Similarity Solution Of Plane Turbulent Mixing Layer

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Abstract: This thesis has been performed for finding similarity solution of plane turbulent mixing layer. Considering the above situation continuity equation and momentum equation have been derived. Then, considering the momentum equation for turbulent fluid flow a third order ordinary differential equation has been derived using similarity transformations which is the governing equation. Finally numerical solution of the governing equation has been achieved by the improvisation of known boundary conditions into the initial boundary conditions. Here, MATLAB has been used to develop a computer program to solve the governing equation using fourth order Runge-Kutta method.

Index Terms:

P	Pressure
u	Velocity component along X axis
v	Velocity component along Y axis
w	Velocity component along Z axis
ρ	Density of the fluid
μ	Absolute viscosity
υ	Kinematic viscosity
υ_t	Eddy viscosity
δ	Mixing layer thickness
ψ	Stream function
U_1	High speed stream velocity
U_2	Low speed stream velocity
$-\overline{u'v'}$	Reynold's shear stress
η	Non dimensional distance
$f(\eta)$	Non dimensional stream function
$y_{1/2}$	Distance along y axis at which $(\overline{u} - U_2)/(U_1 - U_2) = 0.5$

1. Introduction

1.1 General

Mixing phenomena has been of engineering interests since early 1920's. Mixing of two different fluids is much faster in turbulent mixing than in molecular mixing. These occurs high momentum diffusion and rapid variation of pressure and velocity in space and time. The homogeneity in turbulent mixing is not in molecular level but may be to the level of smallest energy containing eddies. Such mixing occurs e.g. in mixing layers which form at the interface of two uniform streams of different velocities. Present endeavor is to study the two dimensional plane mixing layer where the mean flow in planes parallel to a given plane is identical. Plane mixing layer can be classified into different types depending on different bases. Among them single stream and two stream mixing layers are prominent. Single stream mixing layer forms when a single stream discharges into quiescent surrounding fluid and two stream mixing layer forms when two streams of fluids with different velocities are brought in contact. There are many factors that can affect the natural plane mixing layers. The parameters that is known to affect the mixing layer are velocity ratio, trailing edge thickness, presence of trip wire, periodic oscillation force, turbulence level of initial boundary layers, the Reynold's number etc. Among those parameters, velocity ratio is the discernible in the reduced ordinary differential equation.Turbulent mixing layers occur in the flow field of the most engineering applications. Mixing layers are common in combustion chamber, pre-mixer of gas turbine compressors, chemical layers and flow reactors etc. Mixing layer is responsible for most of the broad-band noise generated in propulsion systems. Mixing layer possesses certain universal flow features that have made them very attractive for both experimental and computational studies.

1.2 Objectives

The objectives of this thesis are listed below:

- To derive a third order ordinary differential equation from partial differential momentum equation using similarity transformations for a plane turbulent mixing layer.
- To improvise the known boundary conditions to make it initial conditions.
- To find numerical solution of the governing non linear third order ordinary differential equation using 4th order Runge-Kutta method.
- Study of cross stream mean velocity and Reynold's shear stress.

1.3 Mixing Layer

The mixing layers are produced from two streams of fluids. The development of a typical mixing layer is shown in figure 1 with its nomenclature. The developing region is called as near-field region. The developed region is called as self-preserving region where the flow is fully developed turbulent flow.

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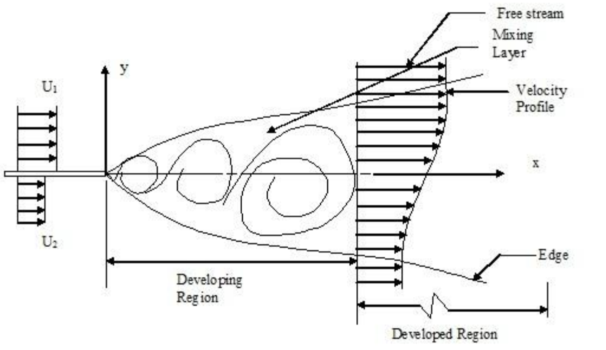

