Multi-Criteria Optimization Techniques In DEA: Methods And Its Applications

Dr B.Venkateswarlu Dr B. Mahaboob, Dr C. Subbaramireddy, Dr C. Narayana

Abstract: This research article explores on multi-objective optimization methods namely multi-objective model optimization, global criterion method, bounded objective function method, lexicon-graphic method etc. The multiplier problem proposed by Chames, Cooper and Rhodes (CCR) problem has been discussed in this paper. Own efficiency and cross efficiency of DMU’s are defined here. Aggressive formulation of cross efficiency and benevolent formulation of cross efficiency have been proposed in this article. Besides MCDEA and MOLP problems are presented. The global efficiency approach of a multi objective model and the concept of super efficiency which works as a tool to develop the discriminating power of DEA in the presence of efficiency DMU’s are depicted here.

Keywords: MCDEAP, MOLPP, DEA, DMU, peer count, own efficiency, cost efficiency, super efficiency, global efficiency.

1. INTRODUCTION

Multi-criteria models have more than one objective function such that these object to functions have to minimize simultaneously subject to a common set of constraints. Taking one objective function at a time, one can optimize it subject to a common set of constraints. The optimal solution of one objective need not be optimal for another. Since the objective functions are conflict in nature this would happen. There are two prominent approaches to solve a multiple objective model. The first approach is Pre-emptive and other is based on weighted sum of objective functions. In the former case the objective functions are arranged according to their importance. Taking into consideration the highest priority objective function it is optimized subject to all constraints. The optimal solution gives the optimal value of the objective function. We select the second highest priority objective function and optimize it subject to all the constraints of the multi-objective model and introduce one constraint extra. This constraint is imposed such that the search for optimal solution should not degrade the optimal value of the highest priority problem. In this way one solves as many optimization problems as there are multi-criteria objective functions. After one objective function has been optimized in pre-emptive processing of a multi criteria model, solutions obtained in the sub- sequent stages turnout to be alternative optima in the first. That is pre-emptive optimization places very great emphasis on the first objective, with all later steps limited to alternative optima in the highest priority objective. This fact is not surprising because in pre-emptive approach to solve a multi-criteria optimization problem, in each stage we impose an additional constraint, consequently, the solution space contracts.

Every feasible solution of a later stage is feasible solution of the former stage. Let us consider a dual criteria optimization problem, for which we can construct an efficiency frontier. Out of the two objective functions, we can force one objective function to be a constraint that depends on a parameter. Varying the values of \( \theta \) on an appropriately formulated range the other objective function can be optimized. Measuring the value of \( \theta \) along one axis, say the horizontal axis and the optimal values of the objective function along vertical axis the piece wise linear efficiency frontier can be obtained. A multi-criteria model possesses numerous efficient solutions; the solutions space is not empty. The priority of efficiency is based on pare to optimality. A feasible solution to a multi objective optimization model is said to be efficient if the feasible solution space if no other feasible solution scores at least as well in all objective functions and strictly better in one. Inferior feasible solutions are said to be dominated by efficient solutions. Andrea Kaim et.al, in 2018, in their research article proposed a review of optimization techniques for land use allocation problems and depicted a crystal clear root map for choosing optimization techniques. In 2015 Xiaoya Ma et.al. In their paper described an LUA model constructed on MOAIQA to get the foremost sustainable MOLUA optimization solution. Guadalupe Azuara Garcia, et.al. in 2017 in their research article proposed a multi-objective model for sustainable land use allocation in which land use allocations are generated and NSGA-II has been reshaped.

Fig: (1)
In the above figure the solution space bound by the line segments BC, OA, AD and OC of a multi criteria optimization problem is convex. Two dotted lines pass through the point E that lies on the line segment BD are objective functions. The shaded area partially bound by these lines gives solutions that are better than those in the shaded area. But, none of them is feasible for the multi-objective model optimization problem. E is an efficient. For an efficient point no distinct feasible solution lies in the region bound by the objective function contours passing through this point where each point of this region gives a better solution than the efficient point of the multi-objective model optimization problem.

3. GLOBAL CRITERION METHOD: A GLOBAL CRITERION IS FORMULATED AS FOLLOWS

\[ F(X) = \sum_{i=1}^{n} \left[ \frac{f_i(x) - f_i(x_0)}{f_i(x)} \right] \]

\[ F(X) \] is minimized subject to the constraints, \( g_j(x) \leq 0, \text{for } j = 1,2, \ldots ,k \)

Thus, the problem is expressed as

\[ F(x) = \min \{ F(X); g_j(x) \leq 0, j = 1,2, \ldots ,k \} \]

\( x_i^{*} \) Refers to optimal solution. For any \( x_i \),

\[ f_i(x_i^{*}) - f_i(x_i) \] measures the deviation of the objective function from ideal value.

