Some Distance Measures Between Intuitionistic Fuzzy Multisets (IFMSS)

P.A. Ejegwa, Edibo O. P.

Abstract: The axiomatic definition of distance measure between intuitionistic fuzzy multisets (IFMSs) is emphasized. We transformed the four existing distance measures between intuitionistic fuzzy sets to intuitionistic fuzzy multisets since the later is the extension of the former. We use a given example of IFMSs A, B, C in X such that $A \subseteq B \subseteq C$; to verified whether the axioms of the definition of distance measure are satisfiedin the four distance measures adapted from IFSs. We also gave some other distance measures, deduced a proposition and proved respectively.

Keywords: distance measures; fuzzy sets; fuzzy multisets; intuitionistic fuzzy sets; intuitionistic fuzzy multisets.

1. Introduction

The theory of fuzzy sets (FS) introduced by [7] has showed meaningful applications in many field of studies. The idea of fuzzy set is welcomed because it handles uncertainty and vagueness which Cantorian set could not address. Fuzzy set has a membership function that assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration. In fuzzy set theory, the membership of an element to a fuzzy set is a single value between zero and one. However in reality, it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there may be some hesitation degree. Therefore, a generalization of fuzzy sets was introduced by [1] as intuitionistic fuzzy sets (IFS) which incorporated the degree of hesitation called hesitation margin (and is defined as 1 minus the sum of membership and non-membership degrees respectively). Later, the concept of intuitionistic fuzzy multisets (IFMS) was proposed by [3]as the combination of IFS in [2] and fuzzy multiset in [6]. As important concept in fuzzy mathematics, distance measure which is the term that describes the difference between intuitionistic fuzzy sets has gained ample attentions by researchers because of its wide applications in real life. Many distance measures between intuitionistic fuzzy sets have been proposed and researched in recent years [5]. Szmidt and Kacprzyk [4] proposed four distance measures between intuitionistic fuzzy sets, which were partly based on the geometric interpretation of intuitionistic fuzzy sets, and have some good geometric properties.In this research, we provide a more generalized definition of distance measure between IFMS, and then adapt the four distance measures between intuitionistic fuzzy sets proposed by [4] to intuitionistic fuzzy multisets. Also, we check (i.e. verifying) whether the axioms are valid with the four distance measures. Some other distance measures between IFMS will be proposed. Finally, we deduce and prove a proposition.

- *P.A. Ejegwa, Edibo O. P.*
- *Department Of Mathematics, University Of Agriculture, P.M.B. 2373, Makurdi Nigeria*
- *Ocholohi@gmail.com; +234(0)7062583323*

2. Intuitionistic Fuzzy Multisets (IFMS)

Definition 1: Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{\langle x, \mu_A(x)\rangle : x \epsilon X\}$, where $\mu_A(x): X \to [0,1]$is the membership function of the fuzzy set A. Fuzzy set is a collection of objects with graded membership i.e. having degrees of membership.

Definition 2: Let X be a nonempty set. A fuzzy multiset (FMS) A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A: X \to Q$ where Q is the set of all crisp multisets (i.e. non- fuzzy multisets) drawn from the unit interval [*0,1*]. Then for any $x \epsilon X$, the value $CM_A(x)$ is a crispmultiset drawn from [*0,1*]. For each $x \epsilon X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by $\mu_A^1(x), \mu_A^2(x)$,...,$\mu_A^n(x)$ where $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^n(x)$. A fuzzy multiset A in X is defined alternatively as the set of ordered sequence given as $A = \{\langle x, \mu_1(x), \mu_2(x), \mu_3(x), \ldots, \mu_n(x), \ldots\rangle : x \in X\}$, where $\mu_n(x): X \to [0,1]$is the membership function of $A$for $n = 1,2, \ldots$.

