Retarded Inequalities With Two Independent Variables For Retarded Partial Differential Equations

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ABSTRACT: In this article some new two dimensional nonlinear integral inequalities are obtained. These inequalities can be used as ready and powerful tools in the analysis of various classes of partial differential, integral and integrodifferential equations. As an application of nonlinear integral inequalities with delay,global existence of nonlinear partial differential is presented.

Keywords: Explicit bounds,retarded integral inequalities,two independent variables,global existence.

1 INTRODUCTION AND PRELIMINARIES

During the past decades, studies of integral inequalities have been greatly enriched by the recognition of their potential applications in various applied sciences [1]-[6]. Recently integral inequalities with delays have received much attention from researchers [7]-[15] and these inequality are successfully applied to prove global existence, uniqueness as well as . In this paper we establish some new retarded integral inequalities with two independent variables and derive explicit bounds on unknown functions.In [15] the present author has established the following useful on the integral inequality.

Lemma 1.1 *Let $\varphi,\varphi',\alpha \in C^1(I,I)$ be increasing functions with $\varphi' \le k, \alpha(t) \le t, \alpha(0)=0, t \in I$, k, u_0 be positive constants.We assume that $u(t)$ and $f(t)$are nonnegative real valued continuous functions defined on I and satisfy the inequality*

$$u(t) \le u_0 + \left(\int_0^{\alpha(t)} f(s)u(s)ds\right)^2 + \int_0^{\alpha(t)}.\left(f(s)\varphi\big(u(s)\big)\big(\varphi\big(u(s)\big) + 2\int_0^s. f(\sigma)\varphi\big(u(\sigma)\big)d\sigma\big)ds, \quad (1)$$

for all $t \in I$.

If $$u_0^{-1} - k\int_0^{\alpha(t)} f(s)exp(4\int_0^s f(\sigma)d\sigma)ds > 0 \quad (2)$$

then
$$u(t) \le \Phi^{-1}(\Phi(u_0) + k\int_0^{\alpha(t)} f(s)B_3(s)ds), \ forall\ t \in I \quad (3)$$

where
$$\Phi(x) = \int_1^x \frac{ds}{\varphi(s)}, \ forall\ x > o \quad (4)$$

$$B_3(t) = exp(4\int_0^{\alpha(t)} f(s)ds)((\varphi(u_0))^{-1} - k\int_0^{\alpha(t)} f(s)exp(4\int_0^s f(\sigma)d\sigma)ds)^{-1} \quad (5)$$

The aim of the present paper is to establish a two independent retarded version of the above inequality which can be used as a tool to study the behavior of a nonlinear differential equation with delay in two independent variables.Applications are also given to convey the importance of our result to the literature.

2 MAIN RESULTS

In this section we present some new nonlinear retarded Gronwall-Bellman type integral inequality in two independent variables which can be used as effective tools in the study on nonlinear partial differential equation with time delay. Throughout this article let $I = [0,\infty)$ and $\Delta = I \times I$.

Theorem 2.1 Let $\varphi \in C^1(I \times I, I)$, be increasing function. $\varphi_{xy} \le k$ for all $x, y \in \Delta$. $\alpha, \beta \in C^1(I,I)$ be increasing functions $\alpha(t) \le t$ and $\beta(t) \le t, \alpha(0) = 0, \beta(0) \le 0$ for all $x \in I$, $y \in I$; k, u_0 be a constants, assume that $u(x,y)$ and $f(x,y)$ are nonnegative real valued continuous functions defined on Δ and satisfy the inequality

$$u(x,y) \le u_0 + (\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t))ds)^2 + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t)) \times (\varphi(u(s,t)) + 2\int_0^s\int_0^t f(\sigma,\tau)\varphi(u(\sigma,\tau))d\tau d\sigma)dtds,$$

for all $x, y \in \Delta$. (6)

If
$$u_0^{-1} - k\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)exp(4\int_0^s\int_0^t f(\sigma,\tau)d\sigma d\tau)dtds > 0 \quad (7)$$

then
$$u(x,y) \le \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B_3(s,t)dtds\right), \quad (8)$$

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$for\ \ all\ \ (s,t)\in I$ (8)

$\Phi(x,y)=\int_1^x \frac{ds}{\varphi(s)},\ for\ \ all\ \ x>o$ (9)

$$B_3(x,y)=exp(4\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds)$$
$$((\varphi(u_0))^{-1}-k\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\times exp(4\int_0^s\int_0^t f(\sigma,\tau)d\tau d\sigma)dtds)^{-1} \quad (10)$$

