results on properties of soft set operations have been established by many authors. We will state without proofs the basic properties on soft set operations, some of which we have established and published.

1. **Idempotent properties**
   - **(i)** $(F,A)\tilde{\cup}(F,A)=(F,A)=(F,A)\cup_R(F,A)$
   - **(ii)** $(F,A)\tilde{\cap}(F,A)=(F,A)=(F,A)\cap_\varepsilon(F,A)$
2. **Identity Properties**
   - **(i)** $(F,A)\tilde{\cup}\tilde{\Phi}=(F,A)=(F,A)\cup_R\tilde{\Phi}$
   - **(ii)** $(F,A)\tilde{\cap}\tilde{U}=(F,A)=(F,A)\cap_\varepsilon\tilde{U}$
   - **(iii)** $(F,A)-_R\tilde{\Phi}=(F,A)=(F,A)\tilde{\Delta}\tilde{\Phi}$
   - **(iv)** $(F,A)-_R(F,A)=\tilde{\Phi}=(F,A)\tilde{\Delta}(F,A)$
3. **Domination Properties**
   - **(i)** $(F,A)\tilde{\cup}\tilde{U}=\tilde{U}=(F,A)\cup_R\tilde{U}$
   - **(ii)** $(F,A)\tilde{\cap}\tilde{\Phi}=\tilde{\Phi}=(F,A)\cap_\varepsilon\tilde{\Phi}$
4. **Complementation Properties**
   - **(i)** $\tilde{\Phi}^C=\tilde{U}=\tilde{\Phi}^r$
   - **(ii)** $\tilde{U}^C=\tilde{\Phi}=\tilde{U}^r$
5. **Double Complementation (Involution) Property**

$$\left((F,A)^C\right)^C=(F,A)=\left((F,A)^r\right)^r$$

6. **Exclusion Properties**

$$(F,A)\tilde{\cup}(F,A)^r=\tilde{U}=(F,A)\cup_R(F,A)^r$$

7. **Contradiction Properties**

$$(F,A)\tilde{\cap}(F,A)^r=\tilde{\Phi}=(F,A)\cap_\varepsilon(F,A)^r$$

Remark 4.1

Exclusion and contradiction properties do not hold with respect to complement in Definition 2.6 [18]

8. **De Morgan's Properties**
   - **(i)** $\left((F,A)\tilde{\cup}(G,B)\right)^C=(F,A)^C\cap_\varepsilon(G,B)^C$
   - **(ii)** $\left((F,A)\cap_\varepsilon(G,B)\right)^C=(F,A)^C\tilde{\cup}(G,B)^C$
   - **(iii)** $\left((F,A)\cup_R(G,B)\right)^r=(F,A)^r\tilde{\cap}(G,B)^r$
   - **(iv)** $\left((F,A)\tilde{\cap}(G,B)\right)^r=(F,A)^r\cup_R(G,B)^r$
   - **(v)** $\left((F,A)\wedge(G,B)\right)^C=(F,A)^C\vee(G,B)^C$
   - **(vi)** $\left((F,A)\vee(G,B)\right)^C=(F,A)^C\wedge(G,B)^C$
   - **(vii)** $\left((F,A)\wedge(G,B)\right)^r=(F,A)^r\vee(G,B)^r$
   - **(viii)** $\left((F,A)\vee(G,B)\right)^r=(F,A)^r\wedge(G,B)^r$
   - **(ix)** $\left((F,A)\tilde{\cup}(G,B)\right)^r=(F,A)^r\cap_\varepsilon(G,B)^r$
   - **(x)** $\left((F,A)\cap_\varepsilon(G,B)\right)^r=(F,A)^r\tilde{\cup}(G,B)^r$

Remark 4.2

De Morgan's Property does not hold for restricted union and restricted intersection with respect to complement in Definition 2.6

i.e. $((F,A)\cup_R(G,B))^c\neq(F,A)^c\tilde{\cap}(G,B)^c$ [18]

$((F,A)\tilde{\cap}(G,B))^c\neq(F,A)^c\cup_R(G,B)^c$ [18]

9. **Absorption Properties**
   - **i.** $(F,A)\tilde{\cup}\left((F,A)\tilde{\cap}(G,B)\right)=(F,A)$
   - **ii.** $(F,A)\tilde{\cap}\left((F,A)\tilde{\cup}(G,B)\right)=(F,A)$
   - **iii.** $(F,A)\cup_R\left((F,A)\cap_\varepsilon(G,B)\right)=(F,A)$
   - **iv.** $(F,A)\cap_\varepsilon\left((F,A)\cup_R(G,B)\right)=(F,A)$

Remark 4.3

- **(i)** $\tilde{\cup}$ and $\cap_\varepsilon$ do not absorb over each other[15]
- **(ii)** $\cup_R$ and $\tilde{\cap}$ do not absorb over each other[15]

10. **Commutative Properties**
    - **(i)** $(F,A)\tilde{\cup}(G,B)=(G,B)\tilde{\cup}(F,A)$
    - **(ii)** $(F,A)\cup_R(G,B)=(G,B)\cup_R(F,A)$
    - **(iii)** $(F,A)\tilde{\cap}(G,B)=(G,B)\tilde{\cap}(F,A)$
    - **(iv)** $(F,A)\cap_\varepsilon(G,B)=(G,B)\cap_\varepsilon(F,A)$
    - **(v)** $(F,A)\tilde{\Delta}(G,B)=(G,B)\tilde{\Delta}(F,A)$

Remark 4.4

$\wedge$ and $\vee$ do not commute.

