Fractionalization Of Fourier Sine And Fourier Cosine Transforms And Their Applications

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Abstract: The fractional Fourier transform (FrFT) is a generalization of classical Fourier transform and received considerable attention of researchers since last four decades due to its wide ranging applicability in various fields such as, electrical engineering, optics, signal processing, signal analysis, optical communication and quantum mechanics. Fractional Fourier sine transform (FrFST) and fractional Fourier cosine transform (FrFCT) are closely related to the fractional Fourier transform (FrFT). In this paper, we introduce the new definition of FrFST and FrFCT of real order $\alpha$ using Mittag-Leffler function in fractional calculus environment. We have derived many algebraic properties including inversion formula, modulation theorem and Parseval’s identity of this new FrFST and FrFCT analytically. In addition, we have discussed FrFST and FrFCT of some standard functions and have applied these transforms to obtain analytical solutions of fractional heat-diffusion and fractional wave equations.

Keywords: Mittag-Leffler function, Fractional Fourier transform, fractional derivative and fractional integral, fractional partial differential equation.

Introduction:

The Fourier transform is one of the most valuable and widely used integral transform because it has wide range of applications in the field of applied mathematics, physics, engineering, science and technology [1-3]. The well-known definition of Fourier transform of the function $f(x)$ is given as [3]

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) \, dx,$$

and, its inversion formula is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{f}(\omega) \, d\omega$$

provided that the above integrals converge.

The fractional Fourier transform (FrFT) is now established as the most extensively used tool in almost every area of science and engineering. It was first originated by Wiener [4] in 1929. In 1980, Namias [5] proposed consolidated definition of FrFT and developed some of its theoretical properties. He applied it in the field of quantum mechanics. His work received huge attention of researchers/engineers. As an extension of Fourier transform, FrFT produced superior results as compared to Fourier transform and generated enormous applications in almost every area of science and engineering specifically in the field of signal processing, signal analysis, optical communication and quantum mechanics [6-11]. Subsequently many other scientists defined fractional Fourier transforms differently and enriched its theory properties [12]. Bolanga and West [13] introduced generalized Fourier transform as extension of Fourier transform in the fractional calculus environment. Recently a novel definition of FrFT of real order $\alpha$, $0 < \alpha \leq 1$ has been introduced by Jumarie [20] using Mittag-Leffler function. This transform plays the same role for the fractional derivatives as Fourier transform plays for the ordinary derivatives. This FrFT is reduced into the Fourier transform particularly for $\alpha = 1$ in usual sense. Hence it is better suited for the definition of FrFT as compared to the other definitions proposed earlier in the literature. Lohmann [15] firstly defined FrFST and FrFCT with their applications in the field of signal processing in 1996. After some time Pei Soo-Chang [16] redefined the fractional sine and cosine transform based on fractional Fourier transform in 2001.

2. Preliminaries

In this section, we present some basic concepts of fractional calculus, which are closely related to the work carried out in this paper.

Definition 2.1. Jumarie [20] proposed following modified Riemann-Liouville derivative

$$D^\alpha x f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_{0}^{x} (x - \xi)^{-\alpha - 1} f(\xi) \, d\xi, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{0}^{x} (x - \xi)^{-\alpha} (f(\xi) - f(0)) \, d\xi, & 0 < \alpha < 1 \\ (f^{(\alpha-n)}(x))^{n}, & n \leq \alpha < n + 1; n \geq 1 \end{cases} \tag{1}$$

where $f : R \rightarrow R$ is continuous function.

Definition 2.2. In 1903, Swedish mathematician Gosta Mittag-Leffler defined following function known as Mittag-Leffler function [21].

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\alpha n + 1)} \tag{2}$$

where $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ and $z \in \mathbb{C}$. The Mittag - Leffler function reduces to exponential function for $\alpha = 1$. Mittag-Leffler function satisfies following identity [14]

$$E_{\alpha}(\lambda u^{\alpha}) = E_{\alpha}(\lambda v^{\alpha}), \quad \lambda \in \mathbb{C} \tag{3}$$

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References: For more details, see [1-11].
Definition 2.4. The Gamma function of fractional order, $\alpha$, is defined by

\[ \Gamma(\alpha) = \int_0^\infty e^{-x} x^{(\alpha-1)} dx \]  

This formula does not apply for classical Riemann-Liouville derivative, but it applies with modified Riemann-Liouville derivative.