Proof: Let z(x,y)denotes the function on the right-hand side of of (6) is nonnegative and nondecreasing function on Δ

$$z(x,y)=u_0+(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t))dtds)^2+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t))\times(\varphi(u(s,t))+2\int_0^s\int_0^t f(\sigma,\tau)\varphi(u(\sigma,\tau))d\tau d\sigma)dtds \quad (11)$$

From (6), we have $u(x,y)\le z(x,y), and$ $z(x,0)=z(0,y)=u_0$

From (6), we have

$$z_{xy}=2f(\alpha(x),\beta(y)))\varphi(u(\alpha(x),\beta(y)))\alpha'(x)\beta'(y)\int_0^{\alpha(s)}\int_0^{\beta(t)} f(s,t)\varphi(u(s,t))dtds$$
$$+f(\alpha(x),\beta(y)))\varphi(u(\alpha(x),\beta(y)))\alpha'(x)\beta'(y)\times(\varphi(u(\alpha(x),\beta(y))))+2\int_0^{\alpha(x)}\int_0^{\beta(y)} f(\sigma,\tau)\varphi(u(\sigma,\tau))d\tau d\sigma)dtds$$

$$z_{xy}=f(\alpha(x),\beta(y)))\varphi(u(\alpha(x),\beta(y)))\alpha'(x)\beta'(y)\times$$
$$(2\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t))dtds+\varphi(u(\alpha(x),\beta(y)))+2\int_0^{\alpha(x)}\int_0^{\beta(y)} f(\sigma,\tau)\varphi(u(\sigma,\tau))d\tau d\sigma)dtds$$

$$z_{xy}\le f(\alpha(x),\beta(y)))\varphi(z(\alpha(x),\beta(y)))\alpha'(x)\beta'(y)\times(\varphi(z(\alpha(x),\beta(y)))+4\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(z(s,t)))dtds \quad (12)$$

Let

$$m(x,y)=\varphi(z(\alpha(x),\beta(y))+4\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(z(s,t))dtds \quad (13)$$

Therefore $m(x,0)=m(0,y)=\varphi(z(\alpha(x),0)=\varphi(0,\beta(y))=\varphi(u_0)$

and

$$\varphi(z(\alpha(x),\beta(y))\le m(x,y) \quad (14)$$

Using (14) in (12), we have

$$z_{xy}\le f(\alpha(x),\beta(y))\varphi(z(\alpha(x),\beta(y)))\alpha'(x)\beta'(y)m(x,y) \quad (15)$$

Differentiating $m(x,y)$ from (13), we have

$$\frac{\partial^2}{\partial x\,\partial y}m(x,y)=\frac{\partial^2}{\partial x\,\partial y}\varphi(z(\alpha(x),\beta(y))\frac{\partial^2}{\partial x\,\partial y}z(\alpha(x),\beta(y))+4f(\alpha(x),\beta(y))\varphi(z(\alpha(x),\beta(y)))\alpha'(x)\beta'(y) \quad (16)$$

Using (15) in (16) and $\frac{\partial^2}{\partial x\,\partial y}\varphi(z(x,y))\le k$, we have

$$\frac{\partial^2}{\partial x\,\partial y}m(x,y)\le kf\big(\alpha(x),\beta(y)\big)\varphi\Big(z\big(\alpha(x),\beta(y)\big)\Big)\alpha'(x)\beta'(y)m(x,y)+4f(\alpha(x),\beta(y))\varphi(z(\alpha(x),\beta(y)))\alpha'(x)\beta'(y) \quad (17)$$

Again using inequality (14) in (17),we deduce

$$\frac{\partial^2}{\partial x\,\partial y}m(x,y)\le kf\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)m^2(x,y)+4f\big(\alpha(x),\beta(y)\big)m(x,y)\alpha'(x)\beta'(y) \quad (18)$$

Dividing inequality (18) by $m^2(x,y)$, on both sides, we have

$$m^{-2}(x,y)\frac{\partial^2}{\partial x\,\partial y}m(x,y)\le 4f\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)m^{-1}(x,y)+kf\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y) \quad (19)$$

Take $v(x,y)=m^{-1}(x,y)$, we have Put $v(x,y)=m^{-1}(x,y)$ in inequality (19),we have

$$-\frac{\partial^2}{\partial x\,\partial y}v(x,y)\le 4f(\alpha(x),\beta(y))\alpha'(x)\beta'(y)v(x,y)+kf(\alpha(x),\beta(y))\alpha'(x)\beta'(y)$$

for all $x,y\in\Delta$

Consider the equation and solving it,we get

$$\frac{\partial^2}{\partial x\,\partial y}(v(x,y)\exp(\int_0^x\int_0^y 4f(\alpha(s),\beta(t))\alpha'(s)\beta'(t)dtds))>-kf\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)\times\exp\big(\int_0^x\int_0^y 4f\big(\alpha(s),\beta(t)\big)\alpha'(s)\beta'(t)dtds\big) \quad (20)$$

