Scaling Of Student T-Distribution And Properties Of Lévy-Student Processes

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Abstract: The Student t-distribution can be applied in financial studies as heavy-tailed substitute to the normal distribution. The aim of this study is to explore the properties of Student t-distribution and Lévy -Student processes under finance. For a suitable modification of the Lévy measure of Student t-distribution, an explicit expression of its Fourier transform was calculated. It was shown that how the Fourier inversion of this function, which yields the density of the Lévy measure. Further, Lévy-student process is derived that nests the Brownian motion with subordinated by GIG distribution as parameters special case.

Index Terms: Student T-distribution, Lévy processes, Heavy-tailed distribution, Modified Lévy measure

1 Introduction

THE distributional form of the returns on the underlying assets plays a key role in finance under valuation theories for derivative securities. Among them, the Student t-distributions plays an important role in financial market return analysis as well as statistics for the analysis of normal samples. One of the major properties of these distributions is heavy-tail behavior compared with other distributions; i.e. the actual kurtosis is higher than the kurtosis of the normal distribution [1]. The probability density function of a general t_υ distribution with υ degrees of freedom is

$$f_X(x) = \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\Gamma\left(\frac{1}{2}\upsilon\right)\sqrt{\pi\ \upsilon}\ \sigma}\left[1+\frac{1}{\upsilon}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-(\upsilon+1/2)}, x \in \mathbb{R}^1$$

(1)

where Γ(x) is the gamma function, μ and σ are mean and standard deviation respectively. The cumulative distribution function of the Student t-distribution is given by the formula

$$F(x) = \int_{-\infty}^{x} f_Y(y)\, dy =$$

$$\int_{-\infty}^{x} \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\pi\,\sigma\,\upsilon}\ \Gamma\left(\frac{1}{2}\upsilon\right)}\left[1+\frac{1}{\upsilon}\left(\frac{y-\mu}{\sigma}\right)^2\right]^{-(\upsilon+1/2)} dy;\ \ y \in \mathbb{R}^1,$$

$$F(x) = \frac{1}{2} + \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\pi\upsilon}\ \Gamma\left(\frac{\upsilon}{2}\right)}\frac{(x-\mu)}{\sigma}\ {}_2F_1\left[\frac{1}{2},\frac{1+\upsilon}{2},\frac{3}{2},-\frac{(x-\mu)^2}{\sigma^2\upsilon}\right].$$

(2)

where ${}_2F_1$ is a Gaussian hyper-geometric function and the inverse cumulative distribution function (iCDF) is given by:

$$F^{-1}(u) = sign\left(u-1/2\right)\sqrt{\upsilon\left(\frac{1}{I^{-1}_{if[u<1/2,2u,2(1-u)]}\left(\frac{n}{2},\frac{1}{2}\right)}-1\right)} \quad .$$

(3)

2 Infinite Divisibility (ID), Self-Decomposability (SD) and Lévy–Khintchine formula

Intensive studies of new criteria for these properties began in last century [2]. Relating student's t-distributions to the Lévy processes; the crucial role are paid the properties of infinite divisibility or self-decomposability [3], [4]. Grosswald [5] proved that the standard student's t-distribution of any degree of freedom is infinitely divisible, by deriving the following formula

$$K_{\upsilon-1}(x) = xK_\upsilon(x)\int_0^\infty \frac{g_\upsilon(u)}{x^2+u}du,\ \upsilon \geq -1,\ x > 0 \tag{4}$$

where, $g_\upsilon(u) = 2[\pi^2 x(J_\upsilon^2(\sqrt{x}) + Y_\upsilon^2(\sqrt{x}))]^{-1}$, $x > 0$, $J_\upsilon(x)$ and $Y_\upsilon(x)$ are the Bessel functions of the first kind and second kind respectively, while Jurek [6] proved that the general student t-distribution is infinite divisible and self-decomposable. That means, the class of all self-decomposable characteristic (or probability distributions) are infinitely divisible, i.e.

