Intersection Matrices Associated With Non Trivial Suborbit Corresponding To The Action Of Rank 3 Groups On The Set Of Unordered Pairs

BettyChepkorir, John K. Rotich, Benard C. Tonui, ReubenC. Langat

Abstract: In this paper we find intersection numbers and intersection matrices associated with each non trivial sub orbit corresponding to the action of rank 3 groups; The symmetric group $S_n$, alternating group $A_n$ and The dihedral group $D_n$ on the set of unordered pairs. We showed that the column sum of the intersection matrices associated with $\Delta_i$ is equal the length of the suborbit $\Delta_i$. They are also square matrices and of order 3x3.

Index Terms: Intersection Matrices,Non Trivial Suborbit, Action of Rank 3 Groups,Set of Unordered Pairs

1 INTRODUCTION
In 1964, Higman introduced the rank of a group when he worked on finite permutation groups of rank 3. He showed that if $G$ is a group acting transitively on a set $X$, where $|X| = n$ and if $G$ is a rank 3 group of degree $n=k^2+1$, where $k$ is the length of a $G_e$-orbit, $x \in X$ then $n = 5,10, 50$ or 3250 .In 1970, Higman calculated the rank and subdegrees of the symmetric group $S_n$ acting on a 2 element subsets from the set $X=\{1,2,\ldots ,n\}$. He showed that the rank is 3 and the subdegrees are $1,2\left( n-2 \right), \left( n-2 \right) \over 2$. In 1972, Cameron worked on suborbits of multiply permutation groups and later in 1974, he studied suborbits of primitive groups. In 1978, he dealt with the orbits of permutation groups of unordered pairs. In 1977, Neuman extended the work of Higman and Sims to primitive groups, edge coloured graphs and also matrices.

2 INTERSECTION MATRICES ASSOCIATED WITH THE ACTION OF $G = S_n$. ON $X\choose 2$.

2.1 Intersection matrix corresponding to $\Delta_i \{1,2\}$. Taking $a = \{1,2\}$ in $X\choose 2$ and $G_{\{1,2\}}$ - orbits arranged as follows

$\Delta_0 \{1,2\} = \{1,2\}$

$\Delta_1 \{1,2\} = \{\{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}\}$

$\Delta_2 \{1,2\} = \{\{3,4\}, \{3,5\}, \{4,5\}\}$

We therefore arrange $G_{\{1,2\}}$ - orbits as follows

$\Delta_0 \{1,3\} = \{1,3\}$

$\Delta_1 \{1,3\} = \{\{1,2\}, \{1,4\}, \{1,5\}, \{3,2\}, \{3,4\}, \{3,5\}\}$

$\Delta_2 \{1,3\} = \{\{2,4\}, \{2,5\}, \{4,5\}\}$

$\Delta_0 \{3,4\} = \{3,4\}$

$\Delta_1 \{3,4\} = \{\{3,1\}, \{3,2\}, \{3,5\}, \{4,1\}, \{4,2\}, \{4,5\}\}$

$\Delta_2 \{3,4\} = \{\{2,1\}, \{2,5\}, \{1,5\}\}$

From definition 6, the intersection numbers relative to the suborbit $\Delta_i \{1,2\}$ are defined by

$\mu_{ij}^{(l)} = |\Delta_i \{b\} \cap \Delta_j \{1,2\}|, \quad b \in \Delta_j \{1,2\}$

Hence we find the intersection numbers relative to $\Delta_i \{1,2\}$ as

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2.2 Intersection matrix corresponding to $\Delta_2 \{1, 2\}$

From definition 6, the intersection numbers relative to the suborbit $\Delta_2 \{1, 2\}$ are defined by

$$\mu^{(2)}_j = |\Delta_2(b) \mid_\Delta_2 \{1, 2\}|, \quad b \in \Delta_j \{1, 2\}.$$ 

