Implication Operator On Pythagorean Fuzzy Set

I.Silambarasan, S.Sriram

Abstract— Pythagorean fuzzy sets,involvoing membership, non membership and hesitancy considerations present mathematically a very general structure. Because of these considerations, if is possible to define several operations/ compositions of these sets. In the existing literature ten different operations on such sets are defined. These ten operations on Pythagorean fuzzy sets bear intersting properties. In this paper, we have identified and proved several of these properties, particularly those involving the operation $A \to B$ defined as pythagorean fuzzy implication with other operations.

Index Terms— Intuitionistic fuzzy set, Pythagorean fuzzy set, Algebraic sum, Algebraic product, Implication operator.

1 INTRODUCTION

YAGER [9] introduced Pythagorean fuzzy set (PFS) characterized by a membership and a nonmembership satisfying the condition that the square sum of its membership and nonmembership is equal to or less than 1, which is a generalization of IFS. The motivation of introducing PFSs is that in the real-life decision process, the sum of the support (membership) degree and the against (nonmembership) degree to which an alternative satisfying a criterion provided by the decision maker may be bigger than 1 but their square sum is equal to or less than 1. Yager [10] gave an example to illustrate this situation: an expert gives his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his against membership is $\frac{1}{2}$. Due to the sum of two values is bigger than 1, they are not available for IFS, but they are feasible for PFS since $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$. Obviously, PFS is more effective than IFS to model the vagueness in the practical multi-criteria decision making (MCDM) problems. In the present communication, we prove some new results associated with standard Pythagorean fuzzy implication ($\to$). The paper is organized as follows: In section 2 some basic definitions related to IFS and PFS theory are presented. In section 3 we define Pythagorean fuzzy implication operator and new results associated with the standard Pythagorean fuzzy implication operator are proved.

2 PRELIMINARIES

Some basic definitions related to IFS theory and PFS theory are presented.

Definition 2.1. [1] An intuitionistic fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is given by,

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

where $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ such that $0 \le \mu_A(x) + \nu_A(x) \le 1$.
The numbers $\mu_A(x)$ and $\nu_A(x)$ denote the degree of membership and degree of non-membership of $x \in X$ to A, respectively.

Definition 2.2. [10] Let a set X be a universe of discourse A PPythagorean fuzzy set (PFS) P is an object having the form,

$$P = \big(\langle x, P(\mu_p(x), \nu_p(x))\rangle | (x \in X)\big) \quad (1)$$

Where the function $\mu_p: X \to [0,1]$ and $\nu_p: X \to [0,1]$ such that $0 \le (\mu_p(x))^2 + (\nu_p(x))^2 \le 1$
The numbers $\mu_P(x)$ and $\nu_P(x)$ denote the degree of membership and degree of non-membership of $x \in X$ to P, respectively.

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Definition 2.3. IFS operators on PFS: [7]

Let PFS (X) denote the family of all PFSs on the universe X, and let $A, B \in PFS(X)$ be given as,
$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$,
$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$.

Then following PFS operators are defined,

$$((i)\ \ A \cup B = \left\{\left\langle \begin{matrix} x, \max(\mu_A(x), \mu_B(x)), \\ \min(\nu_A(x), \nu_B(x)) \end{matrix} \right\rangle | x \in X\right\}$$

$$((ii) A \cap B = \left\{\left\langle \begin{matrix} x, \min(\mu_A(x), \mu_B(x)), \\ \max(\nu_A(x), \nu_B(x)) \end{matrix} \right\rangle | x \in X\right\}$$

$$((iii) A^C = \{\langle x, (\nu_A(x)), (\mu_A(x))\rangle | x \in X\}$$

$$((iv) A \oplus_P B = \left\{\left\langle \begin{matrix} x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \\ \nu_A(x)\nu_B(x) \end{matrix} \right\rangle | x \in X\right\}$$

$$(v) A \odot_P B = \left\{\left\langle \begin{matrix} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{matrix} \right\rangle \\ | x \in X \right\}$$

$$((vi) A @ B = \left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2 + (\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2 + (\nu_B(x))^2}{2}} \right\rangle | x \in X\right\}$$

$$((vii) A \$ B = \left\{\left\langle x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)} \right\rangle | x \in X\right\}$$