4. BOUNDED OBJECTIVE FUNCTION METHOD:

In the bounded objective function method the minimum and maximum acceptable achievement levels are specified for each objective function.

\[ L_i \leq f_i \leq V_i \]

The optimal solution is found minimizing the objective function of top most priority. If \( r^{th} \) objective function is of top most priority, then we solve the following optimization problem:

Min \( f_1(x) \)

Subject to \( g_j(x) \leq 0, \text{for } j = 1,2, \ldots ,m \)

\[ L_i \leq f_i(x) \leq V_i, i = 1,2, \ldots ,n, i \neq r \]

5. LEXICO GRAPHIC METHOD:

In this method the objective functions are assigned with priorities depending upon their importance. Suppose \( f_1(x), f_2(x), \ldots , f_n(x) \) be the objective functions expressed according to their priority.

(i) Min \( f_1(x) \)

Subject to \( g_j(x) \leq 0, \text{for } j = 1,2, \ldots ,m \)

Let \( x^{*} \) be the optimal solution for this problem. The optimal value of the objective function is \( f_1(x^{*}) \)

(ii) Solve the following optimization in the second stage:

Min \( f_2(x) \)

Subject to \( f_1(x) = f_1(x^{*}) \) an \( g_j(x) \leq 0, \text{for } j = 1,2, \ldots ,m \)

Let \( x_2^{*} \) be the optimal solution of the problem then Min \( f_2(x) \)

\( = f_2(x_2^{*}) \)

In this process at \( n^{th} \) stage we solve Min \( f_n(x) \)

Subject to \( f_i(x) = f_i(x^{*}), \text{for } i = 1,2, \ldots ,n-1 \) and \( g_j(x) \leq 0, \text{for } j = 1,2, \ldots ,m \)

The optimal solution of this problem is the optimal solution of the multi-objective programming Problem.

6. DEA-GLOBAL EFFICIENCY:

The multiplier problem proposed by Charnes, Cooper and Rhodes (CCR) problem can be expressed of the form.

\[ h_0 = \max h = \frac{\sum_{j=1}^{n} w_j y_{0j}}{\sum_{j=1}^{n} w_j y_{0j}} \]

Subject to \( \sum_{j=1}^{n} u_j x_{ij} \leq 1, \text{for } j = 1,2, \ldots n \) and \( u_i, v_i \geq 0 \)

The suffix ‘0’ refers to the Decision Making Units (DMU), whose efficiency is under evaluation.

\( y_{rj} : r^{th} \) output by \( j^{th} \) DMU and \( x_{ij} : i^{th} \) input employed by \( j^{th} \) DMU

The numerator and denominator of the objective function refer to the ‘Virtual Output’ and ‘virtual input’ respectively as such.
ratio refers to virtual output per unit of virtual input foregone. \(u_r\) and \(v_i\) are outputs and input multipliers, which are unknown. The multipliers are not known a priori, but determined by solving an optimization problem. These multipliers are expected to be non-negative. The DMU under evaluation chooses the multipliers to the best of its advantage forcing virtual output to virtual input ratios of all DMUs including its own ratio not to be exceeding unity. Exploiting the Charness, Cooper transformation the above FPP is reduced to a LPP expressed as follows:

\[
\text{Max } \sum_{r=1}^{n} u_r y_{r0} \\
\text{Subject to } \sum_{i=1}^{n} v_i x_{i0} = 1, \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{n} v_i x_{ij} \leq 0, \text{for } j = 1,2, \ldots, n \text{ and } u_r, v_i \geq 0
\]

The CCR problem accounts constant RTS only.