Fig. 1 *Formation of mixing layer for two parallel flowing fluids.*

2. Formulation

Assuming the fluid is incompressible and the flow is steady and no pressure variation along perpendicular direction of flat plate, the governing equations for fluid motion are given by –

Continuity Equation for turbulent case:

$(\partial\bar{u}/\partial x) + (\partial\bar{v}/\partial y) = 0$ [Appendix-A]

Momentum Equation for turbulent case:

$\bar{u}(\partial\bar{u}/\partial x) + \bar{v}(\partial\bar{u}/\partial y) = u_t(\partial^2\bar{u}/\partial y^2)$ [Appendix-B]

Using similarity transformations the third ordinary differential equation has been derived from the partial differential momentum equation.

Similarity transformations are

$\eta = (y - y_{1/2})/\delta$

$\bar{u} = \bar{U} + Uf'(\eta)$

The governing ordinary differential equation and the boundary conditions are:

$f''' + \alpha f''[f + \eta\bar{U}/U] = 0$ [Appendix-C]

at $\eta = 0$, $f(0) = 0$

at $\eta = +\infty$, $f'(+\infty) = 1$

at $\eta = -\infty$, $f'(-\infty) = -1$

For simplicity we assume $\alpha = 1.0$

So, the simplified governing equation is

$f''' + f''[f + \eta\bar{U}/U] = 0$

But these boundary conditions cannot be used to solve the above governing differential equation using 4th order Runge-Kutta method. Because, in Runge-Kutta method all the boundary conditions should be at initial value of independent variable. So, we need to improvise the given boundary conditions (at $\eta = +\infty, -\infty$) to initial conditions (at $\eta = 0$).

2.1 Improvisation of known boundary conditions

From the definition of $y_{1/2}$ we know-

$(\bar{u} - U_2) / (U_1 - U_2) = 1/2$

$\Rightarrow \bar{u} = U_2 + (1/2) \times (U_1 - U_2)$

$\Rightarrow \bar{u} = (U_1 + U_2)/2$

Using this equation in the definition of $\bar{u}$

$\bar{u} = \bar{U} + Uf'(0)$ [at $y = y_{1/2}$, $\eta = 0$]

$\Rightarrow (U_1 + U_2)/2 = (U_1 + U_2)/2 + (U_1 - U_2)/2 \times f'(0)$

$\Rightarrow f'(0) = 0$

So, the initial conditions are

at $\eta = 0$, $f(0) = 0$

at $\eta = 0$, $f'(0) = 0$

As three initial conditions are required to solve a third order ordinary differential equation, the value of $f''(0)$ is necessary. But there is no such condition that the value of $f''(0)$ can be found out. For this reason solution has been found out by iterating the computer program for various assumed values of $f''(0)$. The solution is shown in the results and discussion section.

3. Results And Discussion

The obtained results from the numerical solution is shown below –

3.1 Axial Mean Velocity

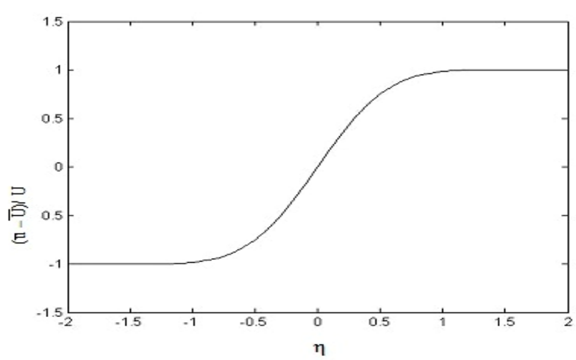

Fig. 2 *Graph of axial mean velocity vs. η*

From the equation of velocity $\bar{u}$ it can be seen that $\bar{u}$ is proportional to $f'(\eta)$. The above graph indicates the profile of axial mean velocity $\bar{u}$ with respect to η. From the graph we can see that the values of $f'(\eta)$ varies between the limit of [-1, 1], which satisfies the boundary conditions as stated above in the formulation section.