11. **Associative Properties**
    - **(i)** $(F,A)\tilde{\cup}\left((G,B)\tilde{\cup}(H,C)\right)=\left((F,A)\tilde{\cup}(G,B)\right)\tilde{\cup}(H,C)$
    - **(ii)** $(F,A)\cup_R\left((G,B)\cup_R(H,C)\right)=\left((F,A)\cup_R(G,B)\right)\cup_R(H,C)$
    - **(iii)** $(F,A)\tilde{\cap}\left((G,B)\tilde{\cap}(H,C)\right)=\left((F,A)\tilde{\cap}(G,B)\right)\tilde{\cap}(H,C)$
    - **(iv)** $(F,A)\cap_\varepsilon\left((G,B)\cap_\varepsilon(H,C)\right)=\left((F,A)\cap_\varepsilon(G,B)\right)\cap_\varepsilon(H,C)$
    - **(v)** $(F,A)\wedge\left((G,B)\wedge(H,C)\right)=\left((F,A)\wedge(G,B)\right)\wedge(H,C)$
    - **(vi)** $(F,A)\vee\left((G,B)\vee(H,C)\right)=\left((F,A)\vee(G,B)\right)\vee(H,C)$

12. **Distributive properties**

(i) $(F,A)\tilde{\cup}\left((G,B)\tilde{\cap}(H,C)\right)=\left((F,A)\tilde{\cup}(G,B)\right)\tilde{\cap}\left((F,A)\tilde{\cup}(H,C)\right)$

(ii) $(F,A)\tilde{\cap}\left((G,B)\tilde{\cup}(H,C)\right)=\left((F,A)\tilde{\cap}(G,B)\right)\tilde{\cup}\left((F,A)\tilde{\cap}(H,C)\right)$

(iii) $(F,A)\cup_R\left((G,B)\cap_\varepsilon(H,C)\right)=\left((F,A)\cup_R(G,B)\right)\cap_\varepsilon\left((F,A)\cup_R(H,C)\right)$

(iv) $(F,A)\cap_\varepsilon\left((G,B)\cup_R(H,C)\right)=\left((F,A)\cap_\varepsilon(G,B)\right)\cup_R\left((F,A)\cap_\varepsilon(H,C)\right)$

(v) $(F,A)\cup_R\left((G,B)\tilde{\cap}(H,C)\right)=\left((F,A)\cup_R(G,B)\right)\tilde{\cap}\left((F,A)\cup_R(H,C)\right)$

(vi) $(F,A)\tilde{\cap}\left((G,B)\cup_R(H,C)\right)=\left((F,A)\tilde{\cap}(G,B)\right)\cup_R\left((F,A)\tilde{\cap}(H,C)\right)$

(vii) $(F,A)-_R\left((G,B)\tilde{\cup}(H,C)\right)=\left((F,A)-_R(G,B)\right)\tilde{\cup}\left((F,A)-_R(H,C)\right)$

(viii) $(F,A)-_R\left((G,B)\cap_\varepsilon(H,C)\right)=\left((F,A)-_R(G,B)\right)\cap_\varepsilon\left((F,A)-_R(H,C)\right)$

(ix) $(F,A)-_R\left((G,B)\cup_R(H,C)\right)=\left((F,A)-_R(G,B)\right)\cup_R\left((F,A)-_R(H,C)\right)$

(x) $(F,A)-_R\left((G,B)\tilde{\cap}(H,C)\right)=\left((F,A)-_R(G,B)\right)\tilde{\cap}\left((F,A)-_R(H,C)\right)$

(xi) $(F,A)\tilde{\cap}\left((G,B)-_R(H,C)\right)=\left((F,A)\tilde{\cap}(G,B)\right)-_R\left((F,A)\tilde{\cap}(H,C)\right)$

(xii) $(F,A)\tilde{\cap}\left((G,B)\tilde{\Delta}(H,C)\right)=\left((F,A)\tilde{\cap}(G,B)\right)\tilde{\Delta}\left((F,A)\tilde{\cap}(H,C)\right)$

Remark 4.5

(i) $\tilde{\cup}$ and $\cap_\varepsilon$ do not distribute over each other

(ii) $\vee$ and $\wedge$ do not distribute over each other

(iii) $\tilde{\cup}, \cup_R$ and $\cap_\varepsilon$ do not distribute over $-_R$

(iv) $\tilde{\cup}, \cup_R, \cap_\varepsilon, \tilde{\cap}$ and $-_R$ do not distribute over $\wedge$ and $\vee$