$$\forall(n \geq 1)\,\exists\,(\psi_n), \forall(u \in \mathbb{R})\ s.t.\ \psi(u) = [\psi_n(u)]^n. \tag{5}$$

The characteristic function of every infinitely divisible distribution can be represented in a very special form, which is called the Lévy–Khintchine formula. The Lévy–Khintchine representation of the characteristic function of the general $ST(\upsilon,\sigma,\mu)$ can be obtained from the results of Halgreen [7] by choosing $\alpha = |\beta| = 0$, $\sigma > 0$ and $\lambda = -\frac{\upsilon}{2} < 0$ as follows

$$\psi(u) = exp\left(iu\mu + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux1_{\{|x|>1\}}\right)g_t(x)dx\right) \quad ,$$

(6)

where, $g_t(x) = \frac{1}{|x|}\int_0^\infty \frac{e^{-|x|\sqrt{2y}}dy}{\pi^2 y(J^2_{\upsilon/2}(\sigma\sqrt{2y}) + Y^2_{\upsilon/2}(\sigma\sqrt{2y}))}$ $J_\upsilon(x)$ and $Y_\upsilon(x)$ respectively denote the Bessel functions of the first and second kind. $g_t(x)$ is the density of the Lévy measure and its numerical evaluation is cumbersome, especially for small values of $|x|$. The decay of the exponential term in the numerator becomes very slow due to two Bessel functions $J_\upsilon(x)$ and $Y_\upsilon(x)$ appearing in the denominator, it is also difficult to examine the density analytically. Therefore, modified Lévy measure is considered according to Raible [8] definition.

3 Lévy Measure, Modified Lévy Measure and its Fourier Transform

The structure of jumps of a Lévy process is determined by its Lévy (or characteristic) measure. In general, the Lévy measure $(\Pi(dx))$ of a t-distribution has infinite mass in the every neighborhood of the origin. This means that the mass is concentrated around $x = 0$. However, condition $\int_{\mathbb{R}}(1 \wedge x)\,\Pi(dx) < \infty$, imposes restrictions on the growth of the Lévy measure around $x = 0$.

Definition 1: Let $\Pi(dx)$be the Lévy measure of an infinitely divisible distribution. The modified Lévy measure $\overline{\Pi}(dx)$ is defined by $\overline{\Pi}(dx) = x^2\Pi(dx)$ on $(\mathbb{R},\mathcal{B})$.

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(Note: Let $\bar{\Pi}(dx)$ be the modified Lévy measure corresponding to the Lévy measure $\Pi(dx)$ of an infinitely divisible distribution that possesses a second moment. Then $\bar{\Pi}(dx)$ is a finite measure.)

Theorem 1: Let $\phi(u)$ denote the characteristic function of an infinitely divisible distribution on $\mathbb{R}$, possessing a second moment. Then the Fourier transform of the modified Lévy measure $x^2\Pi(dx)$ is given by; (This theorem is proven by Raible in Theorem 2.3 [8])

$$\int_{\mathbb{R}} e^{iux}\, x^2 \Pi(dx) = -\sigma^2 - \frac{d}{du}\left(\frac{\phi'(u)}{\phi(u)}\right). \quad (7)$$

A crucial assumption in the proof of Raible (2000, Proposition 2.18) [8] is that the modified Lévy measure $\bar{\Pi}(dx)$ is a finite measure on $(\mathbb{R}, \mathcal{B})$, that is, $\int_{\mathbb{R}} x^2\Pi(dx) < \infty$.

