Self-Adjoint Operator With Triangular Factorization In Hilbert Space

Ahmed Yahya M.H

Abstract: In this paper we examine and apply the issue of triangular factorization of positive self-adjoint operators in Hilbert space; we demonstrate that expansive classes of operators can be factorized.

Keywords: Triangular operators, operators with difference kernels, operator identity, homogeneous kernels.

1. INTRODUCTION

In the Hilbert space $L^2_m(a, a + \epsilon_2)$ we define the orthogonal projectors

$$P_{(x+\epsilon)}f = f(x), \quad a \leq x < (x + \epsilon) \quad \text{and} \quad P_{(x-\epsilon)}f = 0,$$

$$(x - \epsilon) < x \leq a + \epsilon_2,$$ where $f(x) \in L^2_m(a, a + \epsilon_2).

Definition 1.1. A bounded operator $S_-$ on $L^2_m(a, a + \epsilon_2)$ is called lower triangular if for every $(x + \epsilon)$ the relations

$$S_-Q_{(x+\epsilon)} = Q_{(x+\epsilon)}S_-Q_{(x+\epsilon)} \quad \text{(1.1)}$$

are true, where $Q_{(x+\epsilon)} = I - P_{(x+\epsilon)}$.

Definition 1.2. A bounded, positive and invertible operator $S_+$ on $L^2_m(a, a + \epsilon_2)$ is called upper triangular if for every $(x - \epsilon)$ the relations

$$S_+P_{(x-\epsilon)} = P_{(x-\epsilon)}S_+P_{(x-\epsilon)} \quad \text{(1.2)}$$

are true.

Definition 1.3. A bounded, positive and invertible operator $S$ on $L^2_m(a, a + \epsilon_2)$ is said to admit the right triangular factorization if it can be represented in the form

$$S = S_+^2 \quad \text{(1.3)}$$

where $S_+$ and $S_+^{-1}$ are upper triangular, bounded self-adjoint operators.

Definition 1.4. A bounded, positive and invertible operator $S$ on $L^2_m(a, a + \epsilon_2)$ is said to admit the left triangular factorization if it can be represented in the form

$$S = S_-^2 \quad \text{(1.4)}$$

where $S_-$ and $S_-^{-1}$ are lower triangular, bounded self-adjoint operators.

1. Gohberg and M.G. Krein [4] studied the problem of factorization under the assumption

$$S_-^2 - I \in \mathcal{Y}_\infty \quad \text{(1.5)}$$

where $\mathcal{Y}_\infty$ is the set of compact operators. The operators $S_-$ and $S_-$ were assumed to have the form $\sqrt{S} = I + X_+$, $\sqrt{S}^{-1} = I + X_-$; $X_+, X_- \in \mathcal{Y}_\infty$. The factorization method plays an important role in a number of analysis problems.

The factorizing operator $V = S_-^{-1}$ is constructed in an explicit form, also he consider the class of positive operators $S$ which satisfy the operator identity (See [1,2,3,5,6,8,9,11]).

2. TRIANGULAR FACTORIZATION WITH SELF-ADJOINT OPERATOR

Let $S^2_+$ be a linear, bounded, self-adjoint and invertible operator $S^+$ on $L^2_m(a, a + \epsilon_2)$. We introduce the notation

$$[S^2_+]_{(x+\epsilon)} = P_{(x+\epsilon)}S^2_+P_{(x+\epsilon)}, \quad (f_+g)(x+\epsilon) = \int_a^x g(x)f(x)dx$$

where $f(x), g(x) \in L^2_m(a, a + \epsilon_2)$.

We show the following theorem [11].

Theorem 2.1 Let the bounded and invertible operator $S^2_+$ on $L^2_m(a, a + \epsilon_2)$ be positive. For the self-adjoint operator $S^2_+$ to admit the left triangular factorization it is necessary and sufficient that the following assertions are true.

1. There exists an $m \times m$ matrix function $F_0(x)$ such that

$$\text{Tr} \int |F_0(x)|^2dx < \infty, \quad \text{(2.2)}$$

that the $m \times m$ matrix function

$$M(x + \epsilon) = \left(F_0(x), S^2_+^{-1}(x+\epsilon)F_0(x)\right)_{(x+\epsilon)}$$

is absolutely continuous, and almost everywhere

$$\text{det}M(x + \epsilon) \neq 0. \quad \text{(2.4)}$$

2. THE VECTOR FUNCTIONS

$$\int v^*(x,t)f(t)dt \quad \text{(2.5)}$$

are absolutely continuous. Here $f(x) \in L^2_m(a, a + \epsilon_2)$ and

$$v((x + \epsilon), t) = [S^2_+^{-1}(x+\epsilon)]_{(x+\epsilon)} \quad \text{(2.6)}$$

(In (2.3) the self -adjoint operator $[S^2_+]^{-1}_{(x+\epsilon)}$ transforms the matrix column of the original into the corresponding column of the image.)