- *I.Silambarasan, Department of Mathematics, Annamalai University, Tamilnadu, India.E-mail: sksimbuking@mail.com
- S.Sriram, Mathematics Wing DDE, Annamalai University, Tamilnadu, India.*E-mail: ssm_3096@yahoo.co.in.*

$((viii)A\#B = \left\{\left\langle x, \frac{\sqrt{2}\mu_A(x)\mu_B(x)}{\sqrt{(\mu_A(x))^2+(\mu_B(x))^2}}, \frac{\sqrt{2}\nu_A(x)\nu_B(x)}{\sqrt{(\nu_A(x))^2+(\nu_B(x))^2}}\right\rangle | x \in X\right\}$

For which we shall accept that if $\mu_A(x) = \mu_B(x) = 0$ then

$$\frac{\mu_A(x)\mu_B(x)}{\mu_A(x) + \mu_B(x)} = 0$$

aand if $\nu_A(x) = \nu_B(x) = 0$, then $\frac{\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)} = 0$

.

$(ix)\left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2((\mu_A(x))^2+(\mu_B(x))^2+1)}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2((\nu_A(x))^2+(\nu_B(x))^2+1)}}\right\rangle | x \in X\right\}$

$(x)A \to B = \{\langle x, \max\{\nu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \nu_B(x)\}\rangle | x \in X\}$

Lemma 2.4. **[2]** For $A, B \in [0,1]$, then

$a.b \le min\{a, b\} \le \frac{2\,(a.b)}{a+b} \le \sqrt{a.b} \le \frac{a+b}{2} \le max\{a, b\}$
$\le a + b - a.b,\ a.b \le \frac{a+b}{2(a+b+1)} \le \frac{a+b}{2}$.

In the next section, we state and prove some new results involving implication
operation with other PFS operations.

3 RESULTS ON PYTHAGOREAN FUZZY IMPLICATION OPERATOR

Theorem 3.2. For $A, B \in PFS(X)$,
$(i)(A^C \to B)@(A \to B^C)^C = (A@B)$;
$(ii)(A^C \to B) \oplus_P (A \to B^C)^C = (A \oplus_P B)$;
$(iii)(A^C \to B)\odot_P(A \to B^C)^C = (A\odot_P B)$;
$(iv)(A^C \to B)\$(A \to B^C)^C = (A\$B)$;
$(v)(A^C \to B)\#(A \to B^C)^C = (A\#B)$;
$(vi)(A \to B)^C \oplus_P (B \to A) = (A \oplus_P B^C)$;
$(vii)(A \to B)^C@(B \to A) = (A@B^C)$;
$(viii)(A \to B)^C\odot_P(B \to A) = (A\odot_P B^C)$;
$(ix)(A \to B)^C\$(B \to A) = (A\$B^C)$;
$(x)(A \to B)^C\#(B \to A) = (A\#B^C)$;

Proof: We prove (i) and (vi), results (iii),(iv),(v),(vii),(viii),(ix) and (x) can be proved analogously.
Let us recall following simple fact for any two real numbers a and b,
$\max(a, b) + \min(a, b) = a + b$ (2)
$\max(a, b).\min(a, b) = a.b$ (3)
(i) From Definition in (2.3) and Lemma (2.4), we have
$(A^C \to B)@(A \to B^C)^C$
$= \left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}}\right\rangle | x \in X\right\} = (A@B)$.

(vi) Using Definition in (2.3) and Lemma (2.4), we have

$(A \to B)^C \oplus_P (B \to A)$
$= \left\{\left\langle x, \sqrt{(\mu_A(x))^2 + (\nu_B(x))^2 - (\mu_A(x))^2(\nu_B(x))^2}, \nu_A(x)\mu_B(x)\right\rangle | x \in X\right\}$
$= (A \oplus_P B^C)$