Proposition 1: Let $\Pi(dx)$ be the Lévy measure of a Student t-distribution with parameters ν, σ and μ. Then the Fourier transform of the modified measure $x^2\Pi(dx)$ is given by;

$$\int_{\mathbb{R}} e^{iux}\, x^2 \Pi(dx) = \sqrt{\nu}\sigma\left[\beta|u|'' + \sqrt{\nu}\sigma\left(\beta^2 + \beta\left(\frac{\nu-\gamma}{2\gamma}\right) - \frac{1}{2}\right)(|u|')^2\right]\Bigg|_{\beta=\left(\frac{K_{\frac{\nu}{2}-1}(\lambda)}{K_{\frac{\nu}{2}}(\lambda)}\right),\gamma=\sqrt{\nu}\sigma|u|} \quad (8)$$

where the subscripts $\beta = \left(\frac{K_{\frac{\nu}{2}-1}(\lambda)}{K_{\frac{\nu}{2}}(\lambda)}\right)$ and $\gamma = \sqrt{\nu}\sigma|u|$ mean that the given expressions are to be substituted in the term in square brackets. The variable β has to be substituted first.

Proof: According to the Theorem 1 and characteristic function of Student t-distribution, the derivation of $\frac{\phi'(u)}{\phi(u)}$ can be calculated as follows.

Hence, $\phi_{ST}(u) = e^{iu\mu}\frac{2^{1-\frac{\nu}{2}}(\sqrt{\nu}\sigma)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}(|u|)^{\frac{\nu}{2}} K_{\frac{\nu}{2}}\left(\sqrt{\nu}\sigma|u|\right)$ and

$$\phi_{ST}'(u) = i\mu\phi_{ST}(u) + \frac{2^{1-\frac{\nu}{2}}(\sqrt{\nu}\sigma)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\left[(|u|)^{\frac{\nu}{2}} K_{\frac{\nu}{2}}'\left(\sqrt{\nu}\sigma|u|\right)\sqrt{\nu}\sigma|u|' + \frac{\nu}{2}(|u|)^{\frac{\nu}{2}-1}|u|' K_{\frac{\nu}{2}}\left(\sqrt{\nu}\sigma|u|\right)\right]$$

Therefore, $\frac{\phi_{ST}'(u)}{\phi_{ST}(u)} = i\mu + |u|'\frac{K_{\frac{\nu}{2}}'(\sqrt{\nu}\sigma|u|)}{K_{\frac{\nu}{2}}(\sqrt{\nu}\sigma|u|)} + \frac{\nu}{2}\frac{|u|'}{|u|}$

The following relation of modified Bessel functions is used to eliminate the derivatives of the Bessel functions,

$$-2K_\lambda'(x) = K_{\lambda-1}(x) + K_{\lambda+1}(x),$$

$$\frac{\phi_{ST}'(u)}{\phi_{ST}(u)} = i\mu + \left[\sqrt{\nu}\sigma|u|'\frac{\left[K_{\frac{\nu}{2}-1}(\gamma) + K_{\frac{\nu}{2}+1}(\gamma)\right]}{-2K_{\frac{\nu}{2}}(\gamma)} + \frac{\nu}{2}\frac{|u|'}{|u|}\right]\Bigg|_{\gamma=\sqrt{\nu}\sigma|u|}.$$

The following Bessel functions relation is apply to smooth the above equation,

$$K_{\lambda+1}(x) = \frac{2\lambda}{x}K_\lambda(x) + K_{\lambda-1}(x).$$

Then, we have;

$$\frac{\phi_{ST}'(u)}{\phi_{ST}(u)} = i\mu - \left[\sqrt{\nu}\sigma|u|'\frac{K_{\frac{\nu}{2}-1}(\gamma)}{K_{\frac{\nu}{2}}(\gamma)}\right]\Bigg|_{\gamma=\sqrt{\nu}\sigma|u|};\ u \neq 0. \quad (9)$$

Taking the negative derivative of this expression yields:

$$-\frac{d}{du}\left(\frac{\phi_{ST}'(u)}{\phi_{ST}(u)}\right) = \sqrt{\nu}\sigma\left[|u|''\frac{K_{\frac{\nu}{2}-1}(\gamma)}{K_{\frac{\nu}{2}}(\gamma)} + |u|'\frac{d}{d\gamma}\left(\frac{K_{\frac{\nu}{2}-1}(\gamma)}{K_{\frac{\nu}{2}}(\gamma)}\right)\frac{\partial}{\partial u}\left(\sqrt{\nu}\sigma|u|\right)\right]. \quad (10)$$