Definition 2.5. The Fourier sine transform for the function $f(x)$ over the interval $(0, \infty)$ is defined as

\[ \hat{f}_s(\omega) = \int_0^\infty f(x) \sin \omega x \, dx, \quad \omega > 0 \]  

where the function $f(x)$ denotes inverse Fourier sine transform of $\hat{f}_s(\omega)$ and it is defined as follows

\[ f(x) = \frac{2}{\pi} \int_{\pi/2}^{\infty} \hat{f}_s(\omega) \sin \omega x \, d\omega, \quad \omega > 0 \]  

Definition 2.6. The Fourier cosine transform of the function $f(x)$ over the interval $(0, \infty)$ is defined as

\[ \hat{f}_c(\omega) = \int_0^\infty f(x) \cos \omega x \, dx, \quad \omega > 0 \]  

Therefore its inversion formula is given by

\[ f(x) = \frac{1}{2} \int_{\pi/2}^{\infty} \hat{f}_c(\omega) \cos \omega x \, d\omega, \quad \omega > 0 \]  

Definition 2.7. If function $f(x)$, $\mathbb{R} \rightarrow \mathbb{C}$, $x \rightarrow f(x)$ is continuous and piecewise $\alpha$th continuously differentiable, then the fractional Fourier transform of function $f(x)$ is given by

\[ \mathcal{F}_\alpha[f(x)] = \int_{-\infty}^{\infty} E_{\alpha}(i\omega^\alpha x^\alpha) \, f(x) \, dx, \quad 0 < \alpha \leq 1 \]  

and its inversion formula is given by

\[ f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \mathcal{F}_\alpha^{-1}[\hat{f}_\alpha(\omega)] \, d\omega, \quad 0 < \alpha \leq 1 \]  

where $(dx)^\alpha$ denotes the fractional integral expressed as follows [14]

\[ \int_0^\infty u(x) \, dx = \int_0^\infty (v \cdot x)^{-1+\alpha} \, dx \]  

Definition 2.8: The fractionalization of $\sin x$ and $\cos x$ is defined in slightly different form [14] as follows

\[ \cos_{\alpha} x^\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!(\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\a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Provided \( f(x) \) is an even function, when the integral converges and a sufficient condition for convergence is
\[
\int_0^\infty |f(x)| \, (dx)^\alpha < \infty
\]

Similarly, we can establish inversion formula for fractional Fourier cosine transform

**Theorem 3.8.** if \( \mathcal{F}_c^\alpha(\omega) \) is the fractional Fourier cosine transform of the function \( f(x) \), then its inversion formula is given by the following equality
\[
\int_0^\infty \mathcal{F}_c^\alpha(\omega) \, ( dw )^\alpha = \frac{1}{2} \left( \sin \left( \frac{\omega \alpha}{2} \right) \mathcal{F}_c(\omega) - \sin \left( \frac{\omega \alpha}{2} \right) \mathcal{F}_c(\omega) \right)
\]

and
\[
\int_0^\infty \mathcal{F}_c^\alpha(\omega) \, ( dw )^\alpha = \frac{1}{2} \left( \sin \left( \frac{\omega \alpha}{2} \right) \mathcal{F}_c(\omega) + \sin \left( \frac{\omega \alpha}{2} \right) \mathcal{F}_c(\omega) \right)
\]

**Example 5.4.** Let us consider a following function
\[
\begin{align*}
\mathcal{F}_s^\alpha(\omega) &= \frac{x^{am} e^{(-ic)^\alpha x^a}}{\omega^{1+am}} \left( \frac{1}{\omega^{1+am}} \right) \sin \left( \frac{\omega \alpha}{2} \right) \\
\mathcal{F}_c^\alpha(\omega) &= \frac{1}{\omega^{1+am}} \left( \frac{1}{\omega^{1+am}} \right) \sin \left( \frac{\omega \alpha}{2} \right)
\end{align*}
\]

**Applications**

In this section, the analytical solutions of fractional heat diffusion and fractional wave equations are obtained using FrFST and FrFCT.

**Example 6.1:** We consider a fractional heat-diffusion equation in the form of \( \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k \frac{\partial^2 u(x,t)}{\partial x^{2\alpha}} \), \( x \in (0,\infty), t \in (0,\infty), 0 < \alpha \leq 1 \)

with the initial and boundary value problems

\[
\begin{align*}
(i) & \quad u(0,t) = 0 \\
(ii) & \quad u(x,0) = \delta(x - b), b > 0
\end{align*}
\]

where \( u(x,t) \) is the temperature function of the body and \( k \), the thermal diffusivity of the body.