(v) $\cup_R$ distribute over $\tilde{\cup}$ but the converse is false

(vi) $\tilde{\cap}$ distribute over $\cap_\varepsilon$ but the converse is false

5. SOFT SET RELATION AND FUNCTION

Definition 5.1 [3] (Cartesian Product of Soft Set)

Let (*F*,*A*) and (*G*,*B*) be two soft sets over a common universe *U*. Then the Cartesian product of (*F*,*A*) and (*G*,*B*) denoted by $(F,A)\times(G,B)$ is a soft set $(H,A\times B)$ where

$H: A\times B \rightarrow P(U\times U)$ and

$H(a,b)=F(a)\times G(b)\ \forall\ (a,b)\in A\times B$, i.e,

$H(a,b)=\left\{\left(h_i,h_j\right): h_i\in F(a) \text{ and } h_j\in G(b)\right\}$.

Definition 5.2 [3] (Soft Set Relation)

Let (*F*,*A*) and (*G*,*B*) be two soft sets over a common universe *U*. Then a relation from (*F*,*A*) to (*G*,*B*) called a soft set relation (*R*,*C*) or simply *R* is a soft subset of $(F,A)\times(G,B)$ where $C\subseteq A\times B$ and $\forall\ (a,b)\in C$. *R*(*a*,*b*) = *H*(*a*,*b*), where $(H,A\times B)=(F,A)\times(G,B)$.

A soft set relation on (*F*, *A*) is a soft subset of $(F,A)\times(F,A)$. In an equivalent way, we can define a relation *R* on the soft set (*F, A*) in the parameterized form as follows:

If $(F,A)=\{F(a),F(b),\ldots\}$, then

$F(a)RF(b)$ iff $F(a)\times F(b)\in R$.

Definition 5.3

Let *R* be a soft set relation from (*F*, *A*) to (*G*,*B*) such that $(F,A)\times(G,B)=(H,A\times B)$. Then

(a) the domain of *R* (dom*R*) is the soft set $\left(D,A_1\right)\tilde{\subseteq}(F,A)$ where $A_1=\{a\in A: H(a,b)\in R, \text{for some } b\in B\}$ and $D\left(a_1\right)=F\left(a_1\right),\ \forall\ a_1\in A_1$.

(b) the range of *R* (ran *R*) is a soft set $\left(E,B_1\right)\tilde{\subseteq}(G,B)$ where $B_1\subset B$ and $B_1=\{b\in B: H(a,b)\in R \text{ for some } a\in A\}$ and $E\left(b_1\right)=G\left(b_1\right)\ \forall\ b_1\in B_1$

(c) the inverse of *R* denoted by R^{-1} is a soft set relation from (G,B) to (F,A) defined by $R^{-1}=\{G(b)\times F(a): F(a)RG(b)\}$.

Example 5.1

Let *U* denote a set of ten people given by $U=\{p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_9,p_{10}\}$.

Let *A* denote different professionals given by *A* = {Accountants, Doctors, Engineers, Teachers} represented by $A=\{a_1,a_2,a_3,a_4\}$ respectively.

Let *B* denote the qualification of people given by

B = {B.Sc., B.Tech.,MBBS, M.Sc.} represented by $B=\{b_1,b_2,b_3,b_4\}$ respectively.

Then the soft set (*F*,*A*) given by

$$(F,A)=\left\{F\left(a_1\right)=\{p_1,p_2\},F\left(a_2\right)=\{p_4,p_5\},F\left(a_3\right)=\{p_7,p_9\},\right.$$
$$\left.F\left(a_4\right)=\{p_3,p_4,p_7\}\right\}$$

describes people with different professions and the soft set *(G,B)* given by

$$(G,B)=\{G(b_1)=\{p_1,p_2,p_6,p_8,p_{10}\},G(b_2)=\{p_3,p_6,p_7,p_9\},G(b_3)=\{p_3,p_4,p_5,p_8\},$$
$$G(b_4)=\{p_1,p_2,p_3,p_8\}\},$$

describes peoples' qualifications. If we define a relation *R* from (*F*,*A*) to (*G*,*B*) as follows:

$F(a)RG(b)$ iff $F(a)\subset G(b)$, then

(i) $R=\{F(a_1)\times G(b_1),F(a_2)\times G(b_3),F(a_3)\times G(b_2),F(a_1)\times G(b_4)\}$

(ii) $\operatorname{dom} R=(D,A_1)$, where $A_1=\{a_1,a_2,a_3\}\subset A$ and $D(a)=F(a)\ \forall\ a\in A_1$

(iii) $\operatorname{ran} R=(E,B_1)$, where $B_1=\{b_1,b_2,b_3,b_4\}$ and $E(b)=G(b)\ \forall\ b\in B_1$

(iv) $R^{-1}=\left\{\begin{matrix}G(b_1)\times F(a_1),G(b_2)\times F(a_3),\\ G(b_3)\times F(a_2),G(b_4)\times F(a_1)\end{matrix}\right\}.$

Definition 5.4 [3]

Let *R*, *Q* be two soft set relations on a soft set (*F*,*A*)

