A Note On The Radiation Problem of Water Waves In Presence of A Submerged Line Source With A Bottom Having Step Deformation

Uma Basu, Subhabrata Gangopadhyay

Abstract:- Starting from an asymptotic representation of the velocity potential at infinite distances, the radiation problem of water waves due to a line source in presence of a bottom having step deformation is studied. Relations connecting amplitudes of radiated waves at infinite distances are worked out using Green's second identity. The general asymptotic forms of the potential form at infinite distances are written down in a discrete manner. A matrical representation connecting wave amplitudes is arrived at.

Keywords:- asymptotic representation, green's identity, potential function, radiation problem, step deformation, submerged line source, wave amplitude.

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1 INTRODUCTION

In studies of radiation problems concerning water waves the linearised theory of irrotational motion of water due to various types of singularities in water are of significant role. The radiation problems or the scattering problems are of special interest when the fluid region is bounded below but the bottom has a small deformation in the form of steps of varying heights. Davies [1], Davies and Heathershaw [2], Mei[3], Kirby [4], Mandal and Basu [5, 6] have made an extensive study on water waves scattering over a varying bottom. Miles [7] discussed oblique surface wave diffraction in presence of a cylindrical obstacle. Basu and Mandal [8] threw considerable light on the problem of water waves due to a line source in presence of a geometric deformation at the bottom of water. Thorne [9] studied a number of problems involving line and point singularities in deep water or in water of uniform depth while Rhodes – Robinson [10] studied the effect of surface tension in presence of various types of singularities submerged in water. Evans and Linton [11] attempted the scattering problem with step deformation using a transition matrix approach. The present paper deals with the representation of the form of the potential function in terms of amplitudes of radiated water waves at infinite distances. The Green's second identity has been used in various ways as the mathematical tool to connect the wave amplitudes at infinity. The general asymptotic forms of the potential functions at infinite distances are written down. Finally, a matrical connection amongst the wave amplitudes is attempted.

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2 MATHEMATICAL FORMULATION

It is assumed that the motion of the fluid is generated due to an oscillatory line source of unit strength situated at the point (ξ,η). We shall be concerned with the phenomenon of wave propagation in two-dimensional region described by coordinates x,y in which the waves exist for all x with $x < x_i$ and $x > x_i$. Two distinct types of solution, i.e., radiation conditions represented by velocity potential functions φ and ψ can be considered describing waves incident from either $x = -\infty$ or $x = \infty$ and being radiated at $x = x_i$.

In $x < x_i$,

for $x \to -\infty$ we may write

$$\varphi(x,y) \sim A_i(\xi,\eta)\varphi_0(-x,y) \quad 2.1$$

while as $x \to +\infty$

$$\varphi(x,y) \sim B_i(\xi,\eta)\varphi_0(x,y) \quad 2.2$$

In $x > x_i$,

for $x \to +\infty$

$$\psi(x,y) \sim a_i(\xi,\eta)\psi_0(-x,y)\ , \quad 2.3$$

while as $x \to -\infty$

$$\psi(x,y) \sim b_i(\xi,\eta)\psi_0(x,y) \quad 2.4$$

A_i, b_i, are the amplitudes of the radiated waves at $x = -\infty$ and B_i, a_i are those at $x = +\infty$

$$\varphi_0(x,y) = \psi_0(x,y) = \frac{cosh k_0(h_i - y)}{cosh k_0 h_i} e^{ik_0 x}$$

where k_0 is the unique positive root of the equation

$k\, sinh kh_i - K\, cosh kh_i = 0$ where $K = \frac{\sigma^2}{g}$, σ being the circular frequency, h_i is the step height at $x = x_i$, g is the acceleration due to gravity.

We proceed to find out relations connecting A_i, B_i, a_i and b_i.

3 METHOD OF SOLUTION

Both φ and ψ satisfy Laplace's equation and on the infinite boundaries of the region φ and ψ satisfy the same conditions which may be Neumann, Dirichlet or a mixed condition, then it follows from Green's second identity that

$$\int_C \left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right) ds = 0$$

where C is a closed contour bounding the region. It is clear that the only contribution to this integral arises from the lines , , $x = \pm X, X$ large, where the forms (2.1) to (2.4) apply.

