Approximate Analytic Solution For Self-Gravitating Isothermal Gas Sphere

Aswathy Mary Prince, Sanish Thomas

Abstract: In modeling a self gravitating Isothermal Gas Sphere, isothermal Lane Emden Equation is used which cannot be solved analytically. In our present study we have solved the problem using iterative method. The obtained result almost matches with the earlier results. So the method provides an efficient and accurate way to solve this type of problems.

Index terms: Lane Emden Equation, Polytropic model, Isothermal gas sphere, Iterative method

Introduction

Self gravitating gas spheres have been used in diversified area of Astrophysics, such as stellar interiors, star clusters, galaxies and galactic clusters (Binney & Tremaine, 1987, Horedt, 2004). Polytropic stellar structure equation is established by assuming the star to be spherically symmetric and that does not have a magnetic field. Four basic equations- mass conservation, hydrostatic equilibrium, energy conservation and energy transport describes stellar interior for gas spheres. When effects of temperature are integrated in astrophysical phenomena, it can be described by Lane Emden type equations. When temperature is constant the isothermal Lane Emden equation is considered.

Isothermal gas sphere

A polytrope with polytopic index $n=\infty$ represents an isothermal self gravitating gas sphere. For an ideal gas temperature is proportional to $\rho^{\frac{1}{n}}$ .So $n=\infty$ establish a state of constant temperature Consider an isothermal gas sphere in gravitational equilibrium, whose pressure is given by the relation

$$P=\rho K+D \qquad (1)$$

$K=\frac{k}{\mu H}T$ and $D=\frac{a}{3}T^4$

Combining the hydrostatic equilibrium and mass conservation equations and substituting equation 1 for pressure, we obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\log\rho}{dr}\right)=-4\pi G\rho \qquad (2)$$

Introducing dimensionless variables

$$\rho=\rho_c e^{-\varphi}\ ; \quad r=ax=\left(\frac{K}{4\pi G\rho_c}\right)^{1/2}x$$

ρ_c is assumed to be the central density for a complete isothermal configuration

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Equation 2 reduces to the form

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\varphi}{dx}\right)=e^{-\varphi} \qquad (3)$$

This is the Lane Emden equation for isothermal gas sphere having boundary conditions $\varphi=0,\ \frac{d\varphi}{dx}=0$ at $x=0$ In this paper second order differential equation for isothermal gas sphere is solved using New Iterative Method. Approximate analytic solution for the case is given by Iacono(2014), Liu(1996), Mirza(2009).

New Iterative Method

Consider the functional equation

$$t(x)=N(t)+f(x) \qquad (4)$$

Where N is the nonlinear operator from a Banach space B→B and $f(x)$ is a known function.

The solution of equation (4) is assumed to be in the series form (Daftardar, Jafari, 2006)

$$t=\sum_{i=0}^{\infty}t_i$$

The nonlinear operator can be decomposed as

$$N\left(\sum_{i=0}^{\infty}t_i\right)=N(t_0)+\sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i}t_j\right)-N\left(\sum_{j=0}^{i-1}t_j\right)\right]$$

So equation (4) can be re-written as

$$\sum_{i=0}^{\infty}t_i=f+N(t_0)+\sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i}t_j\right)-N\left(\sum_{j=0}^{i-1}t_j\right)\right]$$

We define the reccurence relation

$$\left.\begin{aligned} t_0&=f\\ t_1&=N(t_0)\\ t_2&=N(t_0+t_1)-N(t_0)\\ t_{m+1}&=N(t_0+\cdots+t_m)-N(t_0+\cdots+t_{m-1})\end{aligned}\right\}(5)$$

$$m=1,2\,\ldots..$$

Then

$$(t_1+\cdots+t_{m+1})=N(t_0+\cdots+t_m)$$

$$m=1,2\ \ldots.$$

And $\quad t=f+\sum_{i=1}^{\infty}t_i$

If N is a contraction, that is $\|N(x)-N(t)\|\le K\|x-t\|,0<K<1, then$

$$\|t_{m+1}\|=\|N(t_0+\cdots+t_m)-N(t_0+\cdots+t_{m-1})\|\le K\|t_m\|\le K^m\|t_0\| \qquad m=0,1,2\,\ldots$$

And the series $\sum_{i=1}^{\infty} t_i$ absolutely and uniformly converges to a solution of equation (4) (Cherruault, 1988) which is unique in view of Banach fixed point theorem (Jerri, 1999).