(a) $R\subset Q$ if $\forall\ a,b\in A,\ F(a)\times F(b)\in R\Rightarrow F(a)\times F(b)\in Q$

(b) The **complement of *R*** denoted as R^C is defined by
$$R^C=\{F(a)\times F(b):F(a)\times F(b)\notin R,a,b\in A\}$$

(c) The **union of *R* and *Q***, denoted as $R\cup Q$ is defined by
$$R\cup Q=\{F(a)\times F(b):F(a)\times F(b)\in R \text{ or } F(a)\times F(b)\in Q\}$$

(d) The **intersection of *R* and *Q*** denoted as $R\cap Q$ is defined by
$$R\cap Q=\{F(a)\times F(b):F(a)\times F(b)\in R \text{ and } F(a)\times F(b)\in Q\}.$$

Example 5.2

Consider a soft set (*F*, *A*) over *U*, where $U=\{u_1,u_2,u_3,u_4\}$, $A=\{a_1,a_2\}$ and $F(a_1)=\{u_1,u_2\},F(a_2)=\{u_2,u_3,u_4\}$.

If a soft set relation *R* on (*F*, *A*) is defined as

$$R=\{F(a_1)\times F(a_1),F(a_2)\times F(a_1)\},$$ Then

$$R^C=\{F(a_1)\times F(a_2),F(a_2)\times F(a_2)\}.$$

If another soft set relation *Q* on (*F*, *A*) is defined as

$$Q=\{F(a_1)\times F(a_1),F(a_2)\times F(a_2)\},$$ then

$$R\cup Q=\{F(a_1)\times F(a_1),F(a_2)\times F(a_1),F(a_2)\times F(a_2)\}$$

$$R\cap Q=\{F(a_1)\times F(a_1)\}.$$

It is easy to verify that the union and the intersection of soft set relations satisfy commutative, associative and distributive properties.

Definition 5.5[3]

Let *R* be a soft set relation on (*F*, *A*), then

(i) *R* is **reflexive** if $F(a)\times F(a)\in R\ \forall\ a\in A$

(ii) *R* is **symmetric** if $F(a)\times F(b)\in R\Rightarrow F(b)\times F(a)\in R,\ \forall\ (a,b)\in A\times A$

(iii) *R* is **transitive** if $F(a)\times F(b)\in R$ and $F(b)\times F(c)\in R\Rightarrow F(a)\times F(c)\in R\ \forall\ a,b,c\in A$

(iv) *R* is **equivalence** if it is reflexive, symmetric and transitive

(v) *R* is an **identity** if $a\neq b, F(a)\times F(b)\in R$ but $F(a)\times F(b)\notin R\ \forall\ a,b\in A$, i.e., $F(a)\times F(b)\in R\Rightarrow a=b\ \forall\ a,b\in A$, e.g., $R=\{F(a)\times F(a),F(b)\times F(b),F(c)\times F(c)\},\ \forall\ a,b,c\in A.$

Example 5.3

Consider a soft set (*F*, *A*) over *U*, where $A=\{a_1,a_2\}$. If a relation *R* on (*F*, *A*) is defined by

$$R=\left\{\begin{matrix}F(a_1)\times F(a_2),F(a_2)\times F(a_1),\\ F(a_1)\times F(a_1),F(a_2)\times F(a_2)\end{matrix}\right\},$$ then

R is a soft set equivalence relation.

Note that here $R=(F,A)\times(F,A)$.

Definition 5.6 [3] (Composition of Soft Set Relations)

Let (*F*, *A*), (*G*, *B*) and (*H*, *C*) be three soft sets over a common universe. Let *R* be a soft set relation from (*F*, *A*) to (*G*, *B*) and *S* be a soft set relation from (*G*, *B*) to (*H*, *C*). Then, a new soft set relation from (*F*, *A*) to (*H*,*C*) called the **composition of *R* and *S*** denoted by SoR is defined as follows: If $F(a)\in(F,A)$ and $H(c)\in(H,C)$, then $F(a)\times H(c)\in SoR$

$\Leftrightarrow F(a)\times G(b)\in R$ and $G(b)\times H(c)\in S$, for some $G(b)\in(G,B)$.

Example 5.4

Let $A=\{a_1,a_2,a_3\}$, $B=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$. Let *R* and *S* be soft set relations over a common universe, defined respectively from (*F*,*A*) to (*G*,*B*) and (G,B) to (*H*,*C*) such that $R=\{F(a_1)\times G(b_1),F(a_2)\times G(b_2),F(a_3)\times G(b_2)\}$, and $S=\{G(b_1)\times H(c_1),G(b_2)\times H(c_2)\}$ Then $SoR=\{F(a_1)\times H(c_1),F(a_2)\times H(c_2),F(a_3)\times H(c_2)\}$.