3. THE OPERATOR

$$V^*f = \left[R^*(x)\right]^{-1} \frac{d}{dx} \int_a^x v^*(x,t)f(t)dt \quad \text{(2.7)}$$

is bounded, invertible and lower triangular with its inverse $[V^*]^{-1}$. Here $R^*(x)$ is an $m \times m$ matrix function such that

$$[R^*(x)]^2 = M(x) \quad \text{(2.8)}$$

Proof. Necessity. We suppose that the self-adjoint operator $S^*$ admits the left triangular factorization (1.4). Let $F_0^*(x) \in L^2_m(a, a + \epsilon_2)$ be a fixed $m \times m$ matrix function satisfying relation (2.2). We introduce the $m \times m$ matrix function

$$R^*(x) = V^*F_0^*(x) \quad \text{(2.9)}$$

where $V^* = [S^2_+^{-1}]_{(x+\epsilon)}$. We can choose $F_0^*(x)$ in such a way that almost everywhere then equality

$$\text{det}R^*(x) \neq 0 \quad \text{(2.10)}$$

is true.

From relations (1.4), (2.3) and (2.9) we have

$$M(x + \epsilon) = \int |R^*(x)|^2dx. \quad \text{(2.11)}$$

Hence the function $M^2(x + \epsilon)$ is absolutely continuous and

$$M(x) = [R^*(x)]^2. \quad \text{(2.12)}$$

Now we use the equality
\[
(f, [S]^{-1}_{x+\epsilon} F_0)_{x+\epsilon} = (V^* f, V^* F_0)_{x+\epsilon}. 
\]

Relations (2.9) and (2.13) imply that
\[
\frac{d}{dx} v(x, t) f(t) dt = R^*(x)(V^* f). 
\]

The necessity is proved.

Sufficiency. Let the conditions 1–3 of Theorem 2.1 be fulfilled. It follows from (2.6)–(2.8) that
\[
V^* F_0 = R^*(x). 
\]

We can write \( \hat{M}(x) = [V^* F_0]^2 \).

From relations (2.6), (2.7) and (2.15) we deduce that
\[
(V^* f, V^* F_0)_{x+\epsilon} = (f, [S]^{-1}_{x+\epsilon} P_{x+\epsilon} F_0)_{x+\epsilon}, \quad \text{i.e.,} \\
V^* P_{x+\epsilon} V^* P_{x+\epsilon} F_0 = [S]^{-1}_{x+\epsilon} P_{x+\epsilon} F_0. 
\]

We define \( v(x - \epsilon_1, t) \) in the domain \((x - \epsilon_1) \leq t \leq a + \epsilon_2\) by the equality \( v(x - \epsilon_1, t) = 0 \). It follows from the triangular structure of the self-adjoint operators \( V^* \) and \([V^*]^{-1}\) that
\[
P_{x-\epsilon_1} [V^*]^{-1} P_{x-\epsilon_1} V^* P_{x-\epsilon_1} = P_{x-\epsilon_1}. 
\]

Hence in view of (2.6) and (2.16) we have
\[
P_{x-\epsilon_1} [V^*]^{-2} v((x - \epsilon_1), t) = P_{x-\epsilon_1} F_0. 
\]

It is easy to see that \( P_{x-\epsilon_1} \equiv \mathcal{S} v((x - \epsilon_1), t) = P_{x-\epsilon_1} F_0. \)

Thus according to relations (2.17) and (2.18), the equality \( (V^*)^{-2} v((x - \epsilon_1), t) \) is satisfied.

\[
(V^*)^{-2} v((x - \epsilon_1), t) = \mathcal{S} v((x - \epsilon_1), t), \mu(t) 
\]

is true. If there exists such a vector function \( f_0(x) \in L^2_m(a, a + \epsilon_2) \) if \( f_0, v((x - \epsilon_1), t) = 0 \), then due to (2.7) the relation
\[
V^* f_0 = 0 
\]

is valid. The self-adjoint operator \( V^* \) is invertible. Hence from (2.20) we deduce that \( f_0 = 0 \). This means that \( v((x - \epsilon_1), t) \) is a complete system in \( L^2_m(a, a + \epsilon_2) \). Using this fact and relation (2.19) we obtain the desired equality
\[
[S]^{-1} = [V^*]^{-2}. 
\]

The theorem is proved. \( \square \)

Corollary 2.2. If the conditions of Theorem 2.1 and Theorem 2.3 are fulfilled, then the corresponding self-adjoint operator \( [S]^{-1} \) can be expressed in the form
\[
[S]^{-1} = [V^*]^2 = U^* U. 
\]

We introduce the notation
\[
C_{x-\epsilon_1} = Q_{x-\epsilon_1} S^* Q_{x-\epsilon_1}, 
\]

\[
[f, g]_{x-\epsilon_1} = \int_{x-\epsilon_1} g^*(x) f(x) dx. 
\]

In the same way as Theorem (2.1) we deduce the following result (See [11]).

Theorem 2.3. Let the bounded and invertible self-adjoint operator \( S^* \) on \( L^2_m(a, a + \epsilon_2) \) be positive. For the self-adjoint operator \( S^* \) to admit the right triangular factorization it is necessary and sufficient that the following assertions are true.