We therefore find the intersection numbers relative to $\Delta_2 \{1, 2\}$ as follows:

$$\mu^{(2)}_{00} = |\Delta_2(b) \mid_\Delta_0 \{1, 2\}| = 0,$$

$$\mu^{(2)}_{01} = |\Delta_2(b) \mid_\Delta_1 \{1, 2\}| = 0,$$

$$\mu^{(2)}_{02} = |\Delta_2(b) \mid_\Delta_0 \{1, 2\}| = 3,$$

$$\mu^{(2)}_{10} = |\Delta_2(b) \mid_\Delta_1 \{1, 2\}| = 0,$$

$$\mu^{(2)}_{11} = |\Delta_2(b) \mid_\Delta_1 \{1, 2\}| = 2,$$

$$\mu^{(2)}_{12} = |\Delta_2(b) \mid_\Delta_1 \{1, 2\}| = 0,$$

$$\mu^{(2)}_{20} = |\Delta_2(b) \mid_\Delta_2 \{1, 2\}| = 1,$$

$$\mu^{(2)}_{21} = |\Delta_2(b) \mid_\Delta_2 \{1, 2\}| = 1,$$

$$\mu^{(2)}_{22} = |\Delta_2(b) \mid_\Delta_2 \{1, 2\}| = 0.$$

By definition 7 the intersection matrix $M_1 = (\mu^{(2)}_{ij})_{i,j}$ associated with $\Delta_1 \{1, 2\}$ is obtained as follows:

$$M_1 = \begin{bmatrix}
\mu^{(2)}_{00} & \mu^{(2)}_{01} & \mu^{(2)}_{02} \\
\mu^{(2)}_{10} & \mu^{(2)}_{11} & \mu^{(2)}_{12} \\
\mu^{(2)}_{20} & \mu^{(2)}_{21} & \mu^{(2)}_{22}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
6 & 3 & 4 \\
0 & 2 & 2
\end{bmatrix}$$

2.3 Properties of the intersection matrices associated with $\Delta_1 \{1, 2\}$ and $\Delta_2 \{1, 2\}$

The column sum of the intersection matrix associated with $\Delta_i$ is equal to the length of the suborbit $\Delta_i$. We can see that the column sum of $M_1$ is 6 also the column sum of $M_2$ is 3. $M_1$ and $M_2$ are square matrices of order 3.
3. Intersection matrices associated with the action of $G = A_3$ on $X^{(2)}$

3.1 Intersection matrix corresponding to $\Delta_1 \{1,2\}$
we take $a = \{1,2\}$ in $X^{(2)}$ and $G_{\{1,2\}}$-orbits arranged as follows
\[
\begin{align*}
\Delta_0 \{1,2\} &= \{1,2\} \\
\Delta_1 \{1,2\} &= \{\{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}\} \\
\Delta_2 \{1,2\} &= \{\{3,4\}, \{3,5\}, \{4,5\}\}
\end{align*}
\]
We therefore arrange $G_6$-orbits as follows
\[
\begin{align*}
\Delta_0 \{1,3\} &= \{1,3\} \\
\Delta_1 \{1,3\} &= \{\{1,2\}, \{1,4\}, \{1,5\}, \{3,2\}, \{3,4\}, \{3,5\}\} \\
\Delta_2 \{1,3\} &= \{\{2,4\}, \{2,5\}, \{4,5\}\}
\end{align*}
\]
\[
\begin{align*}
\Delta_0 \{3,4\} &= \{3,4\} \\
\Delta_1 \{3,4\} &= \{\{3,1\}, \{3,2\}, \{3,5\}, \{4,1\}, \{4,2\}, \{4,5\}\} \\
\Delta_2 \{3,4\} &= \{\{2,1\}, \{2,5\}, \{1,5\}\}
\end{align*}
\]
From definition 1.1.6.2 the intersection matrix $M_1 = \left( \mu^{(1)}_{ij} \right)_i j$, associated with $\Delta_1 \{1,2\}$ where $\mu^{(1)}_{ij}$ are the intersection numbers relative to $\Delta_1 \{1,2\}$ is obtained as follows;
\[
M_1 = \begin{bmatrix} \mu^{(1)}_{00} & \mu^{(1)}_{01} & \mu^{(1)}_{02} \\ \mu^{(1)}_{10} & \mu^{(1)}_{11} & \mu^{(1)}_{12} \\ \mu^{(1)}_{20} & \mu^{(1)}_{21} & \mu^{(1)}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}
\]