Suppose $A=B=C=\{e_1,e_2,e_3\}$ such that $R=\{F(e_1)\times F(e_2),F(e_2)\times F(e_3),F(e_3)\times F(e_1)\}$,

$S=\{F(e_1)\times F(e_3),F(e_2)\times F(e_2),F(e_3)\times F(e_3)\}$. Then $SoR=\{F(e_1)\times F(e_2),F(e_2)\times F(e_3),F(e_3)\times F(e_3)\}$ and

$RoS=\{F(e_1)\times F(e_1),F(e_2)\times F(e_3),F(e_3)\times F(e_1)\}$.

Thus in general, $SoR\neq RoS$

Definition 5.7 [3] (Soft Set Function)

Let (*F*,*A*) and (*G*,*B*) be two non-empty soft sets over *U*. Then a soft set relation *f* from (*F*,*A*) to (*G*,*B*) written $f:(F,A)\to(G,B)$ is called a soft set function if every element in the domain of *f* has a unique element in the range of *f*. If *F*(*a*) *f G*(*b*), i.e., $F(a)\times G(b)\in f$, then we write $f(F(a))=G(b)$.

Example 5.5

Let $A=\{a_1,a_2,a_3,a_4\}$ and $B=\{b_1,b_2\}$. Consider the soft sets (*F*,*A*) and (*G*,*B*) over a universe U. Then a soft set function *f* from *(F, A)* to *(G, B)* can be given by

(i) $$f=\begin{Bmatrix} F(a_1)\times G(b_1),F(a_2)\times G(b_1), \\ F(a_3)\times G(b_3),F(a_4)\times G(b_2)\end{Bmatrix}$$

(ii) $$f=\begin{Bmatrix} F(a_1)\times G(b_1),F(a_2)\times G(b_1), \\ F(a_3)\times G(b_1),F(a_4)\times G(b_1)\end{Bmatrix}$$

.

But $\{F(a_1)\times G(b_1),F(a_1)\times G(b_2),F(a_2)\times G(b_2),F(a_3)\times G(b_1)\}$ is not a soft set function.

Definition 5.8 [3]

A function *f* from (*F*,*A*) to (*G*,*B*) is called

- **(i)** **Injective (one-to-one)** if $F(a)\neq F(b)\Rightarrow f(F(a))\neq f(F(b))$
- **(ii)** **Surjective (onto)** if range *f* = (*G*,*B*)
- **(iii)** **Bijective (one- to-one and onto)** if *f* is both injective and surjective.

Example 5.6

Consider the function *f* in Example (5.5), (i) is onto but (ii) is not.

Definition 5.9 [3] (Identity Soft Set Function)

The identity soft set function *I* on a soft set *(F, A)* is defined by $I:(F,A)\to(F,A)$ such that $I(F(a))=F(a)\ \forall\ F(a)\in(F,A)$

6. MATRIX REPRESENTATION OF SOFT SET

We present the matrix representation of soft sets, their basic operations and properties with illustrative examples.

Definition 6.1 [5] (Soft Matrix)

Let *U* be a universe, *E* a set of parameters with respect to *U* and *A* ⊆ *E*. Let (f_A,E) be a soft set over *U*. Then a subset R_A of $U\times E$, uniquely defined as $R_A=\{(u,e):e\in A,u\in f_A(e)\}$, is called a *relation form* of the soft set (f_A,E).

The *characteristic function* χ_{R_A} of R_A is defined as

$\chi_{R_A}:U\times E\to\{0,1\}$, where

$$\chi_{R_A}(u,e)=\begin{cases}1, & (u,e)\in R_A;\\ 0, & (u,e)\notin R_A.\end{cases}$$

Now if $U=\{u_1,u_2,\ldots,u_m\}$ *and* $E=\{e_1,e_2,\ldots,e_n\}$ then the soft set *(f_A , E)* can be represented by a matrix $[a_{ij}]$ called an *m*× *n* "soft matrix" of the soft set *(f_A , E)* over *U* as follows

$$[a_{ij}]_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & A_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$$

where $a_{ij}=\chi_{R_A}(u_i,e_j)$

In other words, a soft set is uniquely represented by its corresponding soft matrix.

Example 6.1

Let $U=\{u_1,u_2,u_3,u_4,u_5\}$ be a universe set, and

$E=\{e_1,e_2,e_3,e_4\}$ be a set of all parameters with respect to *U*.

Let $A=\{e_1,e_3,e_4\}$, $f_A(e_1)=\{u_3,u_4\}$, $f_A(e_3)=\varnothing$, and $f_A(e_4)=\{u_1,u_3,u_5\}$. Then the soft set (f_A,E) is given by

$$(f_A,E)=\{(e_1,\{u_3,u_4\}),\ (e_4,\{u_1,u_3,u_5\})\}.$$

The relation form R_A of (f_A,E) is given by $R_A=\{(u_3,e_1),(u_4,e_1),(u_1,e_4),(u_3,e_4),(u_5,e_4)\}$.

Hence the soft matrix $[a_{ij}]$ of the soft set (f_A,E) is given by

$$[a_{ij}]=\begin{bmatrix}0&0&0&1\\0&0&0&0\\1&0&0&1\\1&0&0&0\\0&0&0&1\end{bmatrix},\ i=1,2,\ldots,5;\ j=1,2,\ldots,4.$$

As noted earlier, $f_A(e_3)=\varnothing$, since there is no element in *U* related to the parameter $e_3\in A$, so it does not appear in the aforesaid description of the soft set (f_A,E).