Own efficiency Cross efficiency: The linear programming problem solved for DMU\(_0\) presents output and input weights to the best of its advantage. The efficiency of DMU\(_0\), computed in this procedure is known as own efficiency. Armed with \(u_{r0}, v_{i0}\) (output and input weights of DMU\(_0\)) DMU\(_0\) evaluates other DMUs. For example DMU\(_1\) as follows

\[
h_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{n} v_i x_{ij}}, j \neq k
\]

\(h_j\) is viewed as cross efficiency of DMU\(_1\) evaluated by DMU\(_0\) with this chosen weights. It is possible that some DMU\(_0\)s emerge to be technically the most efficient. For each one of them efficiency score is unity. For further discrimination of these DMU\(_0\)s and to accord them, one can use the cross efficiency matrix. If \(j^{th}\) DMU is efficient in the standards of its own \(\sum_{k=1}^{n} h_{(j,k)}\) measures mean cross efficiency of DMU\(_j\). Mean cross efficiency reveals how DMU\(_1\) is evaluated by its peers. A decision making unit which is efficient by its standards and whose mean peer appraisal efficiency is the largest may be assigned with rank one. Thus, cross efficiencies can be used for further discriminating the efficient DMU\(_1\). If DMU\(_1\) is efficient and the optimal weights are not unique, so that the \(k^{th}\) column of cross efficiency matrix is generated by the first optimal solution of DMU\(_1\). We can use the aggressive formulation or the benevolent formulation of cross efficiencies and look for unique multiplier estimates. The cross efficiency assigned to DMU\(_1\) by DMU\(_0\) may be expressed as

\[
h(j,k) = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{n} v_i x_{ij}}, j \neq k
\]

If \(j = k\), then \(h(k,k)\) represents self-appraisal efficiency of DMU\(_k\). we propose the following cross efficiency matrix

<table>
<thead>
<tr>
<th>Rated DMU</th>
<th>(h(1,1))</th>
<th>(h(1,2))</th>
<th>(\ldots)</th>
<th>(h(1,k))</th>
<th>(\ldots)</th>
<th>(h(1,n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(h(1,1))</td>
<td>(h(1,2))</td>
<td>(\ldots)</td>
<td>(h(1,k))</td>
<td>(\ldots)</td>
<td>(h(1,n))</td>
</tr>
<tr>
<td>2</td>
<td>(h(2,1))</td>
<td>(h(2,2))</td>
<td>(\ldots)</td>
<td>(h(2,k))</td>
<td>(\ldots)</td>
<td>(h(2,n))</td>
</tr>
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<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\ldots)</td>
<td>(\vdots)</td>
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<td>(\vdots)</td>
</tr>
<tr>
<td>(j)</td>
<td>(h(j,1))</td>
<td>(h(j,2))</td>
<td>(\ldots)</td>
<td>(h(j,k))</td>
<td>(\ldots)</td>
<td>(h(j,n))</td>
</tr>
<tr>
<td>(n)</td>
<td>(h(n,1))</td>
<td>(h(n,2))</td>
<td>(\ldots)</td>
<td>(h(n,k))</td>
<td>(\ldots)</td>
<td>(h(n,n))</td>
</tr>
</tbody>
</table>

\(h(j,j)\): Self-appraisal efficiency of DMU\(_j\) and \(h(j,k)\): Peer appraisal efficiency of DMU\(_j\) by DMU\(_k\). A DMU which is efficient by self-appraisal need not be so by peer appraisal. \(h(j,k)\) is the efficiency of DMU\(_j\) rated by DMU\(_k\). Suppose the CCR problem is solved \(n\) times, one time for one decision making unit and the cross efficiency matrix is obtained.

\[
H(j) = \frac{1}{n-1} \sum_{k=1}^{n} h(j,k)\] and \(H(j)\) measures DMU\(_j\)’s averaged appraisal of its peers against which \(j\) compares itself.

7. AGGRESSIVE FORMULATION OF CROSS EFFICIENCY:

The input and output weights of multiplier problem in which DMU maximizes virtual output per unit of virtual input foregone are not unique since it is likely that the self-appraisal problem possesses multiple optimal solutions. The weights displayed are the weights of the optimal solution that the computer picks up the earliest. One means to get away from the ambiguity is to introduce a secondary objective function so that we formulate and solve a goal programming problem with duals.

Goal-1: Maximize simple efficiency of DMU\(_j\)

Goal-2: Minimize the order DMU\(_j\) cross efficiencies in some manner

The final solution of the goal programming problem leads to the specification of not only unique but also non-zero weights. The goal programming problem is called an aggressive formulation. Benevolent formulation of cross efficiency: The benevolent formulation of cross efficiency is a procedure to formulate a goal programming problem with the primary goal to maximize DMU\(_j\)’s simple efficiency, the secondary goal is to maximize the cross efficiencies of other DMUs. Thus, to further discriminate the decision making units we can use either aggressive or benevolent formulation.