The result of the derivatives appearing in (10) is yielded with application of the above two Bessel formulas

$$\frac{d}{d\gamma}\left(\frac{K_{\frac{\nu}{2}-1}(\gamma)}{K_{\frac{\nu}{2}}(\gamma)}\right) = \left(\frac{K_{\frac{\nu}{2}-1}(\gamma)}{K_{\frac{\nu}{2}}(\gamma)}\right)^2 + \left(\frac{\nu-\gamma}{2\gamma}\right)\frac{K_{\frac{\nu}{2}-1}(\gamma)}{K_{\frac{\nu}{2}}(\gamma)} - \frac{1}{2},$$

and $\frac{\partial}{\partial u}\left(\sqrt{\nu}\sigma|u|\right) = \sqrt{\nu}\sigma|u|'$.

Let, $\beta = \left(\frac{K_{\frac{\nu}{2}-1}(\lambda)}{K_{\frac{\nu}{2}}(\lambda)}\right)$,

then $\frac{d}{d\gamma}\left(\frac{K_{\frac{\nu}{2}-1}(\gamma)}{K_{\frac{\nu}{2}}(\gamma)}\right) = \left[\beta^2 + \left(\frac{\nu-\gamma}{2\gamma}\right)\beta - \frac{1}{2}\right]\Big|_{\gamma=\sqrt{\nu}\sigma|u|}$

The result of equation (8) can be obtained by substituting above three relations into (10). According to the Raible (Corollary 2.4.)[8], the Fourier transform of the modified Lévy measure $x^2\Pi(dx)$, can be represented as $\hat{\vartheta}(u)$ and defined by:

$$\hat{\vartheta}(u) = \sqrt{\nu}\sigma\left[\beta|u|'' + \sqrt{\nu}\sigma\left(\beta^2 + \beta\left(\frac{\nu-\gamma}{2\gamma}\right) - \frac{1}{2}\right)(|u|')^2\right]\Bigg|_{\beta=\left(\frac{K_{\frac{\nu}{2}-1}(\lambda)}{K_{\frac{\nu}{2}}(\lambda)}\right),\ \gamma=\sqrt{\nu}\sigma|u|}. \quad (11)$$

This yields that Gaussian coefficient in the Lévy-Khintchine representation of a Student t-distribution vanishes, and that $\hat{\vartheta}(u)$ is a continuous. This measure has a continuous Lebesgue density $\rho_t(x)$ on $\mathbb{R}$ that can be formed from the function $\hat{\vartheta}(u)$ by Fourier inversion.

$$\rho_t(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\hat{\vartheta}(u)du \quad (12)$$

The density of the Lévy measure $\rho(x)$ is analytically and computationally more tractable than $g_t(x)$ in (6). The Fourier transform (11) is real and symmetric. The integral (12) reduces to the following cosine transform integral for the student t-distribution [8], (refer, Raible (2000, Proposition 2.15 and Corollary 2.16). for detail derivation of generalized hyperbolic distribution)

$$\rho_t(x) = \frac{1}{\pi}\int_0^\infty \cos(ux)\,\hat{\vartheta}(u)du \quad (13)$$

With reference to the Proposition 2.15 and Corollary 2.16 [8] the formula (13) can be reduced to the following form based on asymptotic expansion of $\int_{\mathbb{R}} e^{iux}x^2\Pi(dx)$.

$$\rho_t(x) = \frac{1}{\pi}\int_0^\infty \cos(ux)\left[\sum_{n=2}^{N} q_n\left(\frac{1}{|u|}\right)^n + O\left(\frac{1}{\gamma^{N+1}}\right)\right]\Bigg|_{\gamma=\sqrt{\nu}\sigma|u|} du, \quad (14)$$

for $N = 2, 3, 4, 5, 6$; where,

$q_2 := -\left(-\frac{\nu}{2}+\frac{1}{2}\right), q_3 := -\frac{1}{\sqrt{\nu}\sigma}\left(-\frac{\nu}{2}+\frac{1}{2}\right)\left(-\frac{\nu}{2}-\frac{1}{2}\right),$