Solution of Lane Emden equation for Isothermal Gas Sphere

The volterra integral form of Lane Emden equation for isothermal gas sphere $\varphi'' + \frac{2}{x}\varphi' + e^{-\varphi} = 0$; $\varphi(0) = 0$ $\varphi'(0) = 0$ can be set as (Wazwaz 2001, 2013)

$$\varphi(x) = -\int_0^x t(1 - \tfrac{t}{x})e^{-\varphi}dt \qquad (6)$$

Differentiating the equation, using Leibnitz rule

$$\varphi'(x) = -\int_0^x (\tfrac{t^2}{x^2})e^{-\varphi}dt \qquad (7)$$

Solving equation (7) using New Iterative Method

$$\varphi(x) = -\int_0^x \int_0^x (\tfrac{t^2}{x^2})e^{-\varphi}dtdx \qquad (8)$$

Substituting the expansion in equation (8) and using the recurrence relation

$$\varphi_0 = 0$$

$$\varphi_1 = N(\varphi_0) = -\int_0^x \int_0^x \frac{t^2}{x^2} e^0 dtdx = -\frac{x^2}{6}$$

$$\varphi_2 = N(\varphi_0 + \varphi_1) - N(\varphi_0)$$

$$= -\int_0^x \int_0^x \frac{t^2}{x^2} e^{(\varphi_0+\varphi_1)} dtdx + \int_0^x \int_0^x \frac{t^2}{x^2} e^{\left(0+\frac{x^2}{6}\right)} dtdx = \frac{x^4}{120}$$

Summing up the series,

$$\varphi(x) = \varphi_0 + \varphi_1 + \varphi_2 + \cdots = -\frac{x^2}{6} + \frac{x^4}{120} + \cdots$$

which is the series solution of lane emden equation for polytropic gas sphere.

Results and Discussion

The series solution for Isothermal gas sphere equation, $\varphi'' + \frac{2}{x}\varphi' - e^{-\varphi} = 0;\ \ \varphi(0) = 0, \varphi'(0) = 0$ for small values of x is obtained using New Iterative Method as $\varphi(x) = -\frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots$ (Chandrasekhar S., 1939)

The obtained series gives exact results for values ranging between $0 < x < 1$. The obtained result almost matches with the result earlier obtained (Mirza, 2009). The maximum relative error is 6% in the case. Graph 1 presents the comparison of gravitational potential obtained using iteration and fractional approximation technique (Mirza, 2009). Physical parameters are presented in (Table2). The first column represents the scaled radius, second column represents potential, third column gives the ratio of density to central density, fourth gives acceleration due to gravity, fifth column represents the mass inside the scaled radius.

***Table 1:-** The physical parameters for Isothermal Gas Sphere*

x	φ (author's value)	φ (mirza's value)	Relative error
0.0	0	0	-
0.1	0.00167	0.0016	4.21875
0.2	0.00668	0.0066	1.211924
0.3	0.01507	0.0149	1.123188
0.4	0.02688	0.0266	1.049575
0.5	0.04218	0.0416	1.404808
0.6	0.06107	0.0598	2.124988
0.7	0.08364	0.0813	2.883342
0.8	0.11003	0.106	3.799988
0.9	0.14036	0.1338	4.904376
1.0	0.1748	0.1646	6.197805

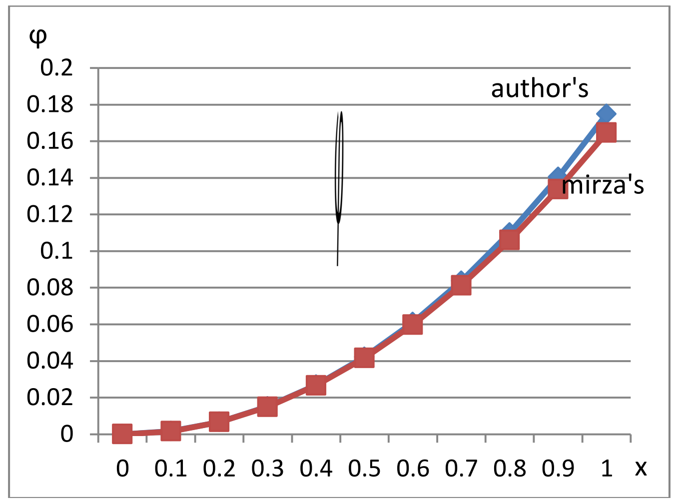

***Figure 1.** Comparison of author's gravitational potential values with Mirza's*

***Table 2:-** The physical parameters for Isothermal Gas Sphere*

x	$-\varphi$	$e^{-\varphi}$	$-\varphi'$	$-x^2\varphi'$
0.0	0.000000000	1.000000000	0.000000000	0.000000000
0.1	0.001667500	0.998333890	0.033300012	0.000333000
0.2	0.006679987	0.993342274	0.066400381	0.002656015
0.3	0.015067355	0.985045589	0.099102893	0.008919260
0.4	0.026879187	0.973478843	0.131212190	0.020993950
0.5	0.042184400	0.958692981	0.162537202	0.040634301
0.6	0.061070743	0.940756686	0.192892571	0.069441326
0.7	0.083644157	0.919758488	0.222100083	0.108829041
0.8	0.110027987	0.895809064	0.249990095	0.159993661
0.9	0.140362055	0.869043537	0.276402964	0.223886401
1.0	0.174801587	0.839623596	0.301190476	0.301190476

Conclusion

We have given analytic approximate solutions to the Lane Emden equation for the Isothermal sphere, found out various physical parameters and numerical results compared with the earlier obtained results.

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