Primary goal:

\[
h_k = \text{Max } h = \sum_{r=1}^{s} u_r y_{r0} \sum_{i=1}^{n} v_i x_{ij}
\]

\[
\text{Subject to } u_r y_{rj} - \sum_{i=1}^{n} v_i x_{ij} \leq 0, j = 1,2, \ldots, n \text{ and } u_r, v_i \geq 0
\]

By CCT the ratio problem is reduced to a LPP.

\[
h_k = \text{Max } h = \sum_{r=1}^{s} u_r y_{r0} \sum_{i=1}^{n} v_i x_{ij}
\]

\[
\text{Subject to } u_r y_{rj} - \sum_{i=1}^{n} v_i x_{ij} \leq 0, j = 1,2, \ldots, n \text{ and } u_r, v_i \geq 0
\]

Let \(H_k = \sum_{j=1}^{n} h_{(j,k)}\) and \(F_k = \sum_{j=1}^{n} h_{(j,k)}\)

Where \(H_k\) measures average appraisal by peers of DMU\(_k\) and \(F_k\) measures average appraisal by peers of DMU\(_k\).

Secondary goal: The \(k^{th}\) DMU maximizes \(F_k\) under benevolent formulation of cross efficiency while it minimizes \(F_k\) under aggressive formulation of cross efficiency.

\[
F_k = \sum_{j=1}^{n} h_{(j,k)} - (n-1) F_k = \sum_{j=1}^{n} h_{(j,k)}\]

The objective function \((n-1)F_k\) is non-linear which cannot be solved by standard linear programming techniques. We replace the non-linear objective function by the following.

\[
F_k^1 = \sum_{j=1}^{n} \left[ \sum_{r=1}^{s} u_r y_{rj} \right] - \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} v_i x_{ij} \right]
\]

\[
F_k^1 = \sum_{j=1}^{n} \left[ \sum_{r=1}^{s} u_r y_{rj} \right] - \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} v_i x_{ij} \right]
\]

\(F_k^1\) is linear in multipliers, its minimization or maximization refers to aggressive or benevolent formulation of cross efficiency. Another reformulation of goal (2) is as follows

\[
(n-1)z_k = \frac{\sum_{j=1}^{n} \sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{n} v_i x_{ij}} = \frac{\sum_{r=1}^{s} u_r \{ \sum_{j=1}^{n} y_{rj} \}}{\sum_{i=1}^{n} v_i \{ \sum_{j=1}^{n} x_{ij} \}}
\]
By CCT the FPP is changed into a LPP.

**Aggressive formulation:**

$$\min M Z_k = \sum_{r=1}^{n} u_r \left( \sum_{j=1}^{n} y_r \right)$$

Subject to

$$\sum_{r=1}^{m} v_{rj} \sum_{j=1}^{n} y_r = 1$$

$$\sum_{r=1}^{m} u_r Y_r - \sum_{j=1}^{n} v_{rj} = 0, j = 1, 2, \ldots, n$$

$$\text{Benevolent formulation:} \max Z_k = \sum_{r=1}^{n} u_r \left( \sum_{j=1}^{n} y_r \right)$$

Subject to

$$\sum_{r=1}^{m} v_{rj} \sum_{j=1}^{n} y_r = 1$$

$$\sum_{r=1}^{m} u_r Y_r - \sum_{j=1}^{n} v_{rj} x_{ij} = 0, j = 1, 2, \ldots, n$$

(i) Average cross efficiency is used to distinguish among the 100% efficient DMUs, establishing a meaningful ranking.

(ii) CE is applied to recognize maverick decision making units. A maverick decision making unit enjoys the greatest relative increment when it moves from peer appraisal to self appraisal. For DMU$\_k$ the maverick index is given as,

$$M_k = \frac{h(k,k) - H_k}{n - 1}$$

The higher $M_k$, the more of a maverick is DMU$\_k$. The maverick index can be calculated to each DMU in competition. The other end of index $M_k$ identifies the all-round performers.

### 8. MULTIPLE CRITERIA APPROACH

**DEA:** Let $d_0$ be the slack associated with the constraint.

$$\sum_{r=1}^{m} u_r Y_r - \sum_{r=1}^{m} u_r \sum_{j=1}^{n} x_{ij} = 0$$

$$d_0 = 0 \Rightarrow \sum_{r=1}^{m} u_r Y_r - \sum_{r=1}^{m} u_r \sum_{j=1}^{n} x_{ij} = 0$$

$\Rightarrow$ DMU$\_0$ is 100% technically efficient. DMU$\_0$ is technically efficient, if and only if $d_0 = 0$

The following linear programs are equivalent.