$q_4 := -\frac{3}{2(\sqrt{\nu}\sigma)^2}\left(-\frac{\nu}{2}+\frac{1}{2}\right)\left(-\frac{\nu}{2}-\frac{1}{2}\right),$

$q_5 := -\frac{1}{2(\sqrt{\nu}\sigma)^3}\left(-\frac{\nu}{2}+\frac{1}{2}\right)\left(-\frac{\nu}{2}-\frac{1}{2}\right)\left(-\frac{\nu}{2}+\frac{5}{2}\right)\left(-\frac{\nu}{2}-\frac{5}{2}\right)$ *and*

$q_6 := -\frac{1}{2(\sqrt{\nu}\sigma)^4}\left(-\frac{\nu}{2}+\frac{1}{2}\right)\left(-\frac{\nu}{2}-\frac{1}{2}\right)\left(5\left(-\frac{\nu}{2}+\frac{5}{2}\right)\left(-\frac{\nu}{2}-\frac{5}{2}\right)+15\right).$

The maximum value of N is 6, due to asymptotic expansion; further expansion of function is not advisable. Thereforethe Fourier transform of the modified Lévy density is of the order $O\left(\frac{1}{|u|}\right)$ as $|u| \to \infty$. This implies that it is absolutely integral. Hence the Fourier inversion integral converges. By considering the estimated parameters of student t-distribution for log returns on the Gold Futures ($\mu = 0.000262$, $\sigma = 0.00938$, $\upsilon = 3.82$), we calculate the modified Lévy density of a student t-distribution as follows. Note that the discontinuity of the modified density at $x = 0$ is captured entirely by the density in Figure 01. This class of Lévy processes matches the empirically observed log return behavior of financial assets very accurately, (see examples in Kumari, 2013). Furthermore, for GH distributions with $0 \le |\beta| < \alpha$ and $\delta > 0$, Raible [8] (2000, Proposition 2.18), found the asymptotic behavior of Modified Lévy density:

$$g_{GH(\lambda,\alpha,\beta,\delta,\mu)}(x) = \frac{\delta}{\pi}x^{-2} + \frac{\lambda+1/2}{2}|x|^{-1} + \frac{\delta\beta}{\pi}x^{-1} + O(|x|^{-1}),$$

when $x \to 0$.

Since the Student t-density $g_{ST(\lambda,\delta,\mu)}$ equal the point-wise limits of $g_{GH(\lambda,\alpha,\beta,\delta,\mu)}$for $\alpha, \beta \to 0$. It is tempting to infer that the above asymptotics are also valid for this limiting case. Therefore, for every t-distribution $ST(\lambda,\delta,\mu)$ with $\lambda < -1$, the asymptotic behavior of the corresponding Lévy density near the origin is given by

$$g_{ST(\lambda,\delta,\mu)}(x) = \frac{\delta}{\pi}x^{-2} + \frac{\lambda+1/2}{2}|x|^{-1} + O(|x|^{-1}),\ x \to 0.$$

(15)

Moreover, $\lambda = -\frac{\upsilon}{2}$ and $\delta = \sigma$ compare with previous parameter definition in Student's t-distribution.

4 LÉVY-STUDENT PROCESSES

The study of Lévy-Student processes has recently given rise to a series of studies [9, 10-11] under different objects. Student t-distribution is infinitely divisible and therefore generate a Lévy processes $X = (X_t)_{t\ge 0}$, such that the distribution of X_1, has symmetric $ST(\upsilon,\sigma,\mu)$ distribution (for simplicity and without loss of generality, we consider $t = 1$ with Student marginal. However, as ST distributions are not closed under convolution Nadarajah & Dey [12], the marginal of these Lévy processes are Student distributed only for one time horizon t_0. On the other time horizons, the marginal are not Student distributed. Their distribution can be derived analytically only for some special cases [11, 13]. Apart from these cases they have to be derived numerically in quite an involved manner [14]. For this reason, only few research papers have covered Lévy Student processes, so far.