Definition 3: Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A(x), \nu_A(x): X \longrightarrow [0,1]$define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of $x$in A. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ i.e.,$\pi_A(x): X \longrightarrow [0,1]$ for every $x \in X$. $\pi_A(x)$, expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 4: Let X be a nonempty set. An IFMS A drawn from X is characterized by two functions: "count membership" of A denoted as CM_A and "count non-membership" of A denoted as CN_Agiven respectively by $CM_A: X \longrightarrow Q$and $CN_A: X \longrightarrow Q$ where Q is the set of all crisp multisets drawn from the unit interval [*0,1*] s.t. for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CM_A(x)$ and it is denoted as $(\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^n(x))$, where $\mu_A^1(x) \geq \mu_A^2(x) \geq \ldots \geq \mu_A^n(x)$ whereas the corresponding non-membership sequence of elements in $CN_A(x)$ is denoted by $(\nu_A^1(x), \nu_A^2(x), \ldots, \nu_A^n(x))$ *s.t.* $0 \leq \mu_A^i(x) + \nu_A^i(x) \leq 1$ for every $x \in X$ and $i = 1, \ldots, n$. We define IFMS alternatively. Let X be nonempty set. An IFMS A drawn from X is given as $A = \{\langle \mu_A^1(x), \ldots, \mu_A^n(x), \ldots, \nu_A^1(x), \ldots, \nu_A^n(x), \ldots \rangle x \in X\}$ where the functions $\mu_A^i(x), \nu_A^i(x): X \to [0,1]$ define the belongingness degrees and the non-belongingness degrees of A in X *s.t.* $0 \leq \mu_A^i(x) + \nu_A^i(x) \leq 1$ for $i = 1, \ldots$ If the sequence of the membership functions and non-membership (belongingness functions and non-belongingness functions) have only n-terms (i.e. finite), n is called the '*dimension*' of A. Consequently $A = \{\langle \mu_A^1(x), \ldots, \mu_A^n(x), \nu_A^1(x), \ldots, \nu_A^n(x) \rangle x \in X\}$ for $i = 1, \ldots, n$. when no ambiguity arises, we define $A = \{\langle \mu_A^i(x), \nu_A^i(x) \rangle x \in X\}$ for $i = 1, \ldots, n$. For each IFMS A in X, $\pi_A^i(x) = 1 - \mu_A^i(x) - \nu_A^i(x)$ is the intuitionistic fuzzy multisets index or hesitation margin of x in A. The hesitation margin $\pi_A^i(x)$ *for each* $i = 1, \ldots, n$ is the degree of non-determinacy of $x \in X$, to the set A and $\pi_A^i(x) \in [0,1]$. The function $\pi_A^i(x)$ expresses lack of knowledge of whether x *in* A or not. In general, an IFMS A is given as $A = \{\langle x, \mu_A^i(x), \nu_A^i(x), \pi_A^i(x) \rangle : x \in X\}$.

Note: 1. $\mu_A^i(x) + \nu_A^i(x) + \pi_A^i(x) = 1$. 2. IFMS is an extension of IFS.

3. Distance Measures between Intuitionistic Fuzzy Multisets

Definition 5: Let X be nonempty such that IFMSs $A, B, C \in X$. Then distance measure d is a mapping $d: X \times X \to [0, 1]$; if $d(A, B)$ satisfies the following axioms:

$A1$ $0 \leq d(A, B) \leq 1$;

$A2$ $d(A, B) = 0$ if and only if $A = B$ (i.e. faithful condition);

$A3$ $d(A, B) = d(B, A)$ (i.e. symmetric);

$A4$ $d(A, C) + d(B, C) \geq d(A, B)$ (i.e. triangle inequality);

$A5$ if $A \subseteq B \subseteq C$, then $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$.

Then $d(A, B)$ is a distance measure between IFMS A and B. Distance measure is a term that describes the difference between intuitionistic fuzzy multisets and can be considered as a dual concept of similarity measure. Many distance measures between IFSs have been proposed and researched in recent years. We make use of the four distance measures proposed by [5] in terms of IFMS since they have some good geometric properties and satisfied the conditions of metric distance. Let $A = \{\langle x, \mu_A^i(x_j), \nu_A^i(x_j), \pi_A^i(x_j) \rangle : x \in X\}$ and $B = \{\langle x, \mu_A^i(x_j), \nu_A^i(x_j), \pi_A^i(x_j) \rangle : x \in X\}$ be two $IFMSs$ in $X = \{x_1, x_2, \ldots, n\}, j = 1,2, \ldots n$; then the following distances, measure the distance between A and B as follow.

The Hamming distance

$$d(A,B) = \frac{1}{2} \sum_{i=1,j=1}^{n} \{ \left|\mu_A^i(x_j) - \mu_B^i(x_j)\right| + \left|\nu_A^i(x_j) - \nu_B^i(x_j)\right| + \left|\pi_A^i(x_j) - \pi_B^i(x_j)\right| \}$$

The Euclidean distance;

$$d(A,B) = \{\frac{1}{2} \sum_{i=1,j=1}^{n} [(\mu_A^i(x_j) - \mu_B^i(x_j))^2 + (\nu_A^i(x_j) - \nu_B^i(x_j))^2 + (\pi_A^i(x_j) - \pi_B^i(x_j))^2]\}^{\frac{1}{2}}$$

The normalized Hamming distance;

$$d(A,B) = \frac{1}{2n} \sum_{i=1,j=1}^{n} \{ \left|\mu_A^i(x_j) - \mu_B^i(x_j)\right| + \left|\nu_A^i(x_j) - \nu_B^i(x_j)\right| + \left|\pi_A^i(x_j) - \pi_B^i(x_j)\right| \}$$