**Min $d_0$**

Subject to

$$\sum_{r=1}^{m} u_r y_r = 1$$

$$\sum_{r=1}^{m} u_r y_r - \sum_{r=1}^{m} u_r \sum_{j=1}^{n} x_{ij} = 0$$

$$\text{Max } h_0 = \sum_{r=1}^{m} u_r y_r$$

Subject to

$$\sum_{r=1}^{m} u_r y_r = 1$$

$$\sum_{r=1}^{m} u_r y_r - \sum_{r=1}^{m} u_r \sum_{j=1}^{n} x_{ij} = 0$$

The form of the multiple criteria DEA model is not unique. It depends on the efficiency criteria used. The MCDEA problem that has the criteria

(i) Min $d_0$, (ii) Minimize the maximum deviation (iii) Minimize the sum of the deviation.

Consider the following optimization problems:

**Min $d_0$**

Subject to

$$\sum_{r=1}^{m} u_r y_r = 1$$

$$\sum_{r=1}^{m} u_r y_r - \sum_{r=1}^{m} u_r \sum_{j=1}^{n} x_{ij} = 0$$

$$\text{Max } h_0 = \sum_{r=1}^{m} u_r y_r$$

Subject to

$$\sum_{r=1}^{m} u_r y_r = 1$$

$$\sum_{r=1}^{m} u_r y_r - \sum_{r=1}^{m} u_r \sum_{j=1}^{n} x_{ij} = 0$$

Thus, the two linear programming problems (8.2) are equivalent. The optimal values of $d_0$ implied by the two problems are also one and same.

Consider the following LPP:

**Min $M$**

Subject to

$$\sum v_i x_{i0} = 1$$

$$\sum u_r y_{rj} - \sum v_i x_{ij} + d_j = 0, \forall r, i, j$$

$$d_j \geq 0 \text{ and } u_r, v_i \geq 0$$

The optimization problem (8.6) is ‘min sum’ problem. The multiplier space is named for (8.3), (8.4) and (8.5). DMU$\_0$ is min sum efficient if and only if $d_0^\ast = 0$ in the optimal solution of (8.5)

$$\sum d_j^\ast = 0 \Rightarrow d_0^\ast = 0 \forall \sum d_j^\ast = 0$$

The value of $d_0^\ast = 0$ may assume different values under different criteria used as objective function.

$$M_0 = 0 \Rightarrow d_0^\ast = 0 \neq M_0 = 0$$

The multi-max and min sum criteria do not give most favourable weights to the multipliers of the DMU under evaluation. Efficiencies implied by multi max and min sum criteria are more restrictive than the technical efficiency of the standard DEA multiplier problem. It is more difficult to achieve min-max or min sum efficiency than to achieve classical DEA efficiency. The above discussion reveals that the multi-max or min sum criterion yields fewer efficient DMUs. To discriminate efficient decision making units like aggressive and benevolent formulations the min-max and min sum criteria can be implemented. A DMU that is self-rated inefficient may turn out to be peer rated efficient when the objective is minimizing the sum of all devitional variables.
Global efficiency: Data Envelopment analysis MOFPP is given as follows:

\[
\max \left\{ \frac{\sum u_j y_{rj}}{\sum v_i x_{ij}} : j = 1, 2, \ldots, n \right\}
\]

Subject to \(\sum u_j y_{rj} / \sum v_i x_{ij} \leq 1\) and \(u_j, v_i \geq 0\)

Optimization of (8.7) \(n\) times, one time for each decision making unit presents the OE point of (8.7), GPP with \(n\) objectives.

Let \(h_j^* = \frac{\sum u_j y_{rj}}{\sum v_i x_{ij}}\) and optimal efficiency point: \((h_1^*, h_2^* \ldots \ldots h_n^*)\)

In classical DEA approach the self-rated DEA uses input and output weights that are most favourable to it in an attempt to evaluate its technical efficiency. One fractional programming problem is solved for each decision making unit. Consequently, the input and output weights change from one decision making unit to another. The DEA approach places a DMU into one of the two mutually exclusive sets of efficient DMU_N. The method has no power in further discriminating the efficient DMU_n. To much flexibility in the choice of weights leads many DMU_n to emerge as efficient. The global efficiency approach provides a multi-objective model that gives common multiplier weights for all DMU_N in such a way that the resultant own efficiencies are as close as possible to the ideal point \((h_1^*, h_2^* \ldots \ldots h_n^*)\) in some sense.