Integrating from $0\ to\ x$ and from $0\ to\ y$, the above inequality in (20), we have

$$v(x,y)>\exp\big(\int_0^x\int_0^y -4f\big(\alpha(s),\beta(t)\big)\alpha'(s)\beta'(t)dtds\big)\times\big[\big(\varphi(u_0)\big)^{-1}-k\int_0^x\int_0^y f\big(\alpha(s),\beta(t)\big)\alpha'(s)\beta'(t)\times\exp\big(\int_0^s\int_0^t 4f\big(\alpha(\sigma),\beta(\eta)\big)\alpha'(\sigma)\beta'(\eta)d\eta d\sigma\big)dtds\big] \quad (21)$$

By making change of variable in (21), we have

$$v(x,y)>exp(\int_0^{\alpha(x)}\int_0^{\beta(y)} -4f(s,t)dtds)\times[(\varphi(u_0))^{-1}-k\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\times\exp(\int_0^s\int_0^t 4f(\sigma,\eta)d\eta d\sigma)dtds] \quad (22)$$

As $v\,(x,y)=m^{-1}(x,y)$, we have

$$m^{-1}(x,y)) > exp(\int_0^{\alpha(x)}\int_0^{\beta(y)} -4f(s,t)dtds) \times [(\varphi(u_0))^{-1} - k\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t) \times \exp(\int_0^s\int_0^t 4f(\sigma,\eta)d\eta d\sigma)dtds] \quad (23)$$

$$m(x,y)) \le \exp(\int_0^{\alpha(x)}\int_0^{\beta(y)} 4f(s,t)dtds) \times [(\varphi(u_0))^{-1} - k\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t) \times \exp(\int_0^s\int_0^t 4f(\sigma,\eta)d\eta d\sigma)dtds]^{-1} \quad (24)$$

From the definition $B_3(x,y)$ and using (24),we have

$$m(x,y)) \le B_3(x,y) \quad (25)$$

Using (25) in inequality (15)

$$z_{xy} \le f(\alpha(x),\beta(y))\varphi(z(\alpha(x),\beta(y)))\alpha'(x)\beta'(y)B_3(x,y) \quad (26)$$

$$\frac{\partial}{\partial y}(\frac{z_x}{\varphi(z(x,y))}) \le f(\alpha(x),\beta(y))\alpha'(x)\beta'(y)B_3(x,y)$$

Integrating above inequality from 0 to y, we have

$$\frac{z_x}{\varphi(z(x,y))} \le \int_0^y f(\alpha(x),\beta(t))\alpha'(x)\beta'(t)B_3(x,t)dt$$

Again integrating above inequality with respect to x, from 0 to x, and using definition of Φ, we have

$$\Phi(z(x,y)) - \Phi(z(0,y)) \le \int_0^x\int_0^y f(\alpha(s),\beta(t))\alpha'(s)\beta'(t)B_3(s,t)dtds$$

$$\Phi(z(x,y)) - \Phi(u_0) \le \int_0^x\int_0^y f(\alpha(s),\beta(t))\alpha'(s)\beta'(t)B_3(s,t)dtds$$

$$\Phi(z(x,y)) - \Phi(u_0) \le \int_0^x\int_0^y f(\alpha(s),\beta(t))\alpha'(s)\beta'(t)B_3(s,t)dtds$$

$$\Phi\big(z(x,y)\big) \le \Phi(u_0) + \int_0^x\int_0^y f\big(\alpha(s),\beta(t)\big)\alpha'(s)\beta'(t)B_3\big(\alpha(s),\beta(t)\big)dtds$$

Making change of variable on both right hand side of the above inequality

$$\Phi(z(x,y)) \le \Phi(u_0) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B_3(s,t)dtds$$

$$z(x,y) \le \Phi^{-1}\{\Phi(u_0) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B_3(s,t)dtds\}$$

As $u(x,y) \le z(x,y)$, we have

$$u(x,y) \le \Phi^{-1}\{\Phi(u_0) + \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)B_3(s,t)dtds\}$$

for all $x,y \in \Delta$

3 APPLICATION

In this section we apply our main result to the following nonlinear partial differential equation.

$$u_{xy} = F(x,y,u(\alpha(x),\beta(y))) + H(x,y,u(\alpha(x),\beta(y),K(x,y,u(\alpha(x),\beta(y))))) \quad (27)$$

for all $x,y \in \Delta$ with the initial boundary conditions

$$u(x,0) = e_1(x), u(0,y) = e_2(y), e_1(0) = e_2(0) \quad (28)$$

where $e_1 \in C^1(I,R)$, $e_2 \in C^1(I,R)$, $F,K \in C(\Delta\times\Delta,R)$, $H \in C(\Delta\times\Delta,R)$ satisfy the follow conditions.