The normalized Euclidean distance ;

$$d(A,B) = \{\frac{1}{2n} \sum_{i=1,j=1}^{n} [(\mu_A^i(x_j) - \mu_B^i(x_j))^2 + (\nu_A^i(x_j) - \nu_B^i(x_j))^2 + (\pi_A^i(x_j) - \pi_B^i(x_j))^2]\}^{\frac{1}{2}}$$

4. Numerical Verification of the Distance Measures

Let A, B and C be $IFMSs$ on $X = \{x\}$, $A = \{(0.1,0.2), (0.3,0.3), (0.6,0.5)\}$, $B = \{(0.2,0.3), (0.4,0.5), (0.4,0.2)\}$, $C = \{(0.3,0.4), (0.5,0.6), (0.2,0.0)\}$. For easy analysis, we convert $IFMS$ to IFS by taking the mean value. Then $A = \{0.15, 0.3, 0.55\}$, $B = \{0.25, 0.45, 0.3\}$, $C = \{0.35, 0.55, 0.1\}$.

Using Hamming distance

A1: $d(A, B) = \frac{1}{2}(|\ 0.15 - 0.25| + |\ 0.3 - 0.45| + |\ 0.55 - 0.3| = 0.25$. Since $0 \leq d(A,B) \leq 1$, $A1$ is verified.
A2: let $A = B$, $d(A, B) = 0$ verifies $A2$.
A3: $d(A, B) = 0.25$, $d(B, A) = 0.25$. This implies that $A3$ is verified.
A4: $d(A, B) = 0.25$, $d(A, C) = 0.45$, $d(B, C) = 0.2$. Since $d(A, C) + d(B, C) \geq d(A, B)$, $A4$ is verified.
A5: Since $A \subseteq B \subseteq C$ from the given example above, then $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$ verifies $A5$.

Using Euclidean distance

A1: $d(A, B) = \sqrt{\frac{1}{2}\{(0.15 - 0.25)^2 + (0.3 - 0.45)^2 + (0.55 - 0.3)^2\}} = 0.2179$ satisfies $A1$.
A2: Take $A = B$, $d(A, B) = 0.0$ justifies $A2$.
A3: $d(A, B) = 0.2179$, $d(B, A) = 0.2179$. This verifies $A3$.
A4: $d(A, B) = 0.2179$, $d(B, C) = 0.1732$, $d(A, C) = 0.3905$. Since $d(A, C) + d(B, C) \geq d(A, B)$, $A4$ is verified.
A5: It is obvious that $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$. $A5$ is verified

Using n –Hamming distance

Since the cardinality of the given set is 2, it means $n = 2$.
A1: $d(A, B) = 0.125$. Since $0 \leq d(A, B) \leq 1$, $A1$ is verified.
A2: let $A = B$, $d(A, B) = 0$ verifies $A2$.
A3: $d(A, B) = 0.25$, $d(B, A) = 0.25$. Then $A3$ is true.

A4: $d(A,B) = 0.25, d(A,C) = 0.45, d(B,C) = 0.2$. Since the sum of $d(A,C)$ and $d(B,C)$ is greater than $d(A,B)$, $A4$ is verified.
A5: Since $A \subseteq B \subseteq C$ from the given example above, then $d(A,C) \geq d(A,B)$ and $d(A,C) \geq d(B,C)$ validates $A5$.
Using n − Euclidean distance
A1: $d(A,B) = 0.1541$ satisfies $A1$ since $0 \leq d(A,B) \leq 1$.
A2: Take $A = B, d(A,B) = 0.0$ satisfies $A2$.
A3: $d(A,B) = 0.1541, d(B,A) = 0.1541$. We conclude that $A3$ is satisfies since $d(A,B) = d(B,A)$.
A4: $d(A,B) = 0.1541, d(B,C) = 0.1225, d(A,C) = 0.2761$. Since $d(A,C) + d(B,C) \geq d(A,B)$, $A4$ is satisfied.
A5: It is clear that $d(A,C) \geq d(A,B)$ and $d(A,C) \geq d(B,C)$. $A5$ is verified.

Proposition: Let X be nonempty and A, B be two IFMSs in $X = \{x\}$. Then $d(A,B) = d(A^c, B^c)$.

Proof: Let $A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B^i(x), \nu_B^i(x)\rangle : x \in X\}$ ignoring any chance of hesitation. The complement of A is defined as $A^c = \{\langle x, 1 - \mu_A^i(x), 1 - \nu_A^i(x)\rangle : x \in X\} = \{\langle x, \nu_A^i(x), \mu_A^i(x)\rangle : x \in X\}$ and consequently $B^c = \{\langle x, \nu_B^i(x), \mu_B^i(x)\rangle : x \in X\}$. Since the only difference between A and A^c (likewise B and B^c) is the issue of arrangement, obviously the distance between $d(A,B)$ and $d(A^c, B^c)$ is the same. Therefore, the result follows. Here we give some other distance measures that conformed with the axioms of distance measure as stated above.