1. There exists an \( m \times m \) matrix function \( F_0(x) \) such that
\[
Tr \int_a^{a + \epsilon_2} |F_0(x)|^2 dx < \infty, 
\]

that the \( m \times m \) matrix function
\[
N(x - \epsilon_1) = [F_0(x), C_{x-\epsilon_1}^{-1} F_0(x)]_{x-\epsilon_1} 
\]

is absolutely continuous, and almost everywhere
\[
det N(x - \epsilon_1) \neq 0. 
\]

2. The vector functions
\[
\int_a^{a + \epsilon_2} u^*(x, t) f(t) dt 
\]

are absolutely continuous. Here \( f(x) \in L^2(a, a + \epsilon_2) \) and \( u((x - \epsilon_1), t) = C_{x-\epsilon_1}^{-1} Q_{x-\epsilon_1} F_0 \).

3. The operator
\[
uf = - [Q^*(x)]^{-1} \int_a^{a + \epsilon_2} u^*(x, t) f(t) dt 
\]

is bounded, upper triangular and invertible together with its inverse \( U^{-1} \).

Here
\[
Q^*(x) Q(x) = -\hat{N}(x), 
\]

The connection between the self-adjoint operators \( V^* \) and \( W \) is given by the following assertion (See [11]).

Proposition 2.4. Let the self-adjoint operator \( V^* \) defined by formula (2.7) be bounded. Then the operator \( W \) is also bounded and
\[
WT^* = V^*. 
\]

Proof. It can be proved by linear algebra methods that (see [9])
\[
T^* Q_{x-\epsilon_1}^{-1} T_{x-\epsilon_1}^{-1} = T^* - [S^*]^{-1} P_{x-\epsilon_1}. 
\]

From relations (2.6), (2.32) and (2.35) we have
\[
T^* w((x - \epsilon_1), t) = T^* F_0 - v((x - \epsilon_1), t). 
\]

Hence the equality
\[
[T^* f, w((x - \epsilon_1), t)]_{x-\epsilon_1} = (T^* f, F_0) - (f, v((x - \epsilon_1), t))_{x-\epsilon_1} 
\]

is true. From formulas (2.7), (2.33) and (2.37) we obtain relation (2.34). The proposition is proved.

Using Proposition 2.4 we deduce [11] the following important assertion.

Proposition 2.5. Let \( S^* \) be a bounded, positive, self-adjoint operator and let the operator \( V^* \) defined by formula (2.7) be bounded. If the relations
\[
V^* F_0 = R^*(x), 
\]

and
\[
V^* f = 0, ||f|| \neq 0 
\]

are true, then the self-adjoint operator \( V^* \) is invertible, the operator \([V^*]^{-1}\) lower triangular, and
\[
T^* = [V^*]^2. 
\]

Proof. The proof is clear from (2.34) and (2.50).

Corollary 2.6. Suppose the hypothesis of Propositions 2.2 and 2.4 are satisfied

(i) \( WT^* F_0 = R^*(x) \),

(ii) \( W[U^* U] F_0 = R^* x \) and hence \( T^* = [U^*]^2 \).

(iii) \( [WT^* F_0]^2 = M(x) \).

Proof:

(i) since \( WT^* = V^* \), \( WT^* F_0 = V^* F_0 = R^* x \).

(ii) \( V^* F_0 = WT^* F_0 = W[S^*]^{-1} F_0 = W[U^* U] F_0 \), which implied that \( T^* = [U^*]^2 \).

(iii) Since \( M(x) = [R^* x]^{-2} = [V^* F_0]^2 = [WT^* F_0]^2 \).

Example 2.7. Let us consider the operator
\[
S^* f = f(x) + \frac{i}{\pi} V^*. P \int_a^{a + \epsilon_2} \frac{c(t)c(x)}{x - t} dt, -\infty < a < \infty, 
\]

\[< a + \epsilon_2 < \infty. \]
where $0 < m < c(t) < 1$. The operator (2.38) does not satisfy condition (1.5) but admits the left triangular factorization (see [7]).

3. SELF-ADJOINT OPERATOR AND FACTORIZATION PROBLEMS

We consider (See [11]) the self-adjoint operators $A^*, S^*, \Pi^*$ and $J$ satisfying the operator identity

$$A^*S^* - S^*A^* = i\Pi^{1/2}. \quad (3.1)$$

We suppose that the self-adjoint operators $A^*$ and $S^*$ act on the Hilbert space $L^2_m(0, a + \varepsilon_2)$, the self-adjoint operator $\Pi^*$ maps $G(\dim G = n < \infty)$ into $L^2_m(0, a + \varepsilon_2)$, the operator $J$ acts on $G$, and $J = J^*$, and $J^2 = I_n$. We note that the operator $\Pi^{1/2}$ has the form

$$\Pi^{1/2}g = [\phi_1(x), \phi_2(x), \ldots, \phi_n(x)]g,$$

where $\phi_k(x)$ are $m \times 1$ vector functions, $g = [g_1, g_2, \ldots, g_m]$, $\phi_k(x) \in L^2_m(0, a + \varepsilon_2)$. Relation (3.1) is fulfilled for the self-adjoint operators $S^*$ which play an important role in the spectral theory of the canonical differential systems (see [9, 11]).