Taking the fractional Fourier sine transform of both sides of equation (27) with respect to \( x \), we have
\[
\frac{\partial^\alpha \mathcal{F}_s(\omega,t)}{\partial t^\alpha} = k \int_0^\infty \frac{\partial^2 \mathcal{F}_s(\omega,u)}{\partial x^{2\alpha}} \sin \left( \frac{\omega \alpha}{2} \right) (dx)^\alpha
\]
Using fractional integration by parts formula in right-hand side of (30), we have
\[
\frac{d^\alpha \tilde{h}_\alpha(x,t)}{d x^\alpha} = k \left[ (\alpha + 1) \sin_a(\omega x)^\alpha \frac{\partial^2 u(x,t)}{\partial x^2} \right]_{x=0} + k \left[ -\omega^\alpha \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \cos_s(\omega x)^\alpha \ (dx)^a \right] = -k\omega^\alpha \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \cos_s(\omega x)^\alpha \ (dx)^a
\]
Again, applying fractional integration by parts formula in right-hand side of above equation and using (28), we have
\[
\frac{d^\alpha \tilde{h}_\alpha(x,t)}{d x^\alpha} = -k\omega^2 \tilde{h}_\alpha(x,t)
\]
Solution of above equation given as
\[
\tilde{h}_\alpha(x,t) = AE_a(-k\omega^2 t^\alpha), \text{ where } A \text{ is an arbitrary constant.}
\]
Using initial condition (29), we have
\[
\tilde{h}_\alpha(x,0) = \alpha \sin_a(\omega b)^\alpha E_a(-k\omega^2 t^\alpha)
\]
Taking inverse fractional Fourier sine transform of above equation, we have
\[
u(x,t) = \frac{2\alpha}{(M^\alpha_\alpha)^\alpha} \int_0^{\infty} \sin_a(\omega x)^\alpha \sin_a(\omega b)^\alpha E_a(-k\omega^2 t^\alpha) (dw)^\alpha
\]
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = -k\omega^2 \tilde{h}_\alpha(x,t)
\]
Which is the required solution of fractional heat diffusion equation.

Example 6.2: we consider a fractional heat-diffusion equation in the form of [23]
\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,+\infty), \ t \in (0,+\infty), \ 0 < \alpha \leq 1
\]
with the initial and boundary value problems
(i) \[
\frac{\partial u(x,0)}{\partial t^\alpha} = 0
\]
(ii) \[
u(x,0) = \delta_s(x)
\]
where \(u(x,0)\) is the temperature function of the body and \(k(\kappa=0, +\infty)\) is the thermal diffusivity of the body.

Applying the fractional Fourier cosine transform of both sides of equation (31) with respect to \(x\), we have
\[
\frac{d^\alpha \tilde{h}_\alpha(x,t)}{d x^\alpha} = \frac{k}{(M^\alpha_\alpha)^\alpha} \int_0^\infty \sin_a(\omega x)^\alpha \sin_a(\omega b)^\alpha E_a(-k\omega^2 t^\alpha) (dw)^\alpha
\]
Using fractional integration by parts formula in right hand side of (34), we have
\[
u(x,t) = \frac{2\alpha}{(M^\alpha_\alpha)^\alpha} \int_0^{\infty} \sin_a(\omega x)^\alpha \sin_a(\omega b)^\alpha E_a(-k\omega^2 t^\alpha) (dw)^\alpha
\]
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = -k\omega^2 \tilde{h}_\alpha(x,t)
\]
whose solution is given by in the term of Mittag-Leffler function as follows
\[
\tilde{h}_\alpha(x,t) = AE_a(-k\omega^2 t^\alpha), \text{ where } A \text{ is an arbitrary constant.}
\]
Using (34), we have
\[
\tilde{h}_\alpha(x,t) = \alpha E_a(-k\omega^2 t^\alpha)
\]
Applying inverse fractional Fourier cosine transform of above equation, we have
\[
u(x,t) = \frac{2\alpha}{(M^\alpha_\alpha)^\alpha} \int_0^{\infty} E_a(-k\omega^2 t^\alpha) \cos_s(\omega x)^\alpha (dw)^\alpha
\]
Using (4) in above equation, we have the required solution in series form
\[
u(x,t) = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(1-\alpha) M^\alpha_\alpha}{2\pi m!} \left( \frac{1}{k\omega^2 t^\alpha} \right)^2 \frac{1}{k\omega^2 t^\alpha}
\]