Definition 6.2 [5](Special Soft Matrices)
Let the set of all *mxn* soft matrices over *U* be denoted *SM (U)*$_{mxn}$ or just *SM(U)*
Let $[a_{ij}]\in SM(U)$. Then $[a_{ij}]$ is called

(a) A ***zero soft matrix***, denoted $[\tilde{0}]$, if $a_{ij}=0\ \ \forall\ \ i \text{ and } j;$

(b) An ***A-universal soft matrix***, denoted $[\tilde{a}_{ij}]$, if $a_{ij}=1\ \ \forall\ \ j\in I_A=\{j:e_j\in A\}$ and *i*.

(Note that it is so called, since $a_{ij}=1$ only for the parameters appearing in the set $A\subset E$); and

(c) A ***universal soft matrix*** denoted $[\tilde{I}]$, if $a_{ij}=1\ \ \forall\ \ i$ and j

Example 6.2
Let $U=\{u_1,u_2,u_3,u_4\}$, $E=\{e_1,e_2,e_3,e_4\}$ and $[a_{ij}],[b_{ij}],[c_{ij}]\in SM(U)_{4\times4}$. If $A=\{e_1,e_2,e_3\}$, $f_A(e_1)=f_A(e_2)=f_A(e_3)=\varnothing$ then $[a_{ij}]=[\tilde{0}]$ is a zero soft matrix given by

$$[\tilde{0}]=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

If $B=\{e_2,e_4\}$, $f_B(e_2)=U=f_B(e_4)$, then $[b_{ij}]$ is a *B*-universal soft matrix given by

$$[\tilde{b}_{ij}]=\begin{bmatrix}0&1&0&1\\0&1&0&1\\0&1&0&1\\0&1&0&1\end{bmatrix}.$$

If $C=E$, $f_e(e_i)=U$ for each *i*, then $[\tilde{c}_{ij}]=[\tilde{I}]$ is a universal soft matrix given by

$$[\tilde{I}]=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}.$$

Definition 6.3 [5](Soft Sub matrices)
Let $M=[a_{ij}],N=[b_{ij}]\in SM(U)$. Then we define the following:

(i) M is a ***soft sub matrix*** of *N*, denoted $M\tilde{\subseteq}N$ if $a_{ij}\le b_{ij}$ for each *i* and *j*.

In this case, we also say that *M* is ***dominated*** by *N* or *N* *dominates M*. Note that similar to $R^k(k>1)$, the *k*-dimensional real space, ≤ holds without the holding of either < or =.
We define *M* and *N* ***comparable***, denoted $M\|N$, iff $M\tilde{\subseteq}N \text{ or } N\tilde{\subseteq}M;$

(ii) *M* is a ***proper soft sub matrix*** of *N*, denoted $M\tilde{\subset}N$, if $[a_{ij}]\tilde{\subseteq}[b_{ij}]$ and for at least one term $a_{ij}<b_{ij}$ for all *i* and *j*. In this case, we say that *M* is ***properly dominated*** by *N*.

(iii) *M* is a ***strictly proper soft sub matrix*** of *N*, denoted $M\tilde{\subsetneq}N$, if $M\tilde{\subset}N$ and $a_{ij}<b_{ij}$, for each *i* and *j*. In this case we say that *M* is ***strictly dominated*** by *N*.

(iv) *M* and *N* are ***soft equal matrices*** denoted $M\tilde{=}N$ if $a_{ij}=b_{ij}$ for each *i* and *j*. Equivalently, if $M\tilde{\subseteq}N$ and $N\tilde{\subseteq}M$, then $M\tilde{=}N$. It is immediate to see that $\tilde{\subseteq}$ is a partial ordering (reflexive, anti-symmetric and transitive) on the class of soft matrices.

6.2 Operations On Soft Matrices

We discuss the operations of union, intersection complement, difference and products of soft matrices and their basic properties.

Definition 6.4 [5] (Soft Matrix Operations)

Let $M=\left[a_{ij}\right], N=\left[b_{ij}\right]\in SM(U)$. Then a soft matrix $P=\left[c_{ij}\right]\in SM(U)$ is called the

(i) ***union*** of *M* and *N*, denoted $M\tilde{\cup}N$, if $c_{ij}=\max\left\{a_{ij},b_{ij}\right\}$ for all i and j;

(ii) ***intersection*** of *M* and *N*, denoted $M\tilde{\cap}N$ $c_{ij}=\min\left\{a_{ij},b_{ij}\right\}$ for all i and j;

(iii) ***complement*** of *M*, denoted M^0, if $c_{ij}=1-a_{ij}$ for all i and j;

(iv) ***difference*** of *N* from *M*, also called the ***relative complement*** of *N* in *M*, denoted *M* – *N* or *M**N* if $P=M\tilde{\cap}N^0$.

In view of the (ii) above, *M* and *N* are said to be ***disjoint*** if $M\tilde{\cap}N=\left[\tilde{0}\right]$.