Theorem 3.1. Let the following conditions be fulfilled.

1. The self-adjoint operator $S^*$ is bounded, positive and invertible.
2. The relations

$$A^*P_{(x-e_1)} = P_{(x-e_1)}A^*, \quad 0 \leq x - e_1 \leq a + \varepsilon_2 \quad (3.2)$$

are true.
3. The spectrum of the self-adjoint operator $S^*$ is concentrated at the origin and there is a constant $M > 0$ such that

$$\|P_{(x-e_1)}A^*(P_{(x-e_1)} + \delta(x-e_1)) - P_{(x-e_1)}\Pi^{1/2}\| \leq M|\delta(x-e_1)|, \quad 0 \leq x - e_1 \leq a + \varepsilon_2. \quad (3.3)$$

Then the $n \times n$ matrix function $W'(x) = I_n + izS^{-1}(x-e_1) - \Pi^{1/2}P_{(x-e_1)}$ satisfies the integral equation

$$W(x, z) = I_n + iz\int_0^x [dB(t)]W(t, z), \quad (3.5)$$

where

$$B(x + e) = [\Pi^{1/2}]^2[S]^{-1}[(x-e_1)P_{(x-e_1)}]. \quad (3.6)$$

From relations (1.4) and (3.6) we obtain (See [11]) the necessary conditions for the self-adjoint operator $S^*$ to admit the left triangular factorization.

Proposition 3.2. Let the self-adjoint operator $S^*$ satisfy the relation (3.1) and let the conditions of Theorem 3.1 be fulfilled. If the self-adjoint operator $S^*$ admits the left triangular factorization, then the matrix function $B(x)$ is absolutely continuous and

$$\frac{dB}{dx}(x) = H(x) = [\beta^*(x)]^2, \quad (3.7)$$

where

$$\beta^*(x) = [h_1(x), h_2(x), \ldots, h_n(x)], \quad h_2(x) = V^*V(x), \quad (3.8)$$

Using relations (3.5) and (3.7) we obtain that

$$\frac{d}{dx}W(x, z) = izH(x)W(x, z). \quad (3.9)$$

Lemma 3.3. Let the conditions of Proposition 3.2 be fulfilled and let the $n \times 1$ vector functions $F_j(x, z) = \{I - A^*\}^{-1}P_{(x-e_1)}$, $1 \leq j \leq n \quad (3.10)$

form a complete system in $L^2_m(a, a + \varepsilon_2)$. Then we have the equality

$$\text{mes}E = 0, \quad (3.11)$$

where the set $E$ is defined by the relation $x \in E$ if $H(x) = 0. \quad (3.12)$

Proof. We use the following relation (see [9]):

$$J - W'((x+\epsilon, \mu))W((x+\epsilon, \lambda)) \quad \begin{cases}
\mu - \lambda = \Pi^*(I - \mu A^*)^{-1}'(S^*)^{-1/2}(I - \lambda A^*)^{-1}P_{(x+\epsilon)}\Pi).
\end{cases} \quad (3.13)$$

Formula (3.13) implies that

$$\left[\left[S^*\right]^{-1/2}F_j(x, \lambda), F_l(x, \mu)\right]_{(x+\epsilon)} = \frac{d}{d(x+\epsilon)}\left[\left[I\right]^{-1}Y_j((x+\epsilon), \lambda) - Y_j((0, \mu))Y_l((0, \lambda))\right] \quad \begin{cases}
\mu - \lambda = \left[S^*\right]^{-1/2}F_j(x, \lambda), F_l(x, \mu)\right]_{(x+\epsilon)} = 0, (x + \epsilon) \in E. \quad (3.15)
\end{cases}$$

From (3.12) and (3.15) it follows that

$$\frac{d}{d(x+\epsilon)}\left[\left[I\right]^{-1}F_j(x, \lambda), V^*F_l(x, \mu)\right]_{(x+\epsilon)} = 0, (x + \epsilon) \in E, \quad (3.16)$$

i.e., the relation

$$[V^*F_j(x, \lambda), \lambda = 0, x \in E, \quad 1 \leq j \leq n, \quad (3.17)$$

is true.

As the self-adjoint operator $V^*$ is invertible and the system of functions $F_j(x, \lambda)$ is complete in $L^2_m(0, a + \varepsilon_2)$, the system of the functions $V^*F_j(x, \lambda)$ is also complete in $L^2_m(0, a + \varepsilon_2)$, the assertion of the lemma follows from this fact and equality (3.17).\]

Further we suppose that the $n \times n$ matrix function $B(x)$ is absolutely continuous and that relations (3.7), (3.8) are true [11]. Let us introduce the $m \times n$ matrix functions $R(x) = k_1(x)\alpha_1 + k_2(x)\alpha_2 + \cdots + k_m(x)\alpha_n, \quad (3.19)$

$F_0(x) = [\alpha_1], \quad (3.20)$

$V(x, x) = [\left[S^*\right]^{-1/2}P_{(x+\epsilon)}F_0(x)], \quad (3.20)$

where $\alpha_k$ are constant $1 \times m$ matrices. From Proposition (3.2) we deduce:

Corollary 3.4. Let the conditions of Theorem 3.1 and Lemma 3.3 be fulfilled. If $m = 1$, then there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that almost everywhere we have the inequality

$$R(x) = 0. \quad (3.21)$$

Now we can verify the formulation of the result obtained by [11].