The forms (2.1) to (2.4) are rewritten explicitly as:

In $x < x_i$,

as $x \to -\infty$

$$\varphi(x,y) \sim A_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x}, \quad 3.1$$

as $x \to +\infty$

$$\varphi(x,y) \sim B_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x}, \quad 3.2$$

In $x > x_i$,

as $x \to +\infty$

$$\psi(x,y) \sim a_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x}, \quad 3.3$$

as $x \to -\infty$

$$\psi(x,y) \sim b_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x}, \quad 3.4$$

We apply the above written Green's second identity using the contour bounded by the lines:

$$y = 0(-X \le x \le X),\ x = X(0 \le y \le h)$$
$$y = h(-X \le x \le X),\ x = -X(0 \le y \le h)$$

We get,

$$\int_0^{h_i} A_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x} . b_i(ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x}]_{x=-X}\, dy$$

$$-\int_0^{h_i} b_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x} . A_i(-ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x}]_{x=-X}\, dy$$

$$+\int_0^{h_i} B_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x} . a_i(-ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x}]_{x=X}\, dy$$

$$-\int_0^{h_i} a_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x} . B_i(ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x}]_{x=X}\, dy = 0$$

or, $(A_i b_i - B_i a_i) \int_0^{h_i} \cos h^2 k_0(h_i - y) dy = 0$

It follows that $A_i b_i = B_i a_i$ or, $\frac{a_i}{A_i} = \frac{b_i}{B_i}$ or, $\frac{a_i}{b_i} = \frac{A_i}{B_i}$ 3.5

Again we use Green's second identity in the form

$$\int_C \left(\varphi\frac{\partial\bar{\varphi}}{\partial n} - \bar{\varphi}\frac{\partial\varphi}{\partial n}\right) ds = 0$$

for the same contour, we get,

$$\int_0^{h_i} A_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x} . \bar{A}_i(ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x}]_{x=-X}\, dy$$

$$-\int_0^{h_i} \bar{A}_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x} . A_i(-ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x}]_{x=-X}\, dy$$

$$+\int_0^{h_i} B_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x} . \bar{B}_i(-ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x}]_{x=X}\, dy$$

$$-\int_0^{h_i} \bar{B}_i \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{-ik_0x} . B_i(ik_0) \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} e^{ik_0x}]_{x=X}\, dy = 0$$

or, $(A_i\bar{A}_i - B_i\bar{B}_i) \int_0^{h_i} \cos h^2 k_0(h_i - y) dy = 0$

which implies $|A_i|^2 - |B_i|^2 = 0$

or, $|A_i| = |B_i|$. 3.6

Again using Green's second identity in the form

$$\int_C \left(\psi\frac{\partial\bar{\psi}}{\partial n} - \bar{\psi}\frac{\partial\psi}{\partial n}\right) ds = 0$$

for the same contour, we obtain in a similar manner the relation

$$|a_i| = |b_i|. \quad 3.7$$

Relations (3.5), (3.6) and (3.7) are the three independent results connecting the amplitudes of the radiated waves at infinite distances and no other independent relation can be arrived at. We now generalize the particular forms of φ and ψ by assuming that:

as $x \to -\infty$,

$$\varphi(x,y) \sim \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} (P_i e^{ik_0x} + Q_i e^{-ik_0x}), \quad 3.8$$

as $x \to +\infty$,

$$\varphi(x,y) \sim \frac{\cosh k_0(h_i - y)}{\cosh k_0 h_i} (P_{i+1} e^{ik_0x} + Q_{i+1} e^{-ik_0x}), \quad 3.9$$

and the values of P_{i+1}, Q_{i+1} can be related to those of P_i, Q_i through the original particular solutions φ and ψ.

Thus we write,

$$\begin{pmatrix} P_{i+1} \\ Q_{i+1} \end{pmatrix} = R_i \begin{pmatrix} P_i \\ Q_i \end{pmatrix} \quad 3.10$$

and note that two special solutions are obtained by choosing

$$\left.\begin{array}{ll} \text{(i)} & P_i = 0, Q_i = A_i, P_{i+1} = B_i, Q_{i+1} = 0 \\ \text{(ii)} & P_i = b_i, Q_i = 0, P_{i+1} = 0, Q_{i+1} = a_i \end{array}\right\} \quad 3.11$$

Using these as column vectors in (3.10) we get

$$\begin{pmatrix} B_i & 0 \\ 0 & a_i \end{pmatrix} = R_i \begin{pmatrix} 0 & b_i \\ A_i & 0 \end{pmatrix}$$

where,

$$R_i = \begin{pmatrix} B_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 0 & b_i \\ A_i & 0 \end{pmatrix}^{-1} = -\frac{1}{A_i b_i} \begin{pmatrix} B_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 0 & -b_i \\ -A_i & 0 \end{pmatrix}$$

$$= \frac{1}{A_i b_i} \begin{pmatrix} 0 & B_i b_i \\ A_i a_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{B_i}{A_i} \\ \frac{a_i}{b_i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{B_i}{A_i} \\ \frac{A_i}{B_i} & 0 \end{pmatrix} \quad \text{[using (3.5)]}$$

It is interesting to observe that

$$|det(R_i)| = 1$$

4 CONCLUSION

The radiation problem in presence of a submerged line source is worked out for a variable bottom topography having step deformation. The potential function is discretely framed in terms of amplitudes of radiated waves at infinite distances. Relations connecting amplitudes of radiated waves come out to be of proportionality nature. The moduli of the wave amplitudes at infinite distances are found to be the same on either side of the point of radiation for a fixed step height. Wave amplitudes at $-\infty$ are related to those at $+\infty$ in a matrix form.

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