Consider the following model:

\[\min \theta \sum d_j + (1 - \theta) z\]

Subject to \(\sum u_j y_{rj} + d_j = h_j^* \)

\(z - d_j \leq 0\) and \(u_j, v_i \geq 0\), \(v_i \geq 0\) Where \(\sum = \sum_{j=1}^{n} d_j / \sum_{i=1}^{n} \)

\(d_j = j^{th}\) deviational variable

\(\frac{\sum u_j y_{rj}}{\sum v_i x_{ij}} = \text{Adjusted global efficiency score of } j^{th} \text{DMU, whose value is known. The real number } d_j \text{ stands for departure of}

\(j^{th}\) DMU_N efficiency score from global efficiency score. The optimal solution of (8.8) provides efficiency scores of various DMU_N with common weights. If \(\theta = 1\), the problem (8.8) reduces to a model in which mean of deviational is minimized, consequently, the objective and objective function are \(\min \sum d_j\). If \(\theta = 0\), the objective and objective function reduces to the following:

Minimize \(z\), which is equivalent to minimize the maximum deviation.

Varying \(\theta\) between 0 and 1 the model provides flexibility to obtain the common weights in different sets, one set for one value of \(\theta\).

(i) Consider the in equation

\[\frac{\sum u_j y_{rj}}{\sum v_i x_{ij}} \leq h_j^*\]

\(\iff \sum u_j y_{rj} - h_j^* \sum v_i x_{ij} \leq 0\)

(ii) \(z - d_j \leq 0 \iff d_j - z \leq 0\), where \(d_j = h_j^* - \frac{\sum u_j y_{rj}}{\sum v_i x_{ij}} - z \leq 0\)

The optimization problem (8.8) for \(\theta = 0\) can be equivalently expressed as

\[\min z\]

Subject to \(\sum u_j y_{rj} - h_j^* \sum v_i x_{ij} \leq 0\)

\(h_j^* - \frac{\sum u_j y_{rj}}{\sum v_i x_{ij}} - z \leq 0\) and \(u_j, v_i, z \geq 0\).

The optimization problem (8.8) for \(\theta = 0\) is non-linear

Stage (1) of multi-criteria problem:

For each decision making unit solve the following multiplier problem

\[\sum v_i x_{i0} = 1, \sum u_j y_{rj} - h_j^* \sum v_i x_{ij} + d_j = 0\]
Let $d_j$ and $\bar{d}_j$ be the deviational variable of the linear programming problems (8.6) and (8.15) respectively.

$$d_j \leq \bar{d}_j, \forall j \iff \frac{1}{n}(\Sigma d_j) \leq \frac{1}{n}(\Sigma d_j)$$

$$\text{Min} \left( \frac{1}{n} \Sigma_d \bar{d}_j \right) \geq \text{Min} \left( \frac{1}{n} \Sigma d_j \right)$$

9. PEER COUNT - DISCRIMINATING POWER OF DEA

When a classical DEA problem is solved any DMU of the choice turns out be efficient or inefficient. For efficient DMUs, efficiency score is one and for inefficient DMUs in terms of their returns to scale, the following multiplier problem may be solved:

$$\text{Max } h_j^* = \Sigma u_r y_{rj} + w$$

Subject to

$$\Sigma v_i x_{i0} = 1, \Sigma u_r y_{rj} - h_j^* \Sigma v_i x_{ij} \leq 0 \text{ and } u_r, v_i \geq 0 \quad (8.15)$$

In the optimal solution of the above problem sign of $w$ reveals the nature of the returns to scale, increasing or constant or decreasing.

\[ w^* < 0 \iff \text{RTS are increasing} \]
\[ w^* = 0 \iff \text{RTS are constant} \]
\[ w^* > 0 \iff \text{Returns to scale are decreasing} \]

The classical CCR multiplier problem is given by

$$\text{Max } h_j^* = \Sigma u_r y_{rj}$$

Subject to

$$\Sigma v_i x_{i0} = 1, \Sigma u_r y_{rj} - \Sigma v_i x_{ij} \leq 0 \text{ and } u_r, v_i \geq 0 \quad (8.17)$$

