Poisson Semigroups With Inverse And Biinverse Square Potentials

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Abstract: In this paper, by means of the principle of subordination, we give explicit formulas for the Poisson kernels with inverse and biinverse square potentials using the well known formulas for the corresponding heat kernels.

Index Terms: Poisson semigroup, inverse square potential, biinverse square potential, Gauss hypergeometric function, Appel hypergeometric functions.

1 INTRODUCTION

The classical Poisson problem on \( \mathbb{R}^n \) is given by
\[
\begin{cases}
\frac{\partial^2 u(y,z)}{\partial y^2} = -\Delta u(y,z), y \in \mathbb{R}^n, y > 0 \\
u(0,x) = u_0(x)
\end{cases}
\]
(1.1)

where \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) is the classical Laplacian on \( \mathbb{R}^n \).

The solution of this problem can be described via the Poisson kernel, where if the functions \( u_0 \) belong to the Lebesgue space \( L^2(\mathbb{R}^n, dx) \) the solution of this problem is given by (Folland [5], p. 93)
\[
u(y,x) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\mathbb{R}^n} \frac{y}{(y^2 + |x-y|^2)^{(n+1)/2}} u_0(x) dx.
\]
(1.2)

In this paper we solve explicitly the following Poisson Problems with inverse and bi-inverse square potentials on \( IR_+ \) and \( IR_+^2 \)
\[
\begin{cases}
\Delta u(z,x) + \frac{\partial^2}{\partial z^2} u(z,x) = 0, (z,x) \in IR_+ \times IR_+, \\
u(0,x) = u_0(x), u_0 \in C_0^0(IR_+),
\end{cases}
\]
(1.3)

and
\[
\begin{cases}
\Delta v(u(z,p) + \frac{\partial^2}{\partial z^2} u(z,p) = 0, (z,p) \in IR_+ \times IR_+^2, \\
u(0,p) = u_0(p), u_0 \in C_0^0(IR_+^2),
\end{cases}
\]
(1.4)

where
\[
\Delta v = \frac{\partial^2}{\partial z^2} + \frac{1+4-v}{2} \frac{1}{z^2},
\]
(1.5)

and
\[
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\]
(1.6)

are the Schrödinger operators with inverse and bi-inverse square potentials.

2 POISSON EQUATION WITH INVERSE SQUARE POTENTIAL

It seems that the first formula for the Poisson kernel with inverse square potential is obtained in (Muckenhoupt-Stein [9]):

\[
u(y,x) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\mathbb{R}^n} \frac{y}{(y^2 + |x-y|^2)^{(n+1)/2}} u_0(x) dx.
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\]
(1.2)
Set $t = z$ and $y = \sqrt{-2}\Lambda$, we obtain
\[ e^{-z^2/2} = \frac{1}{\sqrt{2\pi} \Gamma(\mu)} \int_{\mathbb{R}} e^{-u^2} u^{-1/2} e^{\lambda u/4} du. \]

Where $e^\lambda$ is the heat kernel with the inverse square potential given in (Calin et al. [3], p. 68)
\[ e^{\lambda}(t, x, x') = \frac{2\pi}{(2\pi)^{1/2}} e^{-\frac{(x-x')^2}{4t}} I_0\left(\frac{\lambda x x'}{4t}\right), \]
\[ e^{-z^2/2} = \frac{z}{\sqrt{\pi}} \Gamma(\mu) \int_{\mathbb{R}} e^{-u^2} u^{-1/2} e^{\lambda u/4} du. \]

We use the Lemma 2.1, and the proof of the theorem is Finished.

3 POISSON EQUATION WITH BIINVERSE SQUARE POTENTIAL

The Schrödinger operator with biinverse square potential is considered in (Boyer [2]), for the heat and the wave equation with biinverse square potential see (Ould Moustapha [11], [12]). We recall the following formula for the heat kernel on the quarter plane $R_+^2$.