3.2 Reynold's shear stress

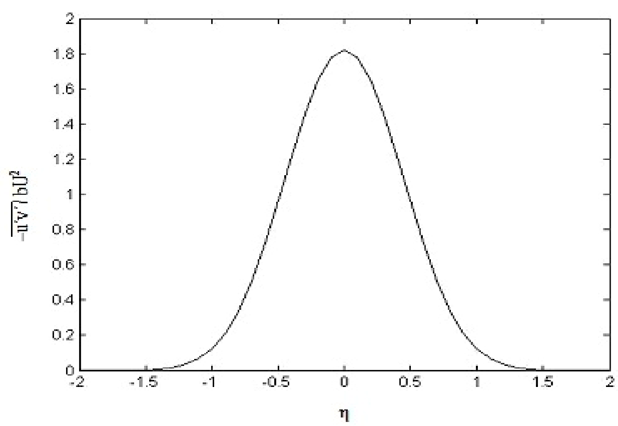

Fig. 3 *Graph of Reynold's shear stress vs. η*

Our simplified governing equation is

$$f''' + f''(f + \eta\bar{U}/U) = 0.$$

In the case $U/\bar{U} << 1$ the equation reduces to

$$f''' + \eta f''\,\bar{U}/U = 0$$

This admits for an error function, with

$f''(0) = 2\sqrt{C}/\sqrt{(2\pi)}$, $C = (\delta'\,U)/(bU)$ [Reference 2]

From the definition of mixing layer thickness, we get at initial condition

$\delta(x) = 2\delta/f''(0)$ which yields $f''(0) = 2.0$

So, the value of $f''(0)$ at initial condition should be 2.0. In our case we have assumed the value of $f''(0) = 1.83$, which is very close to the calculated value. The graph of $f''(\eta)$ vs. η indicates the Reynold's shear stress distribution with respect to non dimensional distance, η. In figure 3 we see that the shear stress is maximum at $\eta = 0$ or at $y = y_{1/2}$. Because, the velocity gradient is maximum at this point and as η increases to +∞ or decreases to -∞ the velocity gradient always decreases and becomes zero at η = +∞ or -∞. So, the Reynold's shear stress is zero at η = +∞ or -∞.

3.3 Cross stream mean velocity

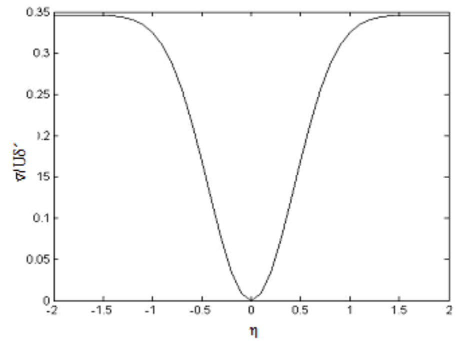

Fig. 4 *Graph of cross stream velocity vs. η*

From the above graph we see that the graph is symmetric with respect to the axis of $v/U\delta'$. Though η tends to be positive or negative value the velocity is always positive and after sometimes it becomes constant with respect to η. From the graph we see that v is zero at η = 0. That means at $y = y_{1/2}$ there is no component of cross stream mean velocity acting along y axis.

4. Appendix

5.1 Appendix-A

Derivation of Continuity Equation

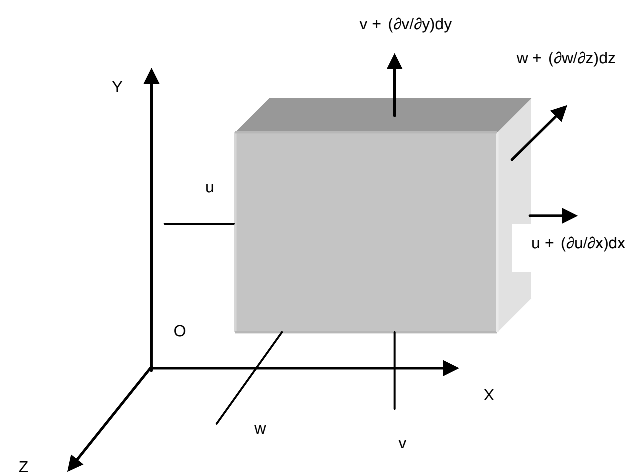

Fig. 5 *Control volume for continuity equation.*

Let us consider, a control volume of length dx, height dy, width dz as shown in figure above is in a flow of a fluid. To simplify the analysis we assume

1. The fluid is incompressible and the flow is steady.
2. There is no pressure variation along y axis.
3. The viscosity is constant.