3.2 Intersection matrix corresponding to $\Delta_2 \{1,2\}$
From definition 1.1.6.1, the intersection numbers relative to the suborbit $\Delta_2 \{1,2\}$ are defined by
\[
\mu^{(2)}_y = |\Delta_2 (b) I \Delta_2 \{1,2\}|, \quad b \in \Delta_2 \{1,2\}
\]
We therefore find the intersection numbers relative to $\Delta_2 \{1,2\}$ as follows
\[
\begin{align*}
\mu^{(2)}_{00} &= |\Delta_2 \{1,2\} I \Delta_0 \{1,2\}| = 0 \\
\mu^{(2)}_{01} &= |\Delta_2 \{1,2\} I \Delta_1 \{1,2\}| = 0 \\
\mu^{(2)}_{02} &= |\Delta_2 \{1,2\} I \Delta_2 \{1,2\}| = 3 \\
\mu^{(2)}_{10} &= |\Delta_2 \{1,2\} I \Delta_0 \{1,2\}| = 0 \\
\mu^{(2)}_{11} &= |\Delta_2 \{1,2\} I \Delta_1 \{1,2\}| = 2 \\
\mu^{(2)}_{12} &= |\Delta_2 \{1,2\} I \Delta_2 \{1,2\}| = 1 \\
\mu^{(2)}_{20} &= |\Delta_2 \{1,2\} I \Delta_0 \{1,2\}| = 0 \\
\mu^{(2)}_{21} &= |\Delta_2 \{1,2\} I \Delta_1 \{1,2\}| = 1 \\
\mu^{(2)}_{22} &= |\Delta_2 \{1,2\} I \Delta_2 \{1,2\}| = 1
\end{align*}
\]
\[ \mu_{02}^{(2)} = \Delta_2 \left( \{3,4\} \right) \Delta_0 \left( \{1,2\} \right) = 1 \]

\[ \mu_{12}^{(2)} = \Delta_2 \left( \{3,4\} \right) \Delta_1 \left( \{1,2\} \right) = 2 \]

\[ \mu_{22}^{(2)} = \Delta_2 \left( \{3,4\} \right) \Delta_2 \left( \{1,2\} \right) = 0 \]

By definition 1.1.6.2 the intersection matrix \( M_2 = \left( \mu_{ij}^{(2)} \right)_{i,j} \), associated with \( \Delta_2 \{1,2\} \), where \( \mu_{ij}^{(2)} \) are the intersection numbers relative to \( \Delta_2 \{1,2\} \) is obtained as follows:

\[ M_2 = \begin{bmatrix} \mu_{00}^{(2)} & \mu_{01}^{(2)} & \mu_{02}^{(2)} \\ \mu_{10}^{(2)} & \mu_{11}^{(2)} & \mu_{12}^{(2)} \\ \mu_{20}^{(2)} & \mu_{21}^{(2)} & \mu_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix} \]

3.3 Properties of the intersection matrices associated with \( \Delta_i \{1,2\} \), and \( \Delta_2 \{1,2\} \).

The column sum of the intersection matrix associated with \( \Delta_i \) is equal to the length of the suborbit \( \Delta_i \). We can see that the column sum of \( M_1 \) is 6 also the column sum of \( M_2 \) is 3. \( M_1 \) and \( M_2 \) are square matrices of order 3.

4. Intersection matrices associated with the action of \( G=\text{D}_5 \) on \( X \).

By Definition 6, given an arrangement of the \( G\)-orbits, the \( G\)-orbits are arranged such that if \( b \in X \) and \( g(a) = b \) then,

\[ g(\Delta_i(a)) = \Delta_i \left( g(b) \right) = \Delta_j(b) \]

4.1 Intersection matrix corresponding to \( \Delta_i \{1\} \).

Taking \( a = 1 \) in \( X \) and \( G\)-orbits arranged as follows,

\[ \Delta_0\{1\} = \{1\} \]
\[ \Delta_1\{1\} = \{2,5\} \]
\[ \Delta_2\{1\} = \{3,4\} \]