In DEA LP problem if more and more decision variables are added, more and more DMUs emerge to be efficient. To improve discriminatory power of DEA the problem (1) is solved for all DMUs such that one problem for one DMU and efficient DMUs are defined. A DMU evaluated efficient by model one need not be efficient, if model (1), model (2) is solved, fewer of the membrane to be efficient. For further discrimination of efficient, DMUs, of model (2), we depend on peer count. The efficient DMUs are leaders and the inefficient DMUs are followers. The Dual problems of (8.18) and (8.19) are called the envelopment problems. These are formulated as follows.

$$\lambda_j = \text{Min } \lambda$$

Subject to

$$\Sigma_{i=1}^{n} \lambda_j x_{ij} \leq \lambda x_{i0}, \quad i = 1, 2, \ldots, m \quad (8.18)$$

$$\Sigma_{i=1}^{n} \lambda_j x_{rj} \geq u_{rj}, \quad r = 1, 2, \ldots, s \text{ and } \Sigma_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0$$

$$\lambda_2 = \text{Min } \lambda$$

Subject to

$$\Sigma_{i=1}^{n} \lambda_j x_{ij} \leq \lambda x_{i0}, \quad i = 1, 2, \ldots, m \quad (8.19)$$

$$\Sigma_{i=1}^{n} \lambda_j x_{rj} \geq u_{rj}, \quad r = 1, 2, \ldots, s \text{ and } \Sigma_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0$$

$$\text{Min} \{ \lambda: \Sigma_j \lambda_j x_{ij} \leq \lambda x_{i0}, \Sigma_j \lambda_j x_{rj} \geq u_{rj}, \Sigma_j \lambda_j = 1, \lambda_j \geq 0 \} \geq$$

$$\text{Min} \{ \lambda: \Sigma_j \lambda_j x_{ij} \leq \lambda x_{i0}, \Sigma_j \lambda_j x_{rj} \geq u_{rj}, \lambda_j \geq 0 \}\]

In $\lambda_2$ the pure technical efficiency is confounded with scale efficiency of the decision making unit whose efficiency is under evaluation. The LP problem (8.18) admits variable returns to scale while (8.19) admits constant returns to scale only.

For an extremely efficient DMU, say DMU $\alpha$, the optimal solution combined with the constraints under constant returns to scale may be expressed as follows.

$$\Sigma_j \lambda_j y_{rj} = y_{r0}, \forall r, \lambda_j \geq 0, \forall j \neq 0$$

and $\lambda_j^* = 1$

If the DMU $\alpha$ is inefficient, the optimal solution combined with the constraints yield the following.

$$\Sigma_j \lambda_j y_{rj} < y_{r0}, \Sigma_j \lambda_j y_{rj} \geq u_{r0}, \lambda_0 = 0 \text{ and } \lambda_j^* 
eq 1$$

The decision making units which correspond to the envelopment weights, not equal to zero are the peers of an inefficient DMU under evaluation. The peers of an inefficient DMU are one or more efficient DMUs, such an efficient DMU that appears in the peer list of efficient DMUs, the largest number of times is the most efficient DMU efficient DMUs under constant returns to scale formulation. However, possible ties may occur.

10. SUPER EFFICIENCY:

The union of ‘super efficiency’ serves as a tool to improve the discriminating power of data envelopment analysis, in the presence of efficiency DMUs.

The above figure shows a part of the piecewise linear production frontier generated by DMUs A, B and C. If the DMU B is removed from reference technology the production possibility set shrinks. The line segments AB and BC are removed from the frontier production function. Instead, in their place the line segment AC is added. For DMU B the following linear programming problem is solved.

$$\text{Max } h_B^* = \Sigma u_r y_{rj}$$

Subject to

$$\Sigma v_i x_{i0} = 1, \Sigma u_r y_{rj} - \Sigma v_i x_{ij} \leq 0 \text{ and } u_r, v_i \geq 0 \quad (8.15)$$

In DEA LP problem if more and more decision variables are added, more and more DMUs emerge to be efficient. To improve discriminatory power of DEA the problem (1) is solved for all DMUs such that one problem for one DMU and efficient DMUs are defined. A DMU evaluated efficient by model one need not be efficient, if model (1), model (2) is solved, fewer of the membrane to be efficient. For further discrimination of efficient, DMUs, of model (2), we depend on peer count. The efficient DMUs are leaders and the inefficient DMUs are followers. The Dual problems of (8.18) and (8.19) are called the envelopment problems. These are formulated as follows.

