Properties And Experimental Of Gaussian And Non Gaussian Time Series Model

A. M. Monem

Abstract: Most of time series that appear in many economical geophysical and other phenomena are driven by non- Gaussian white noise (a), in this paper investigate some probabilistic properties of Gaussian and non- Gaussian mixed with identification methods of ARMA model. We have theoretically derived the characteristic function the first of (four moments) of skeweness and kurtosis coefficients for white noise (a) with Gaussian and non- Gaussian (Poisson) distribution, simulation experiments were used to confirm the accuracy of the theoretical results. Declared the identification sample Autocorrelation function (ESACF) and (Kumar) method (C- table) which depending upon the pad approximation and suggested new method depending upon the extended sample partial Autocorrelation function (ESPACF) and find ascertain efficiency of suggested method.

Index Terms: Gaussian white noise distribution, non- Gaussian white noise distribution, ARMA process, characteristic function of moments for skeweness, characteristic function of moments for kurtosis extended sample autocorrelation, extended sample partial autocorrelation

1 INTRODUCTION

Time series analysis and forecasting methods play an important role in all researchers especially with Gaussian and non Gaussian mixed models. Nelson and Granger (1979), Obeysekera and Yevjevich (1985) reported a procedure for generation of samples of an autoregressive scheme that had an Poisson distribution with given mean and skewness. Fernandez and Salas (1986) developed and tested a new class of time series models capable of reproducing the covariance structure normally found in periodic stream flow time series under non-Gaussian marginal distribution. The general class of forecasting methods involves two basic tasks, analysis of time series data and selection of the forecasting model that best fits the data series. Today, after acrimonious arguments and considerable debates, it is accepted by a large number of researchers that in empirical tests Box-Jenkins is not an accurate method for post-sample time series forecasting, at least in the domains of business and economic applications where the level of randomness is high and where constancy of pattern, or relationships. Sim (1987) considered a time series model which can be used for simulating stationary river flow sequences with high skewness and the long-term correlation structure of an ARMA(1,1) models valid for non-normal distribution have also been developed series of weekly stream flow were used for application and comparison of the proposed method. In this research investigate some probabilistic properties of Gaussian and non-Gaussian mixed with identification methods of ARMA (1, 1) model with derived the characteristic function of moments for Skeweness and Kurtosis coefficients for (a) with Gaussian and non- Gaussian (Poisson) distribution, a Simulation experiments were used to confirm the accuracy of the theoretical results.

2 General theoretical

Let, we have stationary time series (Zt = 0, ±1, ±2, ...), then, we have the following ARMA (P, q) process:

\[ Z_t = \varnothing_0 Z_{t-1} + \varnothing_2 Z_{t-2} + \ldots + \varnothing_p Z_{t-p} + \alpha_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q} \]  

(1)

Where Zt = Z̄t - μ

For each (t), (μ) is the mean of time series and (αi)is a purely random error (white noise) distributed Gaussian with E (αi) = 0, Variance of (αi) = E (αi^2) = σ_a^2 and Covariance (αi,αi+k) = 0, for all k≠0.

Then, ARMA (1, 1) distributed Gaussian or normal distribution. By (B) operator we get:

\[ Z_t[1- \varnothing_1 B - \varnothing_2 B^2 - \ldots - \varnothing_p B^p] = (1- \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q) a_t \]

\[ \varnothing_t = \varnothing_t (B) a_t \]  

(2)

\[ \{ Z_t = \varnothing_t (B) a_t \} \] Where \( \varnothing_t (B) \) and \( \Phi_p (B) \) are the polynomial functions of order (q) and (p) in (B) respectively.

Or \[ \varnothing_t = \psi (B) a_t \]  

(3)

Where; \( \psi (B) = \frac{\varnothing_t (B)}{\varnothing_t (B)^{'}(0)} = \psi_0 B^0 + \psi_1 B^1 + \psi_2 B^2 + \psi_3 B^3 + \ldots \) \( \psi_0 = 1 \) or \( \psi (B) = \frac{\varnothing_t (B)}{\varnothing_t (B)^{'}(0)} = \sum_{j=0}^{\infty} \psi_j B^j \), \( \psi_0 = 1 \), \( \psi_1, \psi_2, \psi_3, \ldots \) the weights

3 Mixed model with normal distribution:

Let we have a mixed model ARMA (1, 1) with normal distribution as follows:

\[ (1- \varnothing_1 B) (Z_t - \mu) = (1- \theta_1 B) a_t, a_t \sim N(0, \sigma_a^2) \]

\[ Z_t - \mu = \frac{1-\theta_1 B}{1-\varnothing_1 B} a_t = \psi (B) a_t \]

Where; \( \psi (B) \) is the polynomial function of order (1) in (B).