Example 6.3: we consider a fractional wave equation in the form of [23]
\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c^2 a \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0, +\infty), \ t \in (0, +\infty), \ 0 < \alpha \leq 1
\]
with the initial and boundary value problems
(i) \[
\frac{\partial u(x,0)}{\partial t^\alpha} = 0
\]
(ii) \[
u(x,0) = \delta_s(x-b), \ b > 0
\]
(iii) \[
\frac{\partial u(x,0)}{\partial x^\alpha} = 0
\]
where \(u(x, t)\) is the displacement function of the string and \(c \in (0, +\infty)\) is the thermal diffusivity of the strings.

Applying the fractional Fourier sine transform of both sides of equation (35) with respect to \(x\), we have
\[
\frac{d^\alpha \tilde{h}_\alpha(x,t)}{d x^\alpha} = c^2 a \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \sin_a(\omega x)^\alpha (dx)^a
\]
Using fractional integration by parts formula in right-hand side of above equation and using (28), we have
\[
\frac{d^\alpha \tilde{h}_\alpha(x,t)}{d x^\alpha} = c^2 a \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \sin_a(\omega x)^\alpha (dx)^a
\]
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = -c^2 a \omega^2 \tilde{h}_\alpha(x,t)
\]
Equation (40) can be rewritten in the following form
\[
(D_t^\alpha + c^2 a \omega^2)(D_t^\alpha - c^2 a \omega^2) \tilde{h}_\alpha(x,t) = 0
\]
We assume that
\[
(D_t^\alpha + c^2 a \omega^2) \tilde{h}_\alpha(x,t) = \tilde{h}_\alpha(x,t)
\]
then equation (41) can be expressed as
\[
D_t^\alpha \tilde{h}_\alpha(x,t) = c^2 a \omega^2 \tilde{h}_\alpha(x,t)
\]
On fractional integration in above equation, we have
\[
\tilde{h}_\alpha(x,t) = A_1 E_a(ic^2 a \omega^2 t^\alpha)
\]
where, \(A_1\) is any arbitrary constant, therefore, we have
\[
(D_t^\alpha + c^2 a \omega^2) \tilde{h}_\alpha(x,t) = A_1 E_a(ic^2 a \omega^2 t^\alpha)
\]
Multiplying both sides by \(E_a(ic^2 a \omega^2 t^\alpha)\) in (24) and subsequently applying fractional integration of both sides, we have
\[
\tilde{h}_\alpha(x,t) = A_1 E_a(ic^2 a \omega^2 t^\alpha) + B E_a(-ic^2 a \omega^2 t^\alpha)
\]
where \(A = \frac{A_1}{2ic^2 a \omega^2}

Using (37) and (38), we have
\[
\tilde{h}_\alpha(x,t) = \frac{\alpha}{2} \sin_a(\omega b)^\alpha E_a(ic^2 a \omega^2 t^\alpha) + E_a(-ic^2 a \omega^2 t^\alpha)
\]
Applying inversion formula of FrFST in above equation, we have
\[
u(x,t) = \frac{2\alpha}{(M^\alpha_\alpha)^\alpha} \int_0^{\infty} \sin_a(\omega x)^\alpha \sin_a(\omega b)^\alpha \cos_s(cot)^\alpha (dw)^\alpha
\]
\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\alpha}{(M^\alpha_\alpha)^\alpha} \int_0^{\infty} \cos_s(cot)^\alpha [\cos_s(\omega(x-y)^\alpha)] (dw)^\alpha
\]
Therefore, using (18), we have the required solution
\[
u(x,t) = \delta_s(x-b+c) - \delta_s(x+b+c)
\]

Example 6.4: we consider a fractional wave equation in the form of [23]
\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c^2 a \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0, +\infty), \ t \in (0, +\infty), \ 0 < \alpha \leq 1
\]
With the initial and boundary value problems
(i) \[
\frac{\partial u(x,0)}{\partial t^\alpha} = 0
\]
(ii) \[
u(x,0) = \delta_s(x), \text{ and}
\]
(iii) \[
\frac{\partial u(x,0)}{\partial x^\alpha} = 0
\]
where \( u(x,t) \) is the displacement function of the string and \( c \) (\( c \in (0, +\infty) \)) is the thermal diffusivity of the strings.