$$\lambda_j = \text{Min } \lambda$$

Subject to

$$\Sigma_{i=1}^{n} \lambda_j x_{ij} \leq \lambda x_{i0}, \quad i = 1, 2, \ldots, m \quad (8.18)$$

$$\Sigma_{i=1}^{n} \lambda_j x_{rj} \geq u_{rj}, \quad r = 1, 2, \ldots, s \text{ and } \Sigma_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0$$

$$\lambda_2 = \text{Min } \lambda$$

Subject to

$$\Sigma_{i=1}^{n} \lambda_j x_{ij} \leq \lambda x_{i0}, \quad i = 1, 2, \ldots, m \quad (8.19)$$

$$\Sigma_{i=1}^{n} \lambda_j x_{rj} \geq u_{rj}, \quad r = 1, 2, \ldots, s \text{ and } \Sigma_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0$$

$$\text{Min} \{ \lambda: \Sigma_j \lambda_j x_{ij} \leq \lambda x_{i0}, \Sigma_j \lambda_j x_{rj} \geq u_{rj}, \Sigma_j \lambda_j = 1, \lambda_j \geq 0 \} \geq$$

$$\text{Min} \{ \lambda: \Sigma_j \lambda_j x_{ij} \leq \lambda x_{i0}, \Sigma_j \lambda_j x_{rj} \geq u_{rj}, \lambda_j \geq 0 \}\]

In $\lambda_2$ the pure technical efficiency is confounded with scale efficiency of the decision making unit whose efficiency is under evaluation. The LP problem (8.18) admits variable returns to scale while (8.19) admits constant returns to scale only. For an extremely efficient DMU, say DMU $\alpha$, the optimal solution combined with the constraints under constant returns to scale may be expressed as follows.

$$\Sigma_j \lambda_j x_{rj} = x_{r0}, \forall r, \lambda_j \geq 0, \forall j \neq 0$$

and $\lambda_j^* = 1$

If the DMU $\alpha$ is inefficient, the optimal solution combined with the constraints yield the following.

$$\Sigma_j \lambda_j x_{rj} < x_{r0}, \Sigma_j \lambda_j x_{rj} \geq u_{r0}, \lambda_0 = 0 \text{ and } \lambda_j^* \neq 1$$

The decision making units which correspond to the envelopment weights, not equal to zero are the peers of an inefficient DMU under evaluation. The peers of an inefficient DMU are one or more efficient DMUs, such an efficient DMU that appears in the peer list of efficient DMUs, the largest number of times is the most efficient DMU efficient DMUs under constant returns to scale formulation. However, possible ties may occur.

The efficiency score $\lambda_B$ may be interpreted follows. To produce the output that is produced by DMU B, the remaining DMUs viewed as a convex combination require using an additional input $\lambda_B = \lambda_{xB}$, where $\lambda_{xB}$ is the input saved by DMU B due to super efficiency. The LP problem postulated above is an input oriented problem. Similarly, we can propose an output oriented DEA problem.

$$\theta_B = \text{Max } \theta$$

$$\Sigma_{j=1}^{n} \lambda_j x_{ij} \leq x_{iB}, \Sigma_{j=1}^{n} \lambda_j y_{rj} \leq \theta y_{rB}, \lambda_j \geq 0, \forall j \neq B$$

DMU B is extremely efficient. Its input vector and output vector removed from the reference technology.
0 ≤ θ_b ≤ 1. The efficiency score θ_b may be interpreted as follows. With the input vector \( X_b \) employed by DMU_b is \((1-θ_b)Y_b\).

The production making units involved in competition can be divided into four groups. (i) Extremely efficient DMUs (ii) Efficient but not extremely efficient DMUs (iii) Weakly efficient DMUs (iv) Inefficient DMUs. The super efficiency problems are solved only for extremely efficient DMUs. The removal of a DMU that is nit extremely efficient from the reference technology will not bring any change to the production possible set. Consequently, the efficient scores remain to be the same.

11 CONCLUSION AND FUTURE RESEARCH:
In the above research paper a brief discussion on multi objective optimization methods has been done. Different types of multi objective models have been proposed. Moreover using discriminating power of DEA peer count is estimated. Finally this article presents the notion of super efficiency which serves as a tool to improve the discriminating power of DEA. In the context of future research one can estimate the super efficiency of extremely efficient DMUs and evaluate the different types of efficiency stability regions and their infeasibility in DEA.

12 REFERENCES