Theorem 3.3: For $A, B \in PFS(X)$,

$(i)((A \oplus_P B) \to (A@B)^C)^C = ((A@B) \to (A \oplus_P B)^C)^C = (A@B)$;
$(ii)((A \oplus_P B)^C \to (A@B)) = ((A@B)^C \to (A \oplus_P B))$
$= (A \oplus_P B)$;
$(iii)((A\odot_P B) \to (A@B)^C)^C = ((A@B) \to (A\odot_P B)^C)^C = (A\odot_P B)$;
$(iv)((A\odot_P B)^C \to (A@B)) = ((A@B)^C \to (A\odot_P B)) = (A@B)$;
$(v)((A \oplus_P B) \to (A\#B)^C)^C = ((A\#B) \to (A \oplus_P B)^C)^C = (A\#B)$;
$(vi)((A \oplus_P B)^C \to (A\#B)) = ((A\#B)^C \to (A \oplus_P B)) = (A \oplus_P B)$;
$(vii)((A\odot_P B) \to (A\#B)^C)^C = ((A\#B) \to (A\odot_P B)^C)^C = (A\odot_P B)$;
$(viii)((A\odot_P B)^C \to (A\#B)) = ((A\#B)^C \to (A\odot_P B)) = (A\#B)$;
$(ix)((A \oplus_P B) \to (A\$B)^C)^C = ((A\$B) \to (A \oplus_P B)^C)^C = (A\$B)$;
$(x)((A \oplus_P B)^C \to (A\$B)) = ((A\$B)^C \to (A \oplus_P B)) = (A \oplus_P B)$;
$(xi)((A\odot_P B) \to (A\$B)^C)^C = ((A\$B) \to (A\odot_P B)^C)^C = (A\odot_P B)$;
$(xii)((A\odot_P B)^C \to (A\$B)) = ((A\$B)^C \to (A\odot_P B)) = (A\$B)$;
$((xiii)((A \oplus_P B) \to (AB)^C)^C = ((AB) \to (A \oplus_P B)^C)^C$
$= (A\odot_P B)$;
$((xiv)((A\odot_P B)^C \to (A \oplus_P B)) =$
$((A \oplus_P B)^C \to (A\odot_P B)) = ((A \oplus_P B)$;

Proof: We prove (i), (iii), (v), (vii), (ix) and (xiii), other results can be proved analogously,

((i) From Definition in (2.3) and Lemma (2.4), we have
$((A \oplus_P B) \to (A@B)^C)^C$
$= \left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}}\right\rangle | x \in X\right\}$
$= A@B$ (4)

And
$((A@B) \to (A \oplus_P B)^C)^C$
$= \left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}}\right\rangle | x \in X\right\}$
$= A@B$ (5)

From (4) and (5), we get the result (i).

(iii) Using Definition in (2.3) and Lemma (2.4), we have
$((A\odot_P B) \to (A@B)^C)^C =$
$\left\{\left\langle \frac{x, \mu_A(x)\mu_B(x),}{\sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2}}\right\rangle | x \in X\right\}$
$= A\odot_P B$ (6)

And
($(A@B) \to (A\odot_P B)^C)^C$
$= \left\{\left\langle \frac{x, \mu_A(x)\mu_B(x),}{\sqrt{(\nu_A(x))^2 + (\nu_A(x))^2 - (\nu_A(x))^2(\nu_A(x))^2}}\right\rangle | x \in X\right\}$
$= A\odot_P B$ (7)

From (6) and (7), we get the result (iii).

(v)From Definition in (2.3) and Lemma (2.4), we have

$((A \oplus_P B) \rightarrow (A\#B)^C)^C$

$= \left\{\left\langle x, \frac{\sqrt{2}\mu_A(x)\mu_B(x)}{\sqrt{(\mu_A(x))^2+(\mu_B(x))^2}}, \frac{\sqrt{2}\nu_A(x)\nu_B(x)}{\sqrt{(\nu_A(x))^2+(\nu_B(x))^2}}\right\rangle | x \in X\right\}$

$= A\#B \quad (8)$

And

$((A\#B) \rightarrow (A \oplus_P B)^C)^C$

$= \left\{\left\langle x, \frac{\sqrt{2}\mu_A(x)\mu_B(x)}{\sqrt{(\mu_A(x))^2+(\mu_B(x))^2}}, \frac{\sqrt{2}\nu_A(x)\nu_B(x)}{\sqrt{(\nu_A(x))^2+(\nu_B(x))^2}}\right\rangle | x \in X\right\}$

$= A\#B \quad (9)$

From (8) and (9), we get the result (v).