Example 6.3

Let

$$M=\begin{bmatrix}0&0&1&1\\1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&1\end{bmatrix} \text{ and } N=\begin{bmatrix}1&0&0&0\\0&1&0&1\\1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}.$$

Then,

(i) $$M\tilde{\cup}N=\begin{bmatrix}1&0&1&1\\1&1&1&1\\1&1&0&1\\0&1&0&0\\0&0&0&1\end{bmatrix};$$

(ii) $$M\tilde{\cap}N=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=\left[\tilde{0}\right]$$ which implies that *M* and *N* are disjoint;

(iii) $$M^0=\begin{bmatrix}1&1&0&0\\0&1&0&1\\1&0&1&0\\1&1&1&1\\1&1&1&0\end{bmatrix} \text{ and } M\tilde{\cup}M^0=\left[\tilde{I}\right], \quad M\tilde{\cap}M^0=\left[\tilde{0}\right];$$

(iv) $$N^0=\begin{bmatrix}0&1&1&1\\1&0&1&0\\0&1&1&1\\1&0&1&1\\1&1&1&1\end{bmatrix};$$

(v) $$M-N=\begin{bmatrix}0&0&1&1\\1&0&1&0\\0&1&0&1\\0&0&0&0\\0&0&0&1\end{bmatrix}=M\tilde{\cap}N^0; \text{and}$$

(vi) $$N-M=\begin{bmatrix}1&0&0&0\\0&1&0&1\\1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=N\tilde{\cap}M^0.$$

Proposition 6.1: Properties of Soft Matrix Operations

Let $M=\left[a_{ij}\right],\ N=\left[b_{ij}\right],\ P=\left[c_{ij}\right]\in SM(U)$.

(i) $M\tilde{\cup}M=M;\ M\tilde{\cap}M=M$ (Idempotent laws)

(ii) $M\tilde{\cup}\left[\tilde{0}\right]=M;\ M\tilde{\cap}\left[\tilde{I}\right]=M$ (Identity laws)

(iii) $M\tilde{\cup}\left[\tilde{I}\right]=\left[\tilde{I}\right];\ M\tilde{\cap}\left[\tilde{0}\right]=\left[\tilde{0}\right]$ (Domination laws)

(iv) $\left[\tilde{0}\right]^0=\left[\tilde{I}\right]$; $\left[\tilde{I}\right]^0=\left[\tilde{0}\right]$ (De Morgan's laws)

(v) $M\tilde{\cup}M^0=\left[\tilde{I}\right]$; $M\tilde{\cap}M^0=\left[\tilde{0}\right]$ (De Morgan's laws)

(vi) $\left(M\tilde{\cup}N\right)^0=M^0\cap N^0$; $\left(M\tilde{\cap}N\right)^0=M^0\tilde{\cup}N^0$ (De Morgan's laws)

(vii) $\left(M^0\right)^0=M$ for all M (Involution law)

(viii) $M\tilde{\cup}N=N\tilde{\cup}M$; $M\tilde{\cap}N=N\tilde{\cap}M$ (Commutative laws)

(ix) $M\tilde{\cup}\left(N\tilde{\cup}P\right)=\left(M\tilde{\cup}N\right)\tilde{\cup}P$; $M\tilde{\cap}\left(N\tilde{\cap}P\right)=\left(M\tilde{\cap}N\right)\tilde{\cap}P$ (Associative laws)

(x) $M\tilde{\cup}\left(N\tilde{\cap}P\right)=\left(M\tilde{\cup}N\right)\tilde{\cap}\left(M\tilde{\cup}P\right)$;

$M \tilde{\cap} (N \tilde{\cup} P) = (M \tilde{\cap} N) \tilde{\cup} (M \tilde{\cap} P)$. (Distributive Laws)

Proof: Most of the proofs follow from definitions.
Let us, for example, prove the first parts of (vi), (ix) and (x).

(vi) For each *i* and *j*,

$$\begin{aligned}(M \tilde{\cup} N)^0 &= \left([a_{ij}] \tilde{\cup} [b_{ij}]\right)^0 \\ &= \left[\max\{a_{ij}, b_{ij}\}\right]^0 \\ &= \left[1 - \max\{a_{ij}, b_{ij}\}\right] \\ &= \left[\min\{1 - a_{ij},\ 1 - b_{ij}\}\right] \\ &= [a_{ij}]^0 \tilde{\cap} [b_{ij}]^0 \\ &= M^0 \tilde{\cap} N^0.\end{aligned}$$

(ix)

$$\begin{aligned}M \tilde{\cup} (N \tilde{\cup} P) &= [a_{ij}] \tilde{\cup} \left([b_{ij}] \tilde{\cup} [c_{ij}]\right) \\ &= \left[\max\{a_{ij}, \max(b_{ij}, c_{ij})\}\right] \\ &= \left[\max\{\max(a_{ij}, b_{ij}), c_{ij}\}\right] \\ &= \left([a_{ij}] \tilde{\cup} [b_{ij}]\right) \tilde{\cup} [c_{ij}] \\ &= (M \tilde{\cup} N) \tilde{\cup} P.\end{aligned}$$