4. Viscous shear forces along Y axis are negligible.

We know, for steady flow-

Rate of mass flow into the control volume = Rate of mass flow out of the control volume

Mass flow rate along X direction at the left face = $\rho udydz$

Again, mass flow rate along X direction at the right face = $\rho\{u + (\partial u/\partial x)dx\}dydz$

Repeating this for Y & Z direction and substituting these into the equation of steady flow –

$\rho udydz+\rho vdxdz+\rho wdydx = \rho\{u+(\partial u/\partial x)dx\}dydz+\rho\{v+(\partial v/\partial y)dy\}dxdz + \rho\{w + (\partial w/\partial z)dz\}dydx$

$\Rightarrow \rho(\partial u/\partial x)dxdydz + \rho(\partial v/\partial y)dydxdz + \rho(\partial w/\partial z)dzdydx = 0$

Dividing both sides by $\rho dxdydz$ we get-

$(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) = 0$

This is the continuity equation for three dimensional fluid flow.

If the velocity along Z axis is zero, then the equation becomes

$(\partial u/\partial x) + (\partial v/\partial y) = 0$

This is the continuity equation for two dimensional fluid flow.

5.2 Appendix- B

Derivation of Momentum Equation

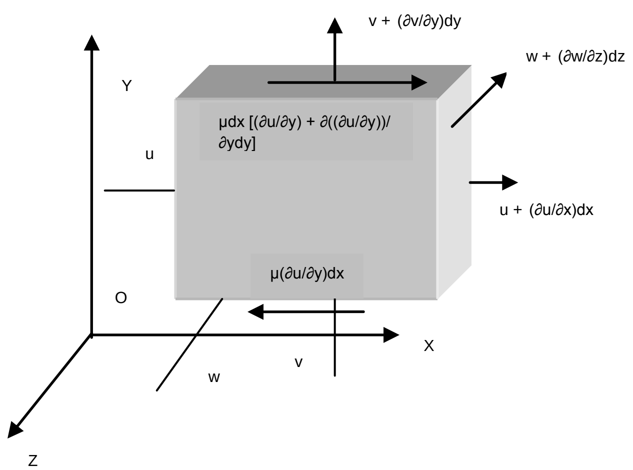

Fig. 6 *Control volume for Momentum equation.*

Let us consider, a control volume of length dx, height dy and unit width at the Z direction. The assumptions are as same as considered in the derivation of continuity equation. The momentum flux in the X direction that enters the bottom face = $\rho vdxu$. The momentum in the X that leaves the top face,

$= \rho\{v + (\partial v/\partial y)dy\}\{u + (\partial u/\partial y)dy\}dx$

The pressure forces on the left face = Pdy

And on right face = $-[P + (\partial P/\delta\partial x)dx]dy$

So, the net pressure force = $Pdy - [P + (\partial P/\partial x)dx]dy$

$= -(\partial P/\partial x)dxdy$

The viscous shear forces at the bottom face = $-\mu(\partial u/\partial y)dx$ and on the top face = $\mu dx[(\partial u/\partial y) + \partial((\partial u/\partial y))/ \partial ydy]$

$= \mu dx[(\partial u/\partial y) + (\partial^2u/\partial y^2)dy]$

So, the net viscous shear force,

$= \mu dx[(\partial u/\partial y) + (\partial^2u/\partial y^2)dy] - \mu (\partial u/\partial y)dx$

$= \mu(\partial^2u/\partial y^2)dxdy$

Equating the sum of the viscous shear and pressure forces to the net momentum transfer in the X direction, we have

$\mu(\partial^2u/\partial y^2)dxdy-(\partial P/\partial x)dxdy = \rho\{v+(\partial v/\partial y)dy\}\{u+(\partial u/\partial y)dy\}dx - \rho uvdx+\rho\{u + (\partial u/\partial x)dx\}^2dy - \rho u^2dy$