We arrange the \( G\)-orbits as follows:

\[ \Delta_0\{2\} = \{2\} \]
\[ \Delta_1\{2\} = \{1,3\} \]
\[ \Delta_2\{2\} = \{4,5\} \]
\[ \Delta_0\{3\} = \{3\} \]
\[ \Delta_1\{3\} = \{1,5\} \]
\[ \Delta_2\{3\} = \{2,4\} \]

From definition 6, the intersection numbers relative to the suborbit \( \Delta_1 \{1\} \) are defined by

\[ \mu_{ij}^{(1)} = \left| \Delta_i \left( b \right) \right| \Delta_j \{1\}, \quad b \in \Delta_j \{1\} \]

Hence we find the intersection numbers relative to \( \Delta_1 \{1\} \) as follows

\[ \mu_{00}^{(1)} = \left| \Delta_1 \{1\} \right| \Delta_0 \{1\} = 0 \]
\[ \mu_{10}^{(1)} = \left| \Delta_1 \{1\} \right| \Delta_1 \{1\} = 2 \]
\[ \mu_{20}^{(1)} = \left| \Delta_1 \{1\} \right| \Delta_2 \{1\} = 0 \]
\[ \mu_{01}^{(1)} = \left| \Delta_1 \{2\} \right| \Delta_0 \{1\} = 1 \]
\[ \mu_{11}^{(1)} = \left| \Delta_1 \{2\} \right| \Delta_1 \{1\} = 0 \]
\[ \mu_{21}^{(1)} = \left| \Delta_1 \{2\} \right| \Delta_2 \{1\} = 1 \]
\[ \mu_{02}^{(1)} = \left| \Delta_1 \{3\} \right| \Delta_0 \{1\} = 1 \]
\[ \mu_{12}^{(1)} = \left| \Delta_1 \{3\} \right| \Delta_1 \{1\} = 1 \]
\[ \mu_{22}^{(1)} = \left| \Delta_1 \{3\} \right| \Delta_2 \{1\} = 0 \]

By definition 7 the intersection matrix \( M_1 = \left( \mu_{ij}^{(1)} \right)_{i,j} \).
associated with $\Delta_i \{1,2\}$ where $\mu_{ij}^{(i)}$ are the intersection numbers relative to $\Delta_i (1)$ is obtained as follows;

$$M_1 = \begin{bmatrix} \mu_{00}^{(1)} & \mu_{01}^{(1)} & \mu_{02}^{(1)} \\ \mu_{10}^{(1)} & \mu_{11}^{(1)} & \mu_{12}^{(1)} \\ \mu_{20}^{(1)} & \mu_{21}^{(1)} & \mu_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

4.2 Intersection matrix corresponding to $\Delta_2 (1)$

From definition 1.1.6.1, the intersection numbers relative to the suborbit $\Delta_2 (1)$ are defined by

$$\mu_{ij}^{(2)} = |\Delta_2 (b) \cap \Delta_1 (1)|, \quad b \in \Delta_j (1),$$

We therefore find the intersection numbers relative to $\Delta_2 (1)$ as follows

$$\mu_{00}^{(2)} = |\Delta_2 (1) \cap \Delta_0 (1)| = 0$$

$$\mu_{01}^{(2)} = |\Delta_2 (1) \cap \Delta_1 (1)| = 0$$

$$\mu_{20}^{(2)} = |\Delta_2 (1) \cap \Delta_2 (1)| = 2$$

$$\mu_{01}^{(2)} = |\Delta_2 (2) \cap \Delta_0 (1)| = 0$$

$$\mu_{11}^{(2)} = |\Delta_2 (2) \cap \Delta_1 (1)| = 1$$

$$\mu_{21}^{(2)} = |\Delta_2 (2) \cap \Delta_2 (1)| = 1$$

$$\mu_{02}^{(2)} = |\Delta_2 (3) \cap \Delta_0 (1)| = 0$$

$$\mu_{12}^{(2)} = |\Delta_2 (3) \cap \Delta_1 (1)| = 1$$

$$\mu_{22}^{(2)} = |\Delta_2 (3) \cap \Delta_2 (1)| = 1$$

By definition 6 the intersection matrix

$$M_2 = \begin{bmatrix} \mu_{00}^{(2)} & \mu_{01}^{(2)} & \mu_{02}^{(2)} \\ \mu_{10}^{(2)} & \mu_{11}^{(2)} & \mu_{12}^{(2)} \\ \mu_{20}^{(2)} & \mu_{21}^{(2)} & \mu_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