(vii) Using Definition in (2.3) and Lemma (2.4), we have

$((A \odot_P B) \rightarrow (A\#B)^C)^C$

$= \left\{\left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{array}\right\rangle | x \in X\right\}$

$= A \odot_P B \quad (10)$

And

$((A\#B) \rightarrow (A \odot_P B)^C)^C$

$= \left\{\left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{array}\right\rangle | x \in X\right\}$

$= A \odot_P B \quad (11)$

From (10) and (11), we get the result (vii).

(ix)From Definition in (2.3) and Lemma (2.4), we have

$((A \oplus_P B) \rightarrow (A\$B)^C)^C$

$= \left\{\left\langle x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)}\right\rangle | x \in X\right\}$

$= A\$B \quad (12)$

And

$((A\$B) \rightarrow (A \oplus_P B)^C)^C$

$= \left\{\left\langle x, \sqrt{\mu_A(x)\mu_B(x)}, \sqrt{\nu_A(x)\nu_B(x)}\right\rangle | x \in X\right\}$

$= A\$B \quad (13)$

From (12) and (13), we get the result (ix),

(xi) Using Definition in (2.3) and Lemma (2.4), we have

$((A \odot_P B) \rightarrow (A\$B)^C)^C$

$= \left\{\left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_A(x))^2 - (\nu_A(x))^2(\nu_A(x))^2} \end{array}\right\rangle | x \in X\right\}$

$= A \odot_P B \quad (14)$

And

$((A\$B) \rightarrow (A \odot_P B)^C)^C$

$= \left\{\left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \\ | x \in X \end{array}\right\rangle\right\}$

$= A \odot_P B \quad (15)$

From (14) and (15) , we get the result (xi).

(xiii) From Definition in (2.3) and Lemma (2.4), we have

$((A \oplus_P B) \rightarrow (A \odot_P B)^C)^C$

$= \left\{\left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{array}\right\rangle | x \in X\right\}$

$= A \odot_P B \quad (16)$

And

$((A \odot_P B) \rightarrow (A \oplus_P B)^C)^C$

$= \left\{\left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{array}\right\rangle | x \in X\right\}$

$= A \odot_P B \quad (17)$

From (16) and (17) , we get the result (xiii).

The proof of the following Corollaries follows from Theorem 3.3,

Corollary 3.4: For $A, B \in PFS(X)$,

$((A \odot_P B) \rightarrow (A@B)^C)^C = ((A@B) \rightarrow (A \odot_P B)^C)^C$
$= ((A \odot_P B) \rightarrow (A\#B)^C)^C = ((A\#B) \rightarrow (A \odot_P B)^C)^C$
$= ((A \odot_P B) \rightarrow (A\$B)^C)^C = ((A\$B) \rightarrow (A \odot_P B)^C)^C$
$= ((A \oplus_P B) \rightarrow (A \odot_P B)^C)^C = ((A \odot_P B) \rightarrow (A \oplus_P B)^C)^C$
$= (A \odot_P B)$

Corollary 3.5: For $A, B \in PFS(X)$,

$((A \oplus_P B)^C \rightarrow (A@B)) = ((A@B)^C \rightarrow (A \oplus_P B))$
$= ((A \oplus_P B)^C \rightarrow (A\#B)) = ((A\#B)^C \rightarrow (A \oplus_P B))$
$= ((A \oplus_P B)^C \rightarrow (A\$B)) = ((A\$B)^C \rightarrow (A \oplus_P B))$
$= ((A \odot_P B)^C \rightarrow (A \oplus_P B)) = ((A \oplus_P B)^C \rightarrow (A \odot_P B))$
$= (A \oplus_P B)$

Theorem 3.6: For $A, B \in PFS(X)$,

$[([(A^C \rightarrow B) \oplus_P (A \rightarrow B^C)^C]@[(A^C \rightarrow B) \odot_P (A \rightarrow B^C)^C]$
$= (A@B)$.