***Lemma 1*:** The law of X_t is determined by the law of X_1which is ID. The independent and stationary of the increments of the Lévy process leads to the cumulant transform$\varkappa(u) = log(\psi(u))$ is given by

$\varkappa_{X_t}(u) = t\varkappa_{X_1}(u)$, where $u \in \mathbb{R}$.

(16)

For $\upsilon > 1$ and $E[X_t] = t\mu,\ t \ge 0$, the process can be split into $X_t = t\mu + X_t^0,\ t \ge 1$, with $E[X_t^0] = 0$. By equation (4.32), the characteristic function of the random variable $X_t^0,\ t \ge 0$ with the characteristic function $\psi(u)$, of the process at time t and

$$\psi_{X_t^0}(u) = (\psi_{X_1^0}(u))^t. \qquad u \in \mathbb{R}.$$

(17)

As the characteristic function of a random variable equals its Fourier transformation up to some constant factors, the inverse Fourier transformation reproduces the density from the characteristic function. In our case, $\psi_{X_1^0}(u) = \phi_{ST}(u;\ \upsilon,\sigma,\mu)$ is a characteristic function of the Student t-distribution; ie. $\psi_{X_t^0}(u) = \phi_{ST}(u;\ \upsilon,\sigma,\mu)^t$. Therefore, the transition PDF of Lévy-student processes is given;

$$f_{L(t)}(x) = \int_{-\infty}^{\infty} e^{iux}\,\psi_{X_t^0}(u)\,du$$

$$= \int_{-\infty}^{\infty} e^{iux}\left(e^{iu\mu}\,2^{1-\frac{\upsilon}{2}}\,\frac{K_{\frac{\upsilon}{2}}(\sqrt{\upsilon}\sigma|u|)}{\Gamma\left(\frac{\upsilon}{2}\right)}\left(\sqrt{\upsilon}\sigma|u|\right)^{\frac{\upsilon}{2}}\right)^t du$$

$$f_{L(t)}(x) = \frac{2^{t(1-\upsilon/2)}}{\pi\Gamma^t\left(\frac{\upsilon}{2}\right)}\int_0^{\infty} cos(ux)e^{iu\mu t}(\sqrt{\upsilon}\sigma|u|)^{\upsilon t/2}K^t{}_{\frac{\upsilon}{2}}\left(\sqrt{\upsilon}\sigma|u|\right)du\ .$$

(18)

The improper integral is always convergent since the asymptotic behavior of the characteristic function is [9]:

$$\phi_{ST}(u;\ \upsilon,\sigma,\mu) = \sqrt{2\pi}\ \frac{(\sqrt{\upsilon}\sigma|u|)^{\frac{\upsilon-1}{2}}e^{-\sqrt{\upsilon}\sigma|u|}[1+O(|u|^{-1})]}{2^{\frac{\upsilon}{2}}\Gamma\left(\frac{\upsilon}{2}\right)},\ |u| \to \infty \quad (19)$$

When $t = 1$ the expression (19) can be exactly calculated and coincides with the PDF (01) of $ST(\upsilon,\sigma,\mu)$.As Student distributions are not closed under convolutions, this distribution at $t = 2$ is not a Student distribution anymore. In general, the convolution of Student distributions and the process integral (18) have to be solved numerically. The integral on the right hand side of (18) can be computed numerically for specific values of υ. The results are shown in Fig 2, where the densities of the convolution semi-group are represented for values of t varying from 1 to 3, with the parameters of $\upsilon = 3$, $\mu = 0$ and unit value of σ. The integral is difficult to evaluate for even υ. Therefore, it is better to see the behavior of Student Lévy process when $\upsilon = 3$ situation (fig. 1).