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We can find the relationship between the parameters (θ) and (φ) as follows:

\[ \psi_i = \phi_1 \psi_{i-1} = \phi_1^{i-1} (\phi_1 - \theta_1), i \geq 1, |\phi_1| < 1 \text{ and } \phi_1 \neq \theta_1 \]

\[ \therefore Z_t = a_t + (\phi_1 - \theta_1) \sum_{i=1}^{\infty} \phi_1^{i-1} a_{t-i} \quad \ldots \; (4) \]

4 New Identification method of ARMA (1, 1) process:

We know there are many methods using for identification the rank of mixed ARMA (1, 1) process, the best of these methods depend on extended sample autocorrelation (ESACF) which suggested by (Tsay and Tiao – 1984), this method depend on the in dependended estimation (OLS) method of autoregressive with order (p) model as follows

\[ Z_t = \sum_{i=1}^{\infty} \phi_i^{(0)} Z_{t-i} + \varepsilon_{p,t}^{(0)} \quad \ldots \; (5) \]

Where \( (Z) \) is the time series of \( n \) observations has ARMA (p, q), \( (0) \) is the ordinary regression; \( (p) \) is the rank of model and \( \varepsilon_{p,t}^{(0)} \) is the purely random error. But the second method of identification ARMA (1, 1) model is suggested by (Kuldeep Kumar – 1987), this method depend on Smoothing approach of (pade) of estimation method of ARMA (1, 1) by using autoregressive or moving average model. The new method of identification ARMA (1, 1) model is suggested to extended sample partial autocorrelation (ESPACF) by using the following formula:

\[ \varphi_{kk} = \begin{cases} \hat{\beta}_k & \text{if } k = 1 \\ \hat{\beta}_k - \sum_{i=1}^{k-1} \hat{\beta}_{k-i} \hat{\beta}_i & \text{if } k = 2, 3, \ldots \end{cases} \quad \ldots \; (6) \]

And \( \varphi_{kj} = \varphi_{k-j} - \varphi_{k-j} \times \varphi_{k-1,j} \) for \( j = 1, 2, \ldots, k-1 \)

We can definite the extended sample partial Autocorrelation function (ESPACF) of time series \( (Z) \) in general and for any positive number \( (m) \) as follows:

\[ \tilde{\varphi}_j(m) = \tilde{\varphi}_j(w_{m,j}(j)) \quad \ldots \; (7) \]

Where \( w_{m,j}(j) = Z_t - \sum_{j=1}^{\infty} \tilde{\varphi}_j(m) Z_{t-j} \)

So if the original model is ARMA (p, q), therefore \( \{w_{m,j}(j)\} \) close to AR (P), because

\[ \tilde{\varphi}_j(m) = \begin{cases} 0 & \text{for } m \geq p, j \geq q \\ 0 & \text{other wise} \end{cases} \quad \ldots \; (8) \]

5 Distribution of time series \( (Z_t = \text{ARMA}\,(1,1))\):

Using the characteristic function to find the distribution of \( (Z) \) depend on The relationship between the characteristic function of \( (Z_t) \) and characteristic function of white noise \( (a_t) \), where the relationship is:

\[ \psi_z(s) = \psi_a(s) \prod_{j=1}^{\infty} \psi_a(\delta^{j-1}(\emptyset - \theta)s) \quad \ldots \; (9) \]

Where \( \psi_z(s) \) is the characteristic function of \( (Z) \) and \( \psi_a(s) \) is the characteristic function of white noise \( (a_t) \).

First: when the white noise \( a_t \sim N(0, \sigma^2) \):

In this case we have:

\[ \psi_a(s) = \exp \left\{ -\frac{1}{2} s^2 \sigma^2 \right\} \quad \ldots \; (10) \]

And by substitute in (10) we gets

\[ \psi_z(s) = \exp \left\{ -\frac{1}{2} s^2 \sigma^2 \sum_{j=1}^{\infty} (\emptyset - \theta)^{2j-1} \right\} \]

\[ = \exp \left\{ -\frac{1}{2} s^2 \sigma^2 \exp \left\{ -\frac{1}{2} s^2 \sigma^2 (\emptyset - \theta)^2 \right\} \right\} \]

\[ = \exp \left\{ -\frac{1}{2} s^2 \sigma^2 (1 + (\emptyset - \theta)^2) \right\} \]

\[ = \exp \left\{ -\frac{1}{2} s^2 \sigma^2 \left( 1 + (\emptyset - \theta)^2 \right) \right\} \]

\[ \therefore \psi_z(s) = \exp \left\{ -\frac{1}{2} s^2 \sigma^2 \right\} \quad \ldots \; (11) \]

Therefore (11) represent the characteristic function of \( (Z) \) normal Distribution with mean (zero),

\[ \text{Variance} = \left\{ \sigma^2 \left( 1 - 2 \cdot \emptyset + \emptyset^2 \right) \right\} \quad \text{and (pdf) as follows:} \]

\[ f(Z) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{2\pi(1-2\emptyset+\emptyset^2)}} \exp \left\{ -\frac{z^2(1-\emptyset)^2}{2(1-2\emptyset+\emptyset^2)} \right\} & -\infty < z < \infty \\ 0 & \text{o.w.} \