$\Rightarrow\mu(\partial^2u/\partial y^2)dxdy - (\partial P/\partial x)dxdy = \rho[u^2+2u(\partial u/\partial x)dx + \{(\partial u/\partial x)dx\}^2]dy - \rho u^2dy + \rho[uv + v(\partial u/\partial y)dy + u(\partial v/\partial y)dy + (\partial u/\partial y)(\partial v/\partial y)dy^2]dx - \rho uvdx$

Clearing terms and neglecting the square terms of derivatives we get,

$\mu(\partial^2u/\partial y^2)dxdy - (\partial P/\partial x)dxdy = 2\rho u(\partial u/\partial x)dxdy + \rho[v(\partial u/\partial y) + u(\partial v/\partial y)]dxdy$

Dividing both sides by dx×dy,

$\Rightarrow \mu(\partial^2u/\partial y^2) - (\partial P/\partial x) = 2\rho u(\partial u/\partial x) + \rho[v(\partial u/\partial y) + u(\partial v/\partial y)]$

$\Rightarrow \mu(\partial^2u/\partial y^2) - (\partial P/\partial x) = \rho u [(\partial u/\partial x) + (\partial v/\partial y)] + \rho u(\partial u/\partial x) + \rho v(\partial u/\partial y)$

Applying continuity equation and if the pressure force is zero then-

$u(\partial u/\partial x) + v(\partial u/\partial y) = (\mu/\rho)(\partial^2u/\partial y^2)$

$\Rightarrow u(\partial u/\partial x) + v(\partial u/\partial y) = \upsilon(\partial^2u/\partial y^2)$

This is the momentum equation for the fluid flowing with constant property.

5.3 Appendix –C

Development of Ordinary Differential Equation Using Similarity Transformation

From Appendix-B we get the momentum equation for turbulent case in the following partial differentiation form -

$\overline{u}(\partial\overline{u}/\partial x) + \overline{v}(\partial\overline{u}/\partial y) = \upsilon_t(\partial^2\overline{u}/\partial y^2)$

Referring to figure 1, definition of $\overline{u}$ can be given as

$$\overline{u} = \overline{U} + Uf'(\eta)$$

Where, η, $\overline{U}$, U can be defined as

$\eta = (y - y_{1/2})/\delta$.

$\overline{U} = (U_1 + U_2)/2$

And $U = (U_1 - U_2)/2$

From the definition of axial mean velocity,

$\overline{u} = \partial\psi/\partial y$

$\Rightarrow \partial\psi = \overline{u}\partial y$;

Reynold's shear stress can be defined as

$$-\overline{u'v'} = \upsilon_t(\partial\overline{u}/\partial y)$$
$$= bU\delta\{(\partial\overline{u}/\partial x)\times(\partial x/\partial y) + (\partial\overline{u}/\partial\eta)\times(\partial\eta/\partial y)\}$$
$$= bU^2\delta f''(\eta)/\delta$$
$$= bU^2 f''(\eta)$$

After integrating this equation we get,

$$\psi = \int\overline{u}dy$$
$$= \int[\overline{U} + Uf'(\eta)]dy$$
$$= \overline{U}y + U\delta f(\eta) + C$$

where, C is an integrating constant.

The cross stream mean velocity is,

$$\overline{v} = -(\partial\psi/\partial x)$$
$$= -[\partial\psi/\partial\delta \times \partial\delta/\partial x + \partial\psi/\partial\eta \times \partial\eta/\partial x] \quad \text{[From chain rule]}$$
$$= -[Uf\,\delta' + U\delta f'(-y/\delta^2)\delta']$$
$$= -[Uf\,\delta' - U\eta f'\delta']$$
$$= U\delta'(\eta f' - f).$$

$$(\partial\overline{u}/\partial x) = \partial\overline{u}/\partial\eta \times \partial\eta/\partial x$$
$$= \partial\{\overline{U} + Uf'(\eta)\}/\partial\eta \times \partial(y/\delta)/\partial x$$
$$= Uf''(-y/\delta^2)\,\delta'.$$