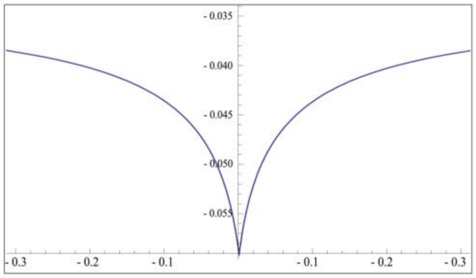

Fig 1. *Modified Lévy density for student t-distribution*

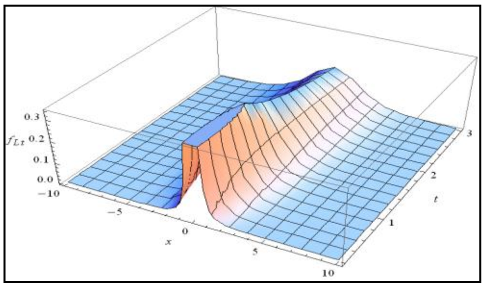

Fig 2. *Convolution Semi-group Densities*

Proposition 2: On the other hand, the Lévy process $X=(X_t)_{t\geq 0}$ is called the Lévy-Student process if, $\mathcal{L}(X_1)=ST(\mu,\sigma^2,\upsilon)$, and it has the following structure; $X_t=L_{G_t}+\mu t$, $t\geq 0$, where $L=(L_t)_{t\geq 0}$ is a arithmetic Brownian motion with the Lévy triplet $(0,\sigma^2,0)$ of Lévy characteristics and $G=(G_t)_{t\geq 0}$ is an independent of L, Lévy subordinator such that; $\mathcal{L}(G_1)=GIG(-\frac{\upsilon}{2},\upsilon,0)$. The characteristic function of X_t can be derived according to the subordination Theorem for the Lévy process as following manner.

Proof: The Lévy exponent of the arithmetic Brownian motion is given by: $\varphi(u)=\frac{1}{2}\sigma^2u^2$, and the Lévy exponent of the GIG process1 is given by

$$\ell(u)=log\left(\frac{2^{1-\frac{\upsilon}{4}}}{\Gamma\left(\frac{\upsilon}{2}\right)}K_{\frac{\upsilon}{2}}\left(\frac{\sqrt{2}\sqrt{-iu}}{\sqrt{\frac{1}{\upsilon}}}\right)(-iu)^{\upsilon/4}(\upsilon)^{\upsilon/4}\right).$$

The characteristic function of the Lévy-student process can be obtained by applying the subordination theorem such that; $\phi_{X_t}(u,t)=exp\{tl(-i\varphi(u))$

$$\phi_x(u,t)=$$
$$exp\left\{t\,log\left(\frac{2^{1-\frac{\upsilon}{4}}}{\Gamma\left(\frac{\upsilon}{2}\right)}K_{\frac{\upsilon}{2}}\left(\frac{\sqrt{2}\sqrt{-i\left(\frac{1}{2}\sigma^2u^2\right)i}}{\sqrt{\frac{1}{\upsilon}}}\right)\left(-i\left(\frac{1}{2}\sigma^2u^2\right)i\right)^{\upsilon/4}(\upsilon)^{\upsilon/4}\right)\right\}$$
$$=\left(\frac{2^{1-\frac{\upsilon}{2}}}{\Gamma\left(\frac{\upsilon}{2}\right)}K_{\frac{\upsilon}{2}}\left(\sqrt{\upsilon}\sigma|u|\right)\left(\sqrt{\upsilon}\sigma|u|\right)^{\frac{\upsilon}{2}}\right)^t.$$

□

[1] The distribution of Generalized Inverse Gaussian, $GIG(-\frac{\upsilon}{2},\upsilon,0)$ is given by:

$f_{GIG}\left(x;-\frac{\upsilon}{2},\upsilon,0\right)=\frac{1}{\Gamma\left(\frac{\upsilon}{2}\right)}x^{-\frac{\upsilon}{2}-1}\exp\left(-\frac{\upsilon^2}{2x}\right)$ for $x>0$.