Taking the fractional Fourier cosine transform of both sides of equation (43) with respect to \( x \), we have
\[
\frac{d^{2\alpha} \hat{u}_c(\omega,t)}{dt^{2\alpha}} = c^{2\alpha} \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^{2\alpha}} \cos_\alpha(\omega x)^a \ (dx)^a
\]
(47)

Using fractional integration by parts formula in right hand side of (47) and using (44), we have
\[
\frac{d^{2\alpha} \hat{u}_c(\omega,t)}{dt^{2\alpha}} = c^{2\alpha} \left[ \Gamma(\alpha + 1) \cos_\alpha(\omega x)^a \frac{\partial^2 u(x,t)}{\partial x^{\alpha}} \right]_{x=0}^{x=\infty}
\]
\[+ c^{2\alpha} \left[ \omega \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^{\alpha}} \sin_\alpha(\omega x)^a \ (dx)^a \right]
\]
\[= c^{2\alpha} \omega \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^{\alpha}} \sin_\alpha(\omega x)^a \ (dx)^a \]
(48)

Again, applying fractional integration by parts formula in right hand side of above equation, we have
\[
\frac{d^{2\alpha} \hat{u}_c(\omega,t)}{dt^{2\alpha}} = -c^{2\alpha} \omega^{2\alpha} \hat{u}_c(\omega,t)
\]
(49)

Equation (48) can be rewritten in the following form
\[(D^\alpha_c + ic^a \omega^n)(D^\alpha_c - ic^a \omega^n) \hat{u}_c(\omega,t) = 0 \quad (50)
\]
We assume that \((D^\alpha_c + ic^a \omega^n) \hat{u}_c(\omega,t) = \hat{v}_c(\omega,t)\) then equation (49) can be expressed as
\[D^\alpha_c \hat{v}_c(\omega,t) = ic^a \omega^n \hat{v}_c(\omega,t)
\]
(51)

On fractional integration in above equation, we have
\[\hat{v}_c(\omega,t) = A \hat{E}_a(ic^a \omega^n t^a) + B \hat{E}_a(-ic^a \omega^n t^a), \quad \text{where} \quad A = \frac{\hat{A}}{2ic^a \omega^n t^a}
\]
where \( A \) is an arbitrary constant, therefore, we have
\[D^\alpha_c \hat{u}_c(\omega,t) + ic^a \omega^n \hat{u}_c(\omega,t) = A \hat{E}_a(ic^a \omega^n t^a)
\]
(52)

Multiplying both sides by \( E_a(ic^a \omega^n t^a) \) in (24) and subsequently applying fractional integration of both sides, we have
\[
\hat{u}_c(\omega,t) = A \hat{E}_a(ic^a \omega^n t^a) + B \hat{E}_a(-ic^a \omega^n t^a)
\]
\[= \frac{\hat{A}}{2ic^a \omega^n t^a}
\]
(53)

Using (45) and (46), we have
\[
\hat{u}_c(\omega,t) = \frac{\hat{A}}{2} [\hat{E}_a(ic^a \omega^n t^a) + \hat{E}_a(-ic^a \omega^n t^a)]
\]

Taking inversion formula of FrFCT in above equation, we have
\[
u(x,t) = 2 \frac{\hat{A}}{[M^a_{2\alpha}]} \int_0^{\infty} \cos_a(\omega x)^a \cos_c(\omega x)^a (dw)^a
\]
\[= \frac{\hat{A}}{[M^a_{2\alpha}]} \int_0^{\infty} \left[ \cos_a(\omega x + \delta a) \right] (dw)^a
\]
\[+ \frac{\hat{A}}{[M^a_{2\alpha}]} \int_0^{\infty} \left[ \cos_a(\omega x - \delta a) \right] (dw)^a
\]
Therefore, using (18), we have the required solution
\[
u(x,t) = \frac{1}{2} [\delta_a(x + ct) + \delta_a(x - ct)]
\]

Conclusion:
In this paper, we have introduced the definitions of fractional Fourier sine transform and fractional Fourier cosine transform. Some of the algebraic properties of these transforms including Parseval’s identity have been established analytically. We have obtained FrSST and FrFCT of some standard functions. In addition, we have obtained analytical solutions of fractional heat diffusion and fractional wave equations using these newly defined fractional transforms.

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References.