Theorem 3.5. Let the following conditions be fulfilled.

1. The self-adjoint operator $S^*$ satisfies relation (3.1).
2. The conditions of Theorem 3.1 are valid.
3. The matrix function $B(x)$ is absolutely continuous and formulas (3.7) and (3.8) are true.
4. The vector functions $F_j(x, \lambda)$ $(1 \leq j \leq n)$ form a complete system in $L^2_m(a, a + \varepsilon_2)$.
5. Almost everywhere the inequality

$$\det R(x) = 0 \quad (3.22)$$

holds. Then the self-adjoint operator $T^* = [S^*]^{-1}$ admits the right triangular factorization.

Proof. We introduce the self-adjoint operator

$$V^*f = \int [R^*(x)]^{-1} \frac{d}{dx} v^*(x, t)f(t)dt. \quad (3.23)$$

From (3.4), (3.22) and (3.23) we deduce the equality

$$V^*F_j = [h_1(x), \ldots, h_n(x)]', \quad (3.24)$$

Relation (3.24) implies that...
\[ \left( V^*F_1(x, \lambda), V^*F_2(x, \mu) \right) \quad \text{for all } \alpha + \varepsilon > 0. \]

Using equality (3.24) and relation
\[ \frac{d}{dx} Y_1(x, z) = izH(x)Y_1(x, z), \]
we have
\[ \left( V^*F_1(x, \lambda), V^*F_2(x, \mu) \right) = \left[ iY_1^*(a + \varepsilon, \mu)Y_1(a + \varepsilon, \lambda) - iY_1^*(0, \mu)Y_1(0, \lambda) \right]. \]

Comparing formulas (3.14) and (3.27) we obtain the equality
\[ T^* = [V^*]^2. \]

This means that the introduced self-adjoint operator \( V^* \) is bounded, \( V^* f \neq 0, \) and \( \| f \|^2 \neq 0. \) Taking into account (3.18), (3.19) and (3.24) when \( z = 0 \) we obtain the relation
\[ V^* F_0^* = R^*. \]

Thus all conditions of Proposition 2.7 are fulfilled. The assertion of the theorem follows from Proposition 2.7. \( \square \)

Proposition 3.6. Let the following conditions be fulfilled.
1. Conditions 1–3 of Theorem 3.5 are valid.
2. The \( m \times m \) blocks \( b_{ij}(x) \) (1 \( \leq j \leq n \)) of the matrix \( B(x) \) are absolutely continuous and
\[ b_{ij}(x) = h_i(x)h_j(x). \]
3. All the entries of the matrices \( h_i(x) \) belong to \( L^2(a, a + \varepsilon). \)
4. Almost everywhere the inequality (3.22) holds. Here \( R^*(x) = h_i(x). \) Then the self-adjoint operator \( V^* \) defined by formula (3.23) and the equality
\[ v(x - \varepsilon), x = [S^{'\dagger}]_{x, x - \varepsilon}P_{x, x - \varepsilon} \phi_1(x) \]
are bounded.
Proof. We introduce the matrix \( H(x) = [\beta^*(x)]^2 \) where \( \beta^*(x) = [h_1(x), h_2(x), \ldots, h_n(x)]. \) Relations (3.23)–(3.25) remain true. We use the formula
\[ \int_0^1 Y_i^*(x, \mu)[dB(x)]Y_j(x, \lambda)dx = \int_0^1 Y_i^*(a + \varepsilon, \mu)Y_j(a + \varepsilon, \lambda) - Y_i^*(0, \mu)Y_j(0, \lambda) \]
\[ \mu - \lambda \]
And the inequality \( H(x)dx \leq dB(x). \) From formulas (3.14), (3.25) and (3.32) we deduce that
\[ [V^*]^2 \leq T. \]

The proposition is proved. \( \square \)

4. SELF-ADJOINT OPERATORS WITH SUM-DIFFERENCE KERNELS
Let us consider the bounded, positive, self-adjoint and invertible operator \( S^* \) with the difference kernel
\[ S^* f = \frac{d}{dx} \int_0^x f(t) s(x - t) dt. \]

Let us put
\[ A^* f = i \int_0^x f(t) dt, \quad f \in L^2(0, a). \]

Equality (3.1) is valid (see [8]). If
\[ J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]
\[ \phi_1(x) = M(x), \phi_2(x) = 1, \]
where \( M(x) = s(x), \) \( 0 \leq x \leq a. \) In the case under consideration the matrix \( B(x + \varepsilon) \) has the form
\[ B(x + \varepsilon) = \begin{bmatrix} (S^{'\dagger})_{x + \varepsilon}M, M & (S^{'\dagger})_{x + \varepsilon}^{-1}M, 1 \\ (S^{'\dagger})_{x + \varepsilon}^{-1}M, 1 \end{bmatrix}. \]

The corresponding function \( F(x, \lambda) \) has the form
\[ F(x, \lambda) = e^{i\lambda x}. \]

The self-adjoint operator \( A^* \) defined by formula (4.2) satisfies all the conditions for Theorem 3.1. The following fact is useful here. (See [11]).

