Identifying Break Points In Root Locus Using Bolzano Theorem

Amin Rezaei, Fariborz Rahimi

Abstract: In linear control systems, Evans' rules are used to manually sketch the root locus. One of these rules concerns identifying break-in or breakaway points. Multitude of analytical methods can provide such break points. However, even to verify the existence of break points, each method requires solving an equation. In this paper, using Bolzano theorem, we propose a method of checking the existence of break points in a specific interval without need to solve any equation.

Index Terms: Bolzano Theorem, Break point, Pole, Root Locus, Zero

1 INTRODUCTION

Any point on a branch of root locus represents a root of the characteristic equation. As shown in Fig.1, when two or multiple branches of root locus join at a point or become separated at a point, that point represents multiple roots of the characteristic equation [5], [6], [7], [8], [9].

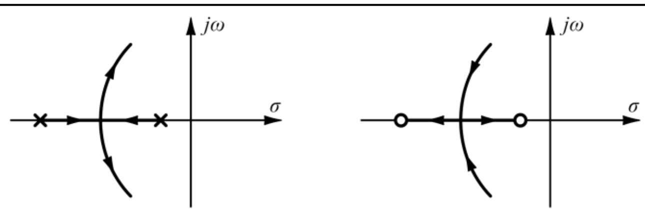

Fig. 1. *A sample break point in the root locus*

Suppose characteristic equation of a system is given as:

$$P(s) + kQ(s) = 0 \quad (1)$$

And the root locus is required with variation in the system parameter k. Then the break points can be found with equation (2).

$$F(s) = P(s)Q'(s) - P'(s)Q(s) = 0 \quad (2)$$

The solutions to the above equation would be considered as the break points if they satisfy equation (1). In other words, those solutions of equation (2) are break points that yield a real-valued k:

$$k = -\frac{P(s)}{Q(s)} \quad (3)$$

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If the resulted k is negative, the break point is on the negative (complimentary) locus, and if the k is positive, the break point would be on the positive root locus. The root locus can have multiple break points and they can be real-valued or complex numbers. All real-valued solutions of the break points equation (2) lay on the complete root locus, as the complete locus covers all the real axis. However, the complex conjugate solutions of equation (2) would be break points only if they satisfy equation (1); that is only if they are part of the complex root locus. In the section 2, we will examine the conditions for existence of the break points and their maximum number. Then in section 3, we will state the Bolzano equation (intermediate value theorem). In section 4, we will demonstrate the method of examination of break points based on this theorem, and clarify it with an example.

2 QUALITATIVE DETERMINATION OF BREAK POINTS IN THE ROOT LOCUS

The following results, that are helpful in qualitative determination of the break points, can be extracted from the definition of these points.

1. If a branch of the root locus resides between two adjacent poles of the open-loop transfer function on the real axis, then there exists at least one break point between these two poles.
2. If a branch of the root locus resides between two adjacent zeros of the open-loop transfer function on the real axis (one can be at infinity), then there exists at least one break point between these two zeros.
3. If a branch of the root locus resides between a zero (finite or infinite) and an adjacent pole (can be at infinity) of the open-loop transfer function on the real axis, then there could exist zero or even number of break points on this section of branch.

Generally, it can be stated that between two adjacent poles or zeros there exist odd number of break points; and between a zero and an adjacent pole there exist even number of break points. Of course, if there is a break point that three branches enter it, it would be considered as two break points as it is a double root of the break point equation [4]. The above mentioned points are depicted in the fig 2. Fig 2-a shows that between the two adjacent finite poles ($s = 0,\ s = -a$), or between a finite zero ($s = -b$) and an infinite zero ($s = -\infty$) there exists a break point. In Fig 2-b, between a zero ($s = -b$) and an adjacent pole ($s = -c$) two distinct break points can be observed. Fig 2-c shows a double break point between a zero and an adjacent pole.

3 Bolzano's Theorem

Bolzano's theorem is well-known in applied mathematics [3], [4] and is used to examine existence of roots for an equation $(F(s) = 0)$ in a specific interval (a, b). The theorem states that:

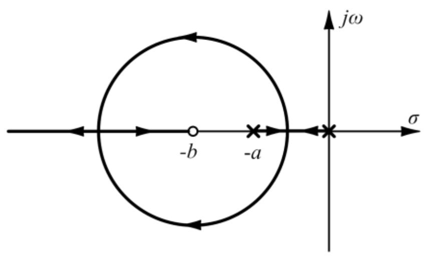

Fig 2-a. Existence of break point between two adjacent poles ($s = 0$, $s = -a$) and between two adjacent zeroes ($s = -b$, $s = -\infty$)

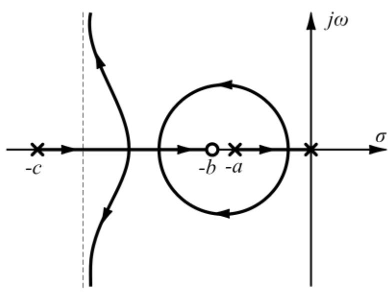

Fig 2-b. Two distinct break points between zero $s = -b$ and adjacent pole $s = -c$.