Proof: From Definition in (2.3) and Lemma (2.4), we have

$[(A^C \rightarrow B) \oplus_P (A \rightarrow B^C)^C] =$

$$\left\{\left\langle \begin{array}{c} x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \\ \nu_A(x)\nu_B(x) \\ | x \in X \end{array}\right\rangle\right\} \quad (18)$$

and

$[(A^C \rightarrow B) \odot_P (A \rightarrow B^C)^C] =$

$$\left\{\left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \\ | x \in X \end{array}\right\rangle\right\} \quad (19)$$

Now with @ of (18) and (19),

$[(A^C \rightarrow B) \oplus_P (A \rightarrow B^C)^C]@[(A^C \rightarrow B) \odot_P (A \rightarrow B^C)^C]$

$$= \left\{ \left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle | x \in X \right\}$$
$$= (A@B)$$

Theorem 3.7: For $A, B \in PFS(X)$,

$$\left[\left((A^C \to B) \oplus_P (A \to B^C)^C\right) \bigcap \left((A^C \to B) \odot_P (A \to B^C)^C\right)\right]$$
@
$$\left[\left((A^C \to B) \oplus_P (A \to B^C)^C\right) \bigcup \left((A^C \to B) \odot_P (A \to B^C)^C\right)\right] = (A@B)$$

Proof:

Taking with $\cap$ of (18) and (19), we get

$$\left[\left((A^C \to B) \oplus_P (A \to B^C)^C\right) \bigcap \left((A^C \to B) \odot_P (A \to B^C)^C\right)\right] = \left\{ \left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{array} \right\rangle |x \in X \right\} \quad (20)$$

Again taking with $\cup$ of (18) and (19),

$$\left[\left((A^C \to B) \oplus_P (A \to B^C)^C\right) \bigcup \left((A^C \to B) \odot_P (A \to B^C)^C\right)\right] = \left\{ \left\langle \begin{array}{c} x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \\ \nu_A(x)\nu_B(x) \end{array} \right\rangle |x \in X \right\} \quad (21)$$

Now wih @ of (20) and (21),

$$\left[\left((A^C \to B) \oplus_P (A \to B^C)^C\right) \bigcap \left((A^C \to B) \odot_P (A \to B^C)^C\right)\right] @ \left[\left((A^C \to B) \oplus_P (A \to B^C)^C\right) \bigcup \left((A^C \to B) \odot_P (A \to B^C)^C\right)\right]$$

$$= \left\{ \left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle | x \in X \right\}$$
$$= A@B$$

Theorem 3.8: For $A, B \in PFS(X)$,

$$\left[((A \oplus_P B) \to (A@B)^C)^C \bigcup ((A \odot_P B) \to (A@B)^C)^C\right] \cup \left[((A \oplus_P B) \to (A@B)^C)^C \cap ((A \odot_P B) \to (A@B)^C)^C\right] = A@B.$$

Proof:

From Theorem 3.3 and Lemma (2.4), we have

$$((A \oplus_P B) \to (A@B)^C)^C = \left\{ \left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle |x \in X \right\} \quad (22)$$

and

$$((A \odot_P B) \to (A@B)^C)^C = \left\{ \left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_A(x))^2 - (\nu_A(x))^2(\nu_A(x))^2} \end{array} \right\rangle |x \in X \right\} \quad (23)$$

Now with $\cup$ of (22) and (23),

$$\left[((A \oplus_P B) \to (A@B)^C)^C \bigcup ((A \odot_P B) \to (A@B)^C)^C\right] = \left\{ \left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle |x \in X \right\} \quad (24)$$

and with $\cap$ of (22) and (23),

$$\left[((A \oplus_P B) \to (A@B)^C)^C \bigcap ((A \odot_P B) \to (A@B)^C)^C\right] = \left\{ \left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{array} \right\rangle |x \in X \right\} \quad (25)$$

Now we consider,

$$\left[((A \oplus_P B) \to (A@B)^C)^C \bigcup ((A \odot_P B) \to (A@B)^C)^C\right] \bigcup \left[((A \oplus_P B) \to (A@B)^C)^C \bigcap ((A \odot_P B) \to (A@B)^C)^C\right]$$
$$= \left\{ \left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle |x \in X \right\}$$
$$= A@B$$

Theorem 3.9: For $A, B \in PFS(X)$,

$$\left[((A \oplus_P B) \to (A@B)^C)^C \bigcup ((A \odot_P B) \to (A@B)^C)^C\right] \bigcap \left[((A \oplus_P B) \to (A@B)^C)^C \bigcap ((A \odot_P B) \to (A@B)^C)^C\right] = A \odot_P B.$$

Proof:

The proof is similar to Theorem 3.8.