Theorem 4.1. Let the self-adjoint operator \( S^* \) be bounded, positive, invertible and have the form (4.1). If the matrix function \( B(x) \) is absolutely continuous and
\[ B(x) = \beta^*(x) \beta(x) = [h_1(x), h_2(x)], \]
Then the equality
\[ h_1(x)h_2(x) + h_2(x)h_1(x) = 1 \]
is true almost everywhere.
Proof. Let us consider the expression
\[ i(x + \varepsilon) = (S^{'\dagger})_{x + \varepsilon}P_{x + \varepsilon}M + (1, (S^{'\dagger})_{x + \varepsilon}^{-1}P_{x + \varepsilon}M). \]

Setting
\[ N_1(x, (x + \varepsilon)) = (S^{'\dagger})_{x + \varepsilon}^{-1}P_{x + \varepsilon}M, \]
we rewrite formula (4.9) in the form
\[ i(x + \varepsilon) = \int_0^{x + \varepsilon} [N_1(x, (x + \varepsilon)) + N_1(x, (x + \varepsilon))] dx, \]
where
\[ N_1(x, (x + \varepsilon)) + N_1(x, (x + \varepsilon) - x, (x + \varepsilon)) = 1. \]

In view of (4.11) and (4.12) we obtain the equality
\[ i(x + \varepsilon) = (x + \varepsilon). \]

Takings into consideration Equations (2.1), (3.8), (4.1) and (4.9) we deduce that
\[ i(x + \varepsilon) = \int_0^{x + \varepsilon} [h_1(x)h_2(x) + h_2(x)h_1(x)] dx. \]

Relation (4.8) follows from (4.13) and (4.14). The theorem is proved. \( \square \)

From equality (4.8) we have
\[ h_1(x) = 0, \quad 0 \leq x \leq a. \] (4.15)

Theorem 4.2. Let the self-adjoint operator \( S^* \) be positive, invertible and have the form (4.1). The self-adjoint operator \( S^* \) admits the left triangular factorization if and only if the matrix \( B(x) \) is absolutely continuous and relation (4.7) is valid.

Example 4.3. Let us consider (See [11]) the self-adjoint operator \( S^*_p \) of the form
\[ S^*_p f = f + \frac{i\beta}{\pi} V^*: \int_0^{\alpha + \varepsilon} f(t) \frac{d}{dx} \int_0^x f(t) s(x - t) dt, \]
where
\[ W_\alpha f = \frac{\alpha}{\Gamma(\alpha - 1)} \int_0^x f(t)(x - t)^{-\alpha - 1} dt. \] (4.17)

Here \( \alpha = \frac{1}{\pi} \text{arctg} \beta, \) and \( \Gamma(z) \) is the gamma function.

We consider the following class of bounded, self-adjoint and positive operators which can be represented in the form \((+, -) - \text{class});\)
acting in the Hilbert space $L^2(0, a + \epsilon_2)$. belongs to the class $R_1^*$ (rank 1) if the following conditions are fulfilled: 

1) $m(f, f) \leq (S^f, f) \leq M(f, f)$, $0 < m < M < \infty$ 

2) $\text{rank}(AS - SA') = 1$, i.e., $(AS - SA')f = i(f, \phi)\phi$, $\phi(x) \in L^2(0, a + \epsilon_2)$.

We associate with the operator $S$ the operator $S_{-f} = \frac{d}{dx} \int f(t)\phi(x - t)dt$.

It is easy to see that $S_{-1} = \phi$. 

Note that if $A'S^* = S'A^*$ then the rank = 0 

Lemma 5.2. Let the bounded self-adjoint operator $S^*$ satisfy relation (5.3). If the corresponding operator $S^*$ is bounded, then the representation

$$S^* = [S^*]^2$$

is true. 

Proof. We consider the operator $X = [S^*]^2$.

Using formula (5.3) and relation $A'S^* = S'A^*$ we deduce the equality 

$$A'X - XX'A^* = S'(A' - A^*)S' = A'S^* - S'A^*.$$ 

The equation $A'X - XX'A^* = F^*$ has no more than one solution $X$ (see [8]). We can deduce that $A'X = XX'A^*$ and $F^* = 0$. Hence we deduce from (5.8) that $S^* = X$. The lemma is proved. 