The characteristic function of $GIG(-\frac{\upsilon}{2},\upsilon,0)$ distribution can be specified as;

$\phi_{GIG}\left(u;-\frac{\upsilon}{2},\upsilon,0\right)=\frac{2^{1-\frac{\upsilon}{4}}}{\Gamma\left(\frac{\upsilon}{2}\right)}K_{\frac{\upsilon}{2}}\left(\frac{\sqrt{2}\sqrt{-iu}}{\sqrt{\frac{1}{\upsilon}}}\right)(-iu)^{\upsilon/4}(\upsilon)^{\upsilon/4}$.

5 CONCLUSIONS

The Student t-distribution has strong reason to be regarded as an alternative model of first choice particularly when the benchmark normal or Black Scholes, model is found to be inadequate. In this study, main financial properties of Student t-distribution is discussed and derived in different ways which are needed for option pricing. Furthermore, the Lévy measure is essential is the study of singularity and absolute continuity of the distribution of Lévy processes. For the case of Student-t Lévy processes, the local behavior of the Lévy measure near *x = 0* is studied. This is the region that is most interesting for the study of singularity and absolute continuity. This knowledge can be applied into a problem in option pricing [9]. (see Kumari, 2013 for more details). The option price is completely undetermined when the stock price models driven by pure-jump Lévy processes with paths of infinite variation. Also, the class of equivalent probability transformations that transform the driving Lévy process into another Lévy process is sufficiently large to generate almost arbitrary option prices consistent with no-arbitrage. For the class of Student-t Lévy processes, we can be specialized this result, by measure transformations such that: transform the driving Student-t Lévy process into a student-t Lévy process.

REFERENCES

[1] B. Mandelbrot, (1963), "The Variation of Certain Speculative Prices," The Journal of Business, 36, 394-419
[2] C. Halgreen, (1977), "Self-Decomposability of the Generalized Inverse Gaussian and Hyperbolic Distribution Functions," Z.Wahrschein. Verw.Gebiete, 47, 13-18.
[3] K. I. Sato, (1999), Lévy Processes and Infinitely Divisible Distributions, Cambridge: Cambridge University Press.
[4] B. Grigelionis, (2013), Student's T-Distribution and Related Stochastic Processes (Vol. 1), London: Springer.
[5] E. Grosswald, (1976), "The Student T-Distribution of Any Degree of Freedom Is Infinity Divisible," Z. Wahrscheinlichkeitstheor.verw.Geb, 36, 103-109.
[6] Z. J. Jurek, (2001), "Remarks on the Self-Decomposability and New Examples," Demonstration Mathematica XXXIV, 2, 241-250.
[7] C. Halgreen, (1989), "Self-Decomposability of the Generalized Inverse Gaussian and Hyperbolic Distributions," Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 47, 13-17 LA - English.
[8] S. Raible, (2000), " Lévy Processes in Finance: Theory, Numerics and Empirical Facts," PhD thesis, Freiburg University, Stockholm School of Economics.
[9] S. N. Kumari and and A. Tan, (2013), "Characterization of Student's T-Distribution with Some Application to Finance," Mathematical Theory and Modeling, 3, 1-9.
[10] N. Cufaro Petroni, S. De Martino, S. De Siena, and F. Illuminati, (2006), "Lévy-Student Processes for a Stochastic Model of Beam Halos," Nuclear Instruments and Methods in Physics Research A, 561, 237-243.
[11] G. Oliver and S. Rafael, (2010), "Scaling of Lévy Student Processes," Physica A, 389, 1455-1463.
[12] S. Nadarajah, and D. K. Dey, (2005), "Convolutions of the T-Distribution," Computers & Mathematics with Applications, 49.

[13] C. C. Heyde, and N. N. Leonenko, (2005), "Student Processes," Advances in Applied Probability, 37, 342-365.
[14] O. E Barndorff-Nielsen and N. Shephard, (2008), Financial Volatility: Stochastic Volatility and Lévy Based Models, Cambridge: Cambridge University Press.