Theorem 3.10: For $A, B \in PFS(X)$,
$((A \oplus_P B)^C \to (A@B))@((A \odot_P B) \to (A@B)^C)^C = (A@B)$

Proof: Using Definition in (2.3) and Lemma (2.4), we have

$$((A \oplus_P B)^C \to (A@B)) = \left\{ \left\langle \begin{array}{c} x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \\ \nu_A(x)\nu_B(x) \end{array} \right\rangle |x \in X \right\} \quad (26)$$

and

$$((A \odot_P B) \to (A@B)^C)^C = \left\{ \left\langle \begin{array}{c} x, \mu_A(x)\mu_B(x), \\ \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \end{array} \right\rangle |x \in X \right\} \quad (27)$$

Now with @ of (26) and (27),

$$((A \oplus_P B)^C \to (A@B))@((A \odot_P B) \to (A@B)^C)^C = \left\{ \left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle |x \in X \right\}$$
$$= A@B$$

Theorem 3.11: For $A, B \in PFS(X)$,
$((A \oplus_P B)^C \to (A\#B))@((A \odot_P B) \to (A\#B)^C)^C = (A@B)$

Proof: Using Definition in (2.3) and Lemma (2.4), we have

$$((A \oplus_P B)^C \to (A\#B))$$

$$= \left\{\left\langle x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \nu_A(x)\nu_B(x) \right\rangle | x \in X\right\} \quad (28)$$

and

$$((A \odot_P B) \to (A\#B)^C)^C =$$

$$\left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \right\rangle | x \in X\right\} \quad (29)$$

Now with @ of (28) and (29),

$$((A \oplus_P B)^C \to (A\#B))@((A \odot_P B) \to (A\#B)^C)^C$$

$$= \left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle | x \in X\right\}$$

$$= A@B$$

Theorem 3.12: For $A, B \in PFS(X)$,
$((A \oplus_P B)^C \to (A\$B))@((A \odot_P B) \to (A\$B)^C)^C = (A@B)$

Proof: Using Definition in (2.3) and Lemma (2.4), we have

$$((A \oplus_P B)^C \to (A\$B))$$

$$= \left\{\left\langle x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \nu_A(x)\nu_B(x) \right\rangle | x \in X\right\} \quad (30)$$

and

$$((A \odot_P B) \to (A\$B)^C)^C$$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \right\rangle | x \in X\right\} \quad (31)$$

Now with @ of (30) and (31),

$$((A \oplus_P B)^C \to (A\$B))@((A \odot_P B) \to (A\$B)^C)^C$$

$$= \left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle | x \in X\right\}$$

$$= A@B$$

Theorem 3.13: For $A, B \in PFS(X)$,
$((A \odot_P B)^C \to (A \oplus_P B))@((A \oplus_P B) \to (A \odot_P B)^C)^C = (A@B)$

Proof: Using Definition in (2.3) and Lemma (2.4), we have

$$((A \odot_P B)^C \to (A \oplus_P B))$$

$$= \left\{\left\langle x, \sqrt{(\mu_A(x))^2 + (\mu_B(x))^2 - (\mu_A(x))^2(\mu_B(x))^2}, \nu_A(x)\nu_B(x) \right\rangle | x \in X\right\} \quad (32)$$

and

$$((A \oplus_P B) \to (A \odot_P B)^C)^C$$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \sqrt{(\nu_A(x))^2 + (\nu_B(x))^2 - (\nu_A(x))^2(\nu_B(x))^2} \right\rangle | x \in X\right\} \quad (33)$$

Now with @ of (32) and (33),

$$((A \odot_P B)^C \to (A \oplus_P B))@((A \oplus_P B) \to (A \odot_P B)^C)^C$$

$$= \left\{\left\langle x, \sqrt{\frac{(\mu_A(x))^2+(\mu_B(x))^2}{2}}, \sqrt{\frac{(\nu_A(x))^2+(\nu_B(x))^2}{2}} \right\rangle | x \in X\right\}$$

$$= A@B$$

4 CONCLUDING REMARKS

The peoperties proved here provide an insight into the PFSs, under the set operations defined earlier in the literature. Our study prompts for further properties as also for defining possibly new operations. Thus there remains scope for studying more properties of these sets arising from those other defining set operations that may be thought of using other ways of combining the functions μ, ν .

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