We show the following result:

Theorem 4.4. Let the self-adjoint operator $S^*$ be positive, invertible and have the form (4.18). The operator $S^*$ admits the left triangular factorization if and only if the matrix $B(x)$ is absolutely continuous and $B(x) = \beta'(x)\beta(x)$, 

$$\beta(x) = [h_1(x), h_2(x), h_3(x), h_4(x)].$$

Example 4.5. Let us consider the equation

$$S^f = f(x) + \frac{i\mu}{\pi} V^*P_1 \int_0^x \frac{f(t)}{x - t} dt - \frac{\lambda}{\pi} \int_0^x \frac{f(t)}{x + t} dt = g(x).$$

where $f(x) \in L^2(0, 1)$, $\lambda = \bar{\lambda}$, $\mu = \bar{\mu}$, and $|\lambda| + |\mu| < 1$. It is well known ([4]) that the self-adjoint operator $S^*$ is bounded, positive and invertible, i.e., the operator $S$ belongs to the $(+, -)$ class. We introduce the functions

$$v(x, \lambda, \mu) = [S^*]^{-1} f(x, \lambda, \mu) = \int_0^x v(x, \lambda, \mu) dx = ([S^*]^{-1})_{1}$$

$$= 0.$$ 

In view of (4.26) and (4.27) the relations 

$$[S^*]_{(x - \epsilon_1)}^{-1} P_{(x - \epsilon_1)} \int_0^x \frac{f(t)}{x - t} dt = \frac{\lambda}{\pi} \int_0^x \frac{f(t)}{x + t} dt = g(x).$$

are true. We introduce the self-adjoint operator

$$V^f = \frac{1}{\sqrt{a(\lambda, \mu)}} \int_0^x f(t) v(t, x, \lambda, \mu) dt.$$ 

Using Proposition 3.6, we deduce that the operator $V^*$ is bounded and $[S^*]^{-1} \geq [V^*]^{-1}$.

5. TRIANGULAR FACTORIZATION, CLASS $R_1^*$

Let us consider the integral operators

$$A^f = i \int_0^x f(t) dt, \quad A^f = -i \int_0^x f(t) dt,$$

where $f(x) \in L^2(0, a + \epsilon_2)$. 

Definition 5.1. We say that the linear bounded operator $A^*$
$P_0S^*P_0$. Operator equation (5.15) has only the trivial solution 
$S_0^* = 0$ (see [8]). The last equality contradicts relation (5.2). It
means that equality (5.13) is impossible when $\|f_0\| \neq 0$. 
Hence in view of (5.6) the self-adjoint operator $S_0^*$ maps 
$L^2(0, b)$ one-to-one onto $L^2(0, a + \varepsilon_2)$. This fact according 
to the classical Banach theorem [1] implies that the self-adjoint 
operator $S_0^*$ is invertible. The self-adjoint operator $[S^*]^{-1}$ is 
defined by formula (see [8])
\[
[S^*]^{-1} f = \frac{d}{dx} \int_0^x f(t)N(x - t)dt, \tag{5.16}
\]
Where $N(x) = [S^*]^{-1} 1$. Thus the self-adjoint operators 
$S_0^*$ and $[S^*]^{-1}$ are bounded and lower triangular. The 
assertion of the theorem now follows directly from Definition 
14.2.

Example 5.5. We consider [11] the case when
\[
\phi(x) = \log(a + \varepsilon_2 - x). \tag{5.17}
\]
In this case we have
\[
S_0^* f = \frac{d}{dx} \int_0^x f(t) \log(a + \varepsilon_2 - x + t) dt
= f(x) \log(a + \varepsilon_2) - \int_0^x \frac{f(t)}{a + \varepsilon_2 - x + t} dt. \tag{5.18}
\]
Let us introduce the operator
\[
Kf = \int_0^x \frac{f(t)}{a + \varepsilon_2 - x + t} dt. \tag{5.19}
\]
It is well known (see [10]) that $\|K\| \leq \pi$. Hence the 
self-adjoint operator $S_0^*$ defined by (5.18) and the operator 
$[S^*]^{-1}$ are bounded, when $\log(a + \varepsilon_2) > \pi$. From Lemma 
5.2 we obtain the assertion.

Proposition 5.6. If $\log(a + \varepsilon_2) > \pi$, then the self-adjoint 
operator $S_0^*$ defined by relations (5.3) and (5.17) admits the 
left triangular factorization (5.6) where the operator $S_0^*$ has 
the form (5.18).

Now we show the following Example 5.7.

Deduce that $\phi(x) = E - Dx$, where $E$ and $D$ are particular 
constants. Since $\log(a + \varepsilon_2) = \pi + \varepsilon_3$ then $a + \varepsilon_2 = Ae^\pi$
where $A = e^\varepsilon_3$, $e^{\phi(x)} = b - x$, where $b = a + \varepsilon_2$, implies 
that $e^{\phi(x)} = a + \varepsilon_2 - x$. By division,
\[
\frac{a + \varepsilon_2 - x}{a + \varepsilon_2} = B e^{\phi(x)},
\]
wher $B = (Ae^\pi)^{-1}$. Hence $B e^{\phi(x)} = C - Dx$. Hence $e^{\phi(x)} = 
C - Dx$. Therefore $\phi(x) = \log(C - Dx)$.