Proposition 3.1 (Ould Moustapha [11], [12]) The Schwartz integral kernel of the heat operator with biinverse square potential $e^{\lambda_{\mu,\nu}}$ on the quarter plane $R_+^2$ is given, for $p = (x, y), p' = (x', y') \in R_+^2$ and $t \in R_+$, by
\[ H_{\lambda,\mu}(t, p, p') = \frac{\Gamma(t/2 + \mu/2)\Gamma(\mu + 1/2)}{\Gamma(2\mu + 1)\Gamma(2\mu + 1/2)} \left(\frac{1}{4\pi t}\right)^{n/2} \Gamma(\mu) \int_{\mathbb{R}} e^{-u^2} u^{-1/2} e^{\lambda u/4} du. \]

Lemma 3.1 For $Rey > -1/2, Rey > -1/2$ and $|c| + |d| < |p - c| - |c'|$, the following formula holds
\[ \int_0^\infty e^{-p x x'} I_0(c x) I_0(d x') dx = \Gamma(\alpha + \nu + \mu)\Gamma(\nu + 1/2)\Gamma(\mu + 1/2) \left(\frac{2\mu}{\pi}\right)^{n/2} \left(\frac{1}{4\pi}\right)^{n/2} \Gamma(\mu) \int_{\mathbb{R}} e^{-u^2} u^{-1/2} e^{\lambda u/4} du. \]

This is a consequence of the formula (2.2), and the proof of the theorem is Finished.

4 APPLICATION TO BESSEL OPERATOR AND SCHRODINGER EQUATION IN NON-INTEGER DIMENSIONS

Note that the Schrödinger operator with inverse square potential $\Delta_v$ is related to the Bessel operator
\[ \Delta_v = \frac{d^2}{dx^2} + \frac{1}{x^2} \frac{\partial}{\partial x}, \]
\[ x^{v-1/2} \Delta_v x^{v-1/2} = \Delta_v, \]
\[ \Delta_v = \frac{d^2}{dx^2} + \frac{2x}{x^2} \frac{\partial}{\partial x}. \]

The Schrödinger operator with biinverse square potential $\Delta_{\mu,\nu}$ is related to the Laplacian in noninteger dimensions given in (Frank-Stillinger[6]) and (Palmer-Stavrinou [13])
\[ \Delta_{\mu,\nu} = \frac{d^2}{dx^2} + \frac{2x}{x^2} \frac{\partial}{\partial x} + \frac{1}{x^2} \frac{\partial}{\partial x} \]
\[ x^{v-1/2} \Delta_{\mu,\nu} x^{v-1/2} = \Delta_{\mu,\nu} \]
\[ \Delta_{\mu,\nu} = \frac{d^2}{dx^2} + \frac{2x}{x^2} \frac{\partial}{\partial x} + \frac{1}{x^2} \frac{\partial}{\partial x}. \]
Bessel operator $\Delta_\nu$, then we have

$$\tilde{H}_\nu(t, x, x') = (xx')^{1/2-\nu} H_\nu(t, x, x').$$
$$\tilde{P}_\nu(y, x, x') = (xx')^{1/2-\nu} P_\nu(y, x, x').$$

Where $H_\nu(t, x, x')$ and $P_\nu(y, x, x')$ are given by (2.7) and (2.4).

We have the following results for the Laplacian in noninteger dimension.

Corollary 4.2 Let $\tilde{H}^{\nu, \mu}(t, p, p')$ and $\tilde{P}^{\nu, \mu}(t, p, p')$ be the heat and the Poisson kernel associated to the Laplacian in noninteger dimension $\Delta_{\nu, \mu}$. Then for $\nu, \mu$ real, we have

$$\tilde{H}^{\nu, \mu}_\nu(t, p, p') = (xx')^{1-2\nu}(yy')^{1-\mu} H^{\nu, \mu}_\nu(t, p, p'),$$
$$\tilde{P}^{\nu, \mu}_\nu(z, p, p') = (xx')^{1-2\nu}(yy')^{1-\mu} P^{\nu, \mu}_\nu(z, p, p').$$

Where $H^{\nu, \mu}_\nu(t, p, p')$ and $P^{\nu, \mu}_\nu(z, p, p')$ are given by (3.1) and (3.5).

Proof. The proof of this corollary is simple and is left to the reader.

5 NUMERICAL EXAMPLES

The Mathematica plotting gives a much better idea of what happens for $p' = (1,1)$ (see figures 1, 2, 3 and 4 below).

![Figure 1: Graphic for the heat kernel on the half real line $\mathbb{R}_+$, $t = 1, \nu = 1/5$.](image1)

![Figure 2: Graphic for the Poisson kernel on the half real line $\mathbb{R}_+$, $y = 1, \nu = 1$.](image2)

![Figure 3: Graphic for the heat kernel on the quarter $\mathbb{R}_+^2$, $t = 1, \nu = 1/5$.](image3)

![Figure 4: Graphic for the Poisson kernel on the half real line $\mathbb{R}_+$, $y = 1, \nu = 3, \mu = 3/2$.](image4)

REFERENCES

18. 1224 (1977); https://doi.org/10.1063/1.523395.