$$(\partial\overline{u}/\partial y) = \partial\overline{u}/\partial\eta \times \partial\eta/\partial y$$
$$= \partial\{\overline{U} + Uf'(\eta)\}/\partial\eta \times \partial(y/\delta)/\partial y$$
$$= Uf''/\delta.$$

$$(\partial^2\overline{u}/\partial y^2) = \partial(\partial\overline{u}/\partial y)/\partial\eta \times \partial\eta/\partial y$$
$$= \partial(Uf''/\delta)/\partial\eta \times \partial(y/\delta)/\partial y$$
$$= U f'''/\delta \times (1/\delta)$$
$$= Uf'''/\delta^2.$$

$$\partial(-\overline{u'v'})/\partial y = \partial(-\overline{u'v'})/\partial\eta \times \partial\eta/\partial y$$
$$= \partial(bU^2f'')/\partial\eta \times \partial(y/\delta)/\partial y$$
$$= bU^2f''' \times 1/\delta$$
$$= bU^2f'''/\delta.$$

From momentum equation,

$$\overline{u}(\partial\overline{u}/\partial x) + \overline{v}(\partial\overline{u}/\partial y) = \upsilon_t(\partial^2\overline{u}/\partial y^2)$$

$$\Rightarrow (\overline{U} + Uf')(-Uf''\eta\delta'/\delta) + U\delta'(\eta f' - f)\,Uf''/\delta = \upsilon_t\,Uf'''/\delta^2$$

$$\Rightarrow (\overline{U} + Uf')(-Uf''\eta\delta'/\delta) + U\delta'(\eta f' - f)Uf''/\delta = (\delta Ub)Uf'''/\delta^2$$

$$\Rightarrow (f' + \overline{U}/U)(-U^2f''\eta\delta'/\delta) + (\eta f' - f)(U^2\delta' f''/\delta) = (bU^2f'''/\delta)$$

$$\Rightarrow (f' + \overline{U}/U)(-f''\eta\delta') + (\eta f' - f)(\delta' f'') = bf'''$$

$$\Rightarrow (\delta' f'')[-\eta f' - \eta U/\overline{U} + \eta f' - f] = bf'''$$

$$\Rightarrow -(\delta' f''/b)[f + \eta U/\overline{U}] = f'''$$

$$\Rightarrow f''' + (\delta'/b)\,f''[f + \eta\overline{U}/U] = 0$$

Let, $(\delta'/b) = \alpha$; where, α is a constant. As δ is the linear function of x, so δ′ is a constant. Then the equation becomes,

$$f''' + \alpha f''[f + \eta U/U] = 0$$

This is the third order ordinary differential equation transformed from the partial differential momentum equation.

5.4 Appendix D

The Algorithm of 4th Order Runge-Kutta Method

The algorithm of 4th order Runge-Kutta method for general solution of a 2nd order differential equation is shown below

$$y_{n+1} = y_n + (k_1 + k_2 + k_3 + k_4)/6$$

where, $k_1 = hf(x_n, y_n)$

$$k_2 = hf(x_n + h/2, y_n + k_1/2)$$

$$k_3 = hf(x_n + h/2, y_n + k_2/2)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

for n = 0, 1, 2, 3 ………… given that $y = y_0$ when $x = x_0$ and for a step size of h.

If there is a third order differential equation, then this equation is reduced to three first order equations. Then, these three first order equations are solved simultaneously using 4^{th} order Runge-Kutta method.

6. Conclusion

Fluid flows are governed by the Navier-Stokes equations, for which there are no practically useful analytical solutions. The equation is therefore usually solved by numerical procedures over a domain discretised into finite elements or volumes. The numerical solutions do not provide a global view of the problem, because in contrast to the general character of analytical solutions each computed case is valid only for a particular set of parameters and boundary condition values, not showing a relation with other cases. This is acceptable for solving particular engineering task, but not for general investigation. From the initial conditions we know that $f(0) = f'(0) = 0$ at $\eta = 0$, which yields that $v = 0$. That means there is no velocity along y axis at $\eta = 0$ or $y = y_{1/2}$. In practical case it may not be zero but it may be very small. We have neglected the pressure gradient by uniform pressure assumption and viscous shear forces being small in deriving the momentum equation. We have also neglected the change of fluid's viscosity.

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