6. HOMOGENEOUS KERNELS OF DEGREE (1)

In this section (See [11]) we consider self-adjoint operators of the form
\[
S^*F = F(x) - \int_0^1 F(y)k(y)dy = G(x), \tag{6.1}
\]

where $F(x) \in L^2(0, 1)$ and
\[
k(y) = \frac{1}{x} \frac{1}{y} \frac{1}{x} dy = G(x), \tag{6.2}
\]

We assume that
\[
A' = 2 \int_0^1 \frac{1}{x} \frac{1}{y} \frac{1}{x} x^2 dx < \infty. \tag{6.3}
\]

From condition (6.3) we deduce that the operator
\[
KF = \int_0^1 F(y) k(y) dy, \tag{6.4}
\]
is bounded and (see [4])
\[
\|k\| \leq A'. \tag{6.5}
\]
We have the following (See [11])

Theorem 6.1. Let conditions (6.2) and (6.3) be fulfilled and let 
the corresponding self-adjoint operator $S^*$ be positive and invertible, then the operator $S^*$ admits the left triangular factorization.

Proof. We introduce the change of variables $x = e^{u}$ and $y = e^{-u}$. Hence equation (6.1) takes the form
\[
lf = f(u) - \int_0^u f(u - e)H(e) du - e = g(u). \tag{6.6}
\]

Where
\[
f(u) = F(e^{-u})e^{\frac{u}{2}}, \quad g(u) = G(e^{-u})e^{\frac{u}{2}} \quad u \geq 0. \tag{6.7}
\]

It follows from relation (6.3) that
\[
\int_0^\infty |H(u)| du = A'. \tag{6.9}
\]

We denote by $\gamma(u)$ the solution of Equation (6.6) when $g(u) = H(u)$. In the theory of equations (6.6) the following function plays an important role (see [11])
\[
\gamma(u) = 1 + \int_0^u \gamma(v)e^{tv} dv, \quad \gamma(0) = 0. \tag{6.10}
\]

Let us consider the solution $\gamma(x + e)(u)$ of equation (6.6) when $g(u) = e^{iu(x+e)}$ and $\gamma(x + e) = 0$. We use the formula (see [11])
\[
\gamma(x + e)(u) = \gamma^u(x + e) + \int_0^\infty \gamma(r)e^{-ir(x+e)} dr \gamma^{u} e^{iu}(x+e). \tag{6.11}
\]

Further we need the particular case of $\gamma(x + e)(u)$ when $x + e = i/2$. In this case we have
\[
\gamma(x + e)(u) = \beta [1 + \int_0^u \gamma(r)e^{iz} dr] e^{-\frac{u}{2}}, \tag{6.12}
\]

where
\[
\beta = \gamma^u (i/2). \tag{6.13}
\]

Let us introduce the function $v(x)$, which satisfies Equation (6.1) when $G(x) = 1$. It is easy to see that
\[
v(x) = \gamma(x)e^{\frac{x}{2}}, \tag{6.14}
\]

From (6.11) and (6.13) we deduce that
\[
v(x) = -\beta \gamma(t)e^{-\frac{t}{2}} \tag{6.15}
\]

Using relations (6.11) and (6.13) we can calculate the integral
\[
\alpha = \int_0^1 \gamma(x)dx = \beta \left[ 1 + \int_0^\infty \gamma(r)e^{iz} dr \right]. \tag{6.16}
\]

Hence the equalities
\[
\alpha = \beta [1 + \int_0^\infty \gamma(r)e^{-iz} dr] = \beta \beta \tag{6.17}
\]

are true. The self-adjoint operator $V^*$ in (6.1) has the form
\[
V^*f = \frac{1}{\beta} \int_0^x f(t) \gamma^u (\frac{t}{x}) dt. \tag{6.18}
\]
In view of (6.14) and (6.15) we can represent the self-adjoint operator $V^*$ in the form

$$V^* f = f(x) + \int_0^x f(t)L\left(\frac{t}{x}\right)\frac{1}{t} dt,$$

(6.18)

where

$$L(x) = \gamma(t)e^{-t}.$$  

(6.19)

Now the assertion of the theorem follows from Proposition 2.2. \hfill \Box

Corollary 6.2. Let the conditions of Theorem 6.1 be fulfilled. Then we have the equality

$$[S^{-1}]^{-1} = [V^*]^2,$$

(6.20)

where the self-adjoint operator $V^*$ is defined by relations (6.18) and (6.19).

Example 6.3. We obtain an interesting example when

$$k(u) = \frac{\lambda}{1 - |u|^{(1 + u)^2}},$$

(6.21)

where $\lambda = \bar{\lambda}, \alpha \geq 0, \beta > 0,$ and $\alpha + \beta = 1.$ We note that $k(u)$ satisfies conditions (6.2) and (6.3). Equations (6.1) and (6.21) coincide with the Dixon equation when $\alpha = 0.$

Now we have the following example.

Example 6.4. If $-\frac{t}{2} = u$ in the proof of Theorem 6.1 the equation (6.19) becomes

$$L(x) = \gamma(2u)e^{-u}$$

then we can deduce that

$$k\left[\frac{1}{2} \gamma^{-1}[L(x)e^{x}]\right] = \frac{\lambda}{1 + \gamma^{-1}[L(x)e^{x}]},$$

we can show that is easily

$$\|V^*W\| \leq \|\gamma(t)\|\|e^x\|.$$  

7. CONCLUSION

We studied and applied The Matter of Triangular Factorization of Positive Self-Adjoint Operators with sum-difference kernels in class $R_1$ and Homogeneous Kernels of degree (-1) in Hilbert Space;

REFERENCES