1. $d(A,B) = \frac{\sum_{i,j=1}^{n} |S_A^i(x_j) - S_B^i(x_j)|}{2n}$ where $S_A^i(x_j) = \mu_A^i(x_j) - \nu_A^i(x_j)$ and $S_B^i(x_j) = \mu_B^i(x_j) - \nu_B^i(x_j)$ are called core of A and B or degree of support of A and B respectively.
2. $d(A,B) = \frac{\sum_{i,j=1}^{n} |S_A^i(x_j) - S_B^i(x_j)|}{4n} - \frac{\sum_{i,j=1}^{n} |\mu_A^i(x_j) - \mu_B^i(x_j)| + |\nu_A^i(x_j) - \nu_B^i(x_j)|}{4n}$ where $S_A^i(x_j)$ and $S_B^i(x_j)$ are as above.
3. $d(A,B) = \sqrt[p]{\frac{\sum_{i,j=1}^{n} |\varphi_A^i(x_j) - \varphi_B^i(x_j)|^p}{n}}$ where p is a positive integer parameter i.e. $1 \leq p \leq \infty$, $\varphi_A^i(x_j) = \frac{\mu_A^i(x_j) + 1 - \nu_A^i(x_j)}{2}$ and $\varphi_B^i(x_j) = \frac{\mu_B^i(x_j) + 1 - \nu_B^i(x_j)}{2}$.
4. $d(A,B) = \sqrt[p]{\frac{\sum_{i,j=1}^{n} (\phi_t^i(x_j) - \phi_f^i(x_j))^p}{n}}$ where $\phi_t^i(x_j) = \frac{|\mu_A^i(x_j) - \mu_B^i(x_j)|}{2}$, $\phi_f^i(x_j) = \left| \frac{[1 - \nu_A^i(x_j)]}{2} - \frac{[1 - \nu_B^i(x_j)]}{2} \right|$. This measure makes use of two end points of the subinterval in IFMSs.
5. $d(A,B) = \sqrt[p]{\frac{\sum_{i,j=1}^{n} (\eta_1^i(j) + \eta_2^i(j) + \eta_3^i(j))^p}{3n}}$ where $\eta_1^i(j) = \phi_t^i(x_j) + \phi_f^i(x_j)$, $\eta_2^i(j) = \varphi_A^i(x_j) - \varphi_B^i(x_j)$, $\eta_3^i(j) = \max(l_A^i(j), l_B^i(j)) - \min(l_A^i(j), l_B^i(j))$ denote degree of the length among which $l_A^i(j) = \frac{(1 - \nu_A^i(x_j) - \mu_A^i(x_j))}{2}$, $l_B^i(j) = \frac{(1 - \nu_B^i(x_j) - \mu_B^i(x_j))}{2}$. This is relative powerful method that considers more information to measure the distance degree of IFMSs.
6. $d(A,B) = \frac{1}{n} \sum_{i,j=1}^{n} \max(|\mu_A^i(x_j) - \mu_B^i(x_j)|, |\nu_A^i(x_j) - \nu_B^i(x_j)|)$
7. $d(A,B) = \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i,j=1}^{n} [\phi_t^i(j) + \phi_f^i(j)]^p}$ where $\phi_t^i(j) = \frac{|\mu_A^i(x_j) - \mu_B^i(x_j)|}{2}$, $\phi_f^i(j) = \frac{|\nu_A^i(x_j) - \nu_B^i(x_j)|}{2}$, and p is a positive integer i.e. $0 \leq p \leq \infty$.

5. Conclusion

We have seen that the four distance measures transformed from IFS to IFMS are valid because they satisfied the five axioms of the definition of distance measure introduced by [5] and also, the seven distance measures proposed seem to be lucrative in applications.

References

[1]. Atanassov K., Intuitionistic fuzzy sets, VII ITKR's Session, Sofia, 1983.

[2]. Atanassov K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87.

[3]. Shinoj T. K, Sunil J. J, Intuitionistic fuzzy multisets and its application in medical diagnosis, Int. J. of Mathematical and Computational Sciences 6 (2012).

[4]. Szmidt, E., Kacprzyk, J., Distances between intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, vol. 114, No. 3(2000) 505-518.

[5]. Wang, W., Xin, X., Distances between intuitionistic fuzzy sets, Pattern Recognition Letters 26 (2005) 2063-2069.

[6]. Yager R. R., On the theory of bags, Int. J. of General Systems 13 (1986) 23-37.

[7]. Zadeh L. A, Fuzzy sets, Inform. and Control 8 (1965) 338-353.