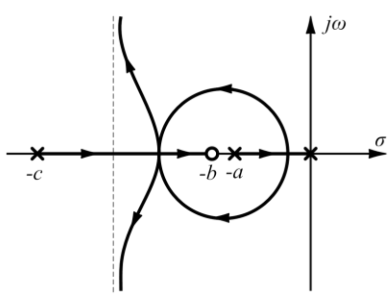

Fig 2-c. *Double break point the between zero $s = -b$ and adjacent pole $s = -c$.*

1. If $\mathrm{F(a)F(b)} < 0$ then $\mathrm{F(s)} = 0$ has at least one root in the (a, b) interval.
2. If $\mathrm{F(a)F(b)} > 0$ then $\mathrm{F(s)} = 0$ has no roots in the (a, b) interval.
3. If $\mathrm{F(a)F(b)} < 0$ and $\mathrm{F(s)}$ is monotone in the (a, b) interval, then $\mathrm{F(s)} = 0$ would only have one root in the (a, b) interval.

In fact, Bolzano's theorem states that to investigate the existence of solution in an interval, there is no need to solve the equation $F(s) = 0$.

4 Investigating the Break Points Using Bolzano's Theorem

To investigate existence of the break points in an interval, equation (2) should be solved which becomes extremely difficult for higher order systems. Remember that if $P(s)$ and $Q(s)$ in the characteristic equation are of orders n and m respectively, then $P'(s)$ and $Q'(s)$ would be of orders $n-1$ and $m-1$. Then the maximum order of equation (2), or the maximum number of the break points, would be $n+m-1$. To illustrate value of Bolzano's theorem in investigating the break points in the root locus, consider the loop transfer function of the following system as an example [4]:

$$GH(s) = \frac{k(s+3)(s+4)}{s(s+1)(s+2)(s+5)(s+6)} \tag{4}$$

We are asked to investigate the presence of break points on the real axis for both positive and negative k values. First, we mark the intervals on the real axis for $k > 0$ and $k < 0$ according to Fig. 3.

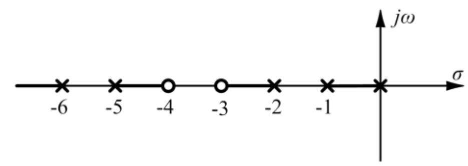

Fig. 3-a. *The root locus of (4) on the real axis for $k > 0$*

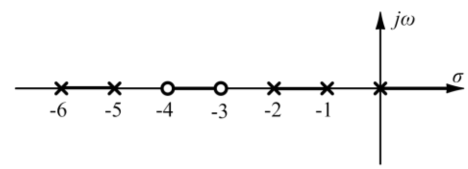

Fig. 3-b. *The root locus of (4) on the real axis for $k < 0$*

Recalling that between two adjacent poles or adjacent zeros there exists a break point, there should be one break point for $k > 0$, and three break points for $k < 0$. Yet, the maximum number of break points for this system is $n+m-1 = 6$. Therefore, we will possibly have two more break points. We know that between a zero and adjacent pole there is either no break point or an even number of them. If the other two break points cannot be in the intervals of $(-2, -3)$ or $(-4, -5)$ then the two break points should be complex numbers, or the system has a maximum of four break points. We examine the existence of break points in the mentioned two intervals using Bolzano's theorem. Full calculation of $P'(s)$ is lengthy and since in applying Bolzano's theorem we only need $P'(-2)$ and $P'(-5)$, we consider $P'(s)$ only in the following two cases:

$$P'(s) = s(s+1)(s+5)(s+6) + (s+2)[A_1(s)] \tag{5}$$
$$P'(s) = s(s+1)(s+2)(s+6) + (s+5)[A_2(s)] \tag{6}$$

The derivative of $Q(s)$ can be easily found:

$$Q'(s) = 2s + 7 \tag{7}$$

Now using equation (2) for investigation of the break points in $(-2, -3)$ interval, we need the following two steps:

$$\begin{aligned} F(-2) &= P'(-2)Q(-2) - P(-2)Q'(-2) \\ &= 24 \times 2 - 0 \times Q'(-2) = 48 \end{aligned} \tag{8}$$
$$\begin{aligned} F(-3) &= P'(-3)Q(-3) - P(-3)Q'(-3) \\ &= P'(-3) \times 0 - (-36) \times 1 = 36 \end{aligned} \tag{9}$$

According to Bolzano's theorem, since$F(-2)F(-3) > 0$, the equation $F(s) = 0$ does not have any roots in $(-2, -3)$ interval. Similarly, we can examine the existence of break points in the other interval:

$$\begin{aligned} F(-4\,) &= P'(-4)Q(-4) - P(-4)Q'(-4) \\ &= P'(-4) \times 0 - 48 \times (-1) = 48 \end{aligned} \tag{10}$$

$$\begin{aligned} F(-5\,) &= P'(-5)Q(-5) - P(-5)Q'(-5) \\ &= (-60) \times 2 - 0 \times Q'(-5) = -60 \end{aligned} \tag{11}$$

Since $F(-4)F(-5) < 0$, the equation $F(s) = 0$ has at least one root in $(-4, -5)$ interval. Considering that this interval is between a zero and an adjacent pole, the number of break points in this interval should be even. Hence, we have two break points in this interval. Therefore, all six break points lie on the real axis out of which three would be on the positive root locus, and three on the complimentary root locus.

5 Conclusion

In this paper, using the theorem of Bolzano was studied to the existence of breakpoints and the approach fully explained using an example. When using this method monotonic check of the function $F(s)$ can also provide useful information about breakpoints. By solving numerous examples, the performance of the new method expressed in this article, can be better understood.

6 References

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