(x)

$$\begin{aligned}M \tilde{\cup} (N \tilde{\cap} P) &= [a_{ij}] \tilde{\cup} \left([b_{ij}] \tilde{\cap} [c_{ij}]\right) \\ &= \left[\max\{a_{ij}, \min(b_{ij}, c_{ij})\}\right] \\ &= \left[\min\{\max\{a_{ij}, b_{ij}\}, \max\{a_{ij,c_{ij}}\}\}\right] \\ &= \left([a_{ij}] \tilde{\cup} [b_{ij}]\right) \tilde{\cap} \left([a_{ij}] \tilde{\cup} [c_{ij}]\right) \\ &= (M \tilde{\cup} N) \tilde{\cap} (M \tilde{\cup} P).\end{aligned}$$

Definition 6.5 [5] (Product of Soft Matrices)
Let $M = [a_{ij}], N = [b_{ik}] \in SM(U)_{m \times n}$. Then

(i) ***AND-product*** of *M* and *N*, denoted $M \wedge N$ is defined $\wedge : SM(U)_{m\times n} \times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that $[a_{ij}] \wedge [b_{ik}] = [c_{ip}]$, where $c_{ip} = \min\{a_{ij}, b_{ik}\}$ and $P = n(j-1)+k$.

(ii) ***OR-product*** of *M* and *N*, denoted $M \vee N$ is defined $\vee : SM(U)_{m\times n} \times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that $[a_{ij}] \vee [b_{ik}] = [c_{ip}]$, where $c_{ip} = \max\{a_{ij}, b_{ik}\}$ and $P = n(j-1)+k$.

(iii) ***AND-NOT-product*** of *M* and *N*, denoted $M \barwedge N$ is defined $\barwedge : SM(U)_{m\times n} \times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that $[a_{ij}] \barwedge [b_{ik}] = [c_{ip}]$, where $c_{ip} = \min\{a_{ij}, 1-b_{ik}\}$ and $P = n(j-1)+k$.

(iv) ***OR-NOT-product*** of *M* and *N*, denoted $M \veebar N$ is defined $\veebar : SM(U)_{m\times n} \times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that $[a_{ij}] \veebar [b_{ik}] = [c_{ip}]$, where $c_{ip} = \max\{a_{ij}, 1-b_{ik}\}$ and $P = n(j-1)+k$

Note: Products of soft matrices hold if the two matrices are of the same order or have the
same number of rows

Proposition 6.2 [5] (Properties of Product of Soft Matrices)
Let $M = [a_{ij}],\ N = [b_{ik}] \in SM(U)$. Then the following hold:

(i) $(M \vee N)^0 = M^0 \wedge N^0;\ (M \wedge N)^0 = M^0 \vee N^0$
(De Morgan's laws)

(ii) $(M \veebar N)^0 = M^0 \barwedge N^0;\ (M \barwedge N)^0 = M^0 \veebar N^0$
(De Morgan's laws)

***Proof**:* The proofs follow from definitions.

Example 6.4
Let $M = [a_{ij}],\ N = [b_{ik}] \in SM(U)_{4\times 4}$ given by

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Then

$$M \wedge N = [a_{ij}] \wedge [b_{ik}] = [c_{ip}]_{4\times16} = \begin{bmatrix} 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&1&0&0&1&1&0&0&1&1&0&0&1 \\ 0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Similarly, the other products $M \vee N, M \barwedge N$ and $M \veebar N$ can be found.
Also,

$$(M \wedge N)^{\circ} = \begin{bmatrix} 1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0 \\ 1&1&1&1&0&1&1&0&0&1&1&0&0&1&1&0 \\ 1&1&1&1&1&1&0&0&1&1&0&0&1&1&1&1 \\ 1&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1 \end{bmatrix},$$

$$M^{\circ} \vee N^{\circ} = \begin{bmatrix} 1&0&1&0 \\ 1&0&0&0 \\ 1&0&0&1 \\ 1&0&1&1 \end{bmatrix} \vee \begin{bmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 1&1&0&0 \\ 1&1&0&0 \end{bmatrix} = \begin{bmatrix} 1&1&1&1&1&1&0&0&1&1&1&1&1&1&0&0 \\ 1&1&1&1&0&1&1&0&0&1&1&0&0&1&1&0 \\ 1&1&1&1&1&1&0&0&1&1&0&0&1&1&1&1 \\ 1&1&1&1&1&1&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

Thus, $(M \wedge N)^0 = M^0 \vee N^0$. Note that the commutative laws are not valid for products of soft matrices.

CONCLUSION AND FUTURE WORK

In this paper, we have discussed in detail the fundamentals of soft set theory such as soft subsets, soft set operations and their properties, soft set relation and function, soft matrices among others, and exemplified them. It was observed that some properties on classical sets do not hold for soft set operations. Similar study could be extended to related concepts such as fuzzy soft set, intuitionistic fuzzy soft set, soft multi set among others.

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