$$|e_1(x) + e_2(y)| < u_0 \quad (29)$$

$$\Big|F\Big(x,y,u\big(\alpha(x),\beta(y)\big)\Big)\Big| \le f^2(\alpha(x),\beta(y)))|\varphi(u(\alpha(x),\beta(y)))|^2 \quad (30)$$

$$|K(x,y,u(\alpha(x),\beta(y))| \le f(\alpha(x),\beta(y)))|\varphi(u(\alpha(x),\beta(y)))| \quad (31)$$

$$|H(x,y,u,v)| \le |v|(\varphi|u|) + 2\int_0^t |v|ds \quad (32)$$

$f(x,y)$ is nonnegative real valued continuous functions defined on Δ.

Corollary 1: Consider the nonlinear system (27) - (28) and suppose that F,K, and H satisfy the conditions (29),(30) , (31) , (32) . Let $\varphi \in C^1(\Delta,I)$ be increasing functions $\varphi_{xy} \le k$, $\alpha,\beta \in C^1(I,I)$ be increasing functions $\alpha(x) \le x, \beta(y) \le y, \alpha(0) = 0, \beta(0) = 0$, for all $x,y \in \Delta$. k is positive constant, then all solutions of equation (27) - (28) exist on I and satisfy the following estimation

$$|u(x,y)| \le \Phi^{-1}\{\Phi(|u_0|) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \frac{f(s,t)B_3(s,t)}{\alpha'(\alpha^{-1}(s))\beta'(\beta^{-1}(t))}dtds\} \quad (33)$$

for all $x,y \in \Delta$

$$\Phi(x,y) = \int_0^r \frac{ds}{\varphi(s)},\ for\ all\ x > o \quad (34)$$

$$B_3(x,y) = \exp\Big(4\int_0^{\alpha(x)}\int_0^{\beta(y)} \frac{f(s,t)}{\alpha'(\alpha^{-1}(s))\beta'(\beta^{-1}(t))}dtds\Big)\times ((\varphi(|u_0|)^{-1}) - \int_0^{\alpha(x)}\int_0^{\beta(y)} \frac{f(s,t)}{\alpha'(\alpha^{-1}(s))\beta'(\beta^{-1}(t))} \times exp(4\int_0^s\int_0^t \frac{f(\sigma,\eta)}{\alpha'(\alpha^{-1}(\sigma))\beta'(\beta^{-1}(\eta))}d\eta d\sigma)dtds) \quad (35)$$

Proof: Integrating both sides of the equation (27) from 0 to x and 0 to y, we get

$$u(x,y) = e_1(x) + e_2(y) + \int_0^x\int_0^y F(s,t,u(\alpha(s)),\beta(t))dtds + \int_0^x\int_0^y H\Big(s,t.u(s,t),K\Big(s,t,u\big(\alpha(s),\beta(t)\big)\Big)\Big)dtds \quad (36)$$

$|u(x,y)| \le$

$$|e_1(x)+e_2(y)|+\int_0^x\int_0^y \left|F\left(s,t,u\big(\alpha(s)\big),\beta(t)\right)dtds\right|+\int_0^x\int_0^y \left|H\left(s,t.u\big(\alpha(s),\beta(t)\big),K\left(s,t,u\big(\alpha(s),\beta(t)\big)\right)\right)\right|dtd \quad (37)$$

Using (29), (30), (31) and (32), in (3.4) and making change of variable,we deduce

$$|u(x,y)|\le |u_0|+\int_0^{\alpha(x)}\int_0^{\beta(y)} f^2(s,t)\varphi(|u(s,t)|)^2\frac{1}{\alpha'(\alpha^{-1}(m),\beta'(\beta^{-1}(n)}dtds+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)|\varphi(|u(s,t)|)|\frac{1}{\alpha'(\alpha^{-1}(s))\beta'(\beta^{-1}(t))}\times(\varphi(|u(s,t)|)+2\int_0^s\int_0^t f(\sigma,\eta)\,|\varphi(|u(\sigma,\eta)|)\frac{1}{\alpha'(\alpha^{-1}(\sigma)),\beta'(\beta^{-1}(\eta))}d\eta d\sigma)dtds \quad (38)$$

Applying Theorem (2.1) to the above inequality , we have

$$|u(x,y)|\le \Phi^{-1}\{\Phi(|x_0|)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\frac{f(s,t)B_3(s,t)}{\alpha'(\alpha^{-1}(s))\beta'(\beta^{-1}(t))}dtds\}$$

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