4.3 Properties of the intersection matrices associated with $\Delta_2 (1)$, and $\Delta_2 (1)$.

The column sum of the intersection matrix associated with $\Delta_i$ is equal to the length of the suborbit $\Delta_i$. We can see that the column sum of $M_1$ is 6 also the column sum of $M_2$ is 3. $M_1$ and $M_2$ are square matrices of order 3.

5 CONCLUSION

We conclude that intersection matrices associated with the action of rank 3 on $X^{(2)}$ are square matrices of order 3x3 and that the column sum of the intersection matrices associated with $\Delta_i$ is equal the length of the suborbit $\Delta_i$.

REFERENCES


iii). $X^{(2)}$, the set of unordered pairs from the set $X \{1,2,...,n\}$

iv). $\{t,q\}$, Unordered pair.

v). $\Delta_i$, the $i^{th}$ orbit or suborbit.

vi). $\mu^{(i)}$, the intersection number relative to a suborbit $\Delta_i(a)$.

vii). $M_i$, intersection matrix of a suborbit $\Delta_i(a)$.

2. DEFINITION AND PRELIMINARY RESULTS

Definition 1
The alternating group $A_n$ is the subgroup of $S_n$ comprising of all even permutations. Its order is $\frac{n!}{2}$.

Definition 2
Let $X$ be a finite set $\{1,2,..,n\}$, then a symmetric group of regular $n$-gon is called a dihedral group denoted by $\Delta_n$.

Definition 3
When $G$ act on a set $X$, $X$ is divided into disjointed equivalence classes of the action called orbits. The orbits containing $X$ is called the orbit of $x$, denoted by $\text{Orb}_G(x)$.

Definition 4
Let $G$ be transitive on $X$ and let $G_x$ be the stabilizer of a point $x \in X$. The orbits $\Delta_0, \Delta_1, \Delta_2,...,\Delta_{r-1}$ of $G_x$ on $X$ are the suborbit of $G$.

Definition 5
The rank $r$ of $G$ is the number of the suborbits of $G$ while the lengths of the suborbits of $G$ are known as the subdegrees of $G$.

Note: The cardinalities of the suborbits $\Delta_i$ and rare independent of the choice of $x \in X$.

INTERSECTION NUMBERS AND INTERSECTION MATRICES

Definition 6
Let $G$ be a finite group acting on a finite set $X$ and $\Delta_i(a)$ be the $i^{th}$ orbit of $a \in X$ and for a given arrangement of the $G_x$-orbits. The $G_x$-orbit, $b \in X$, are also arranged such the $g(a)=b$, then $g(\Delta_i(a))=\Delta_i(g(a))=\Delta_i(b)$. The intersection numbers relative to a suborbit $\Delta_i(a)$ are defined by

$$\mu^{(i)}_{ij} = |\Delta_i(b) \cap \Delta_i(a)|, \quad (b \in \Delta_i(a))$$

Definition 7
If the rank of $G$ is $r$, then the $r \times r$ matrix $M_i = (\mu^{(i)}_{ij})_{r \times r}$ is
called the intersection matrix of $\Delta_i(\alpha)$.

**Theorem 1 [Higman, [11]]**

If $|\Delta_i| = n_i$ and $\Delta_i^* = \Delta_i^{*'}$ is the suborbit paired with $\Delta_i$, then

a) $\mu^{(i)}_{ii} = \begin{cases} n_i & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$

b) $\mu^{(i)}_{i'i'} = \begin{cases} 1 & \text{if } j = l' \\ 0 & \text{if } j \neq l' \end{cases}$

c) $n_i\mu^{(i)}_{ij} = n_i\mu^{(j')}_{ji}$ and $n_i\mu^{(i')}_{ij} = n_i\mu^{(i)}_{i'j} = n_i\mu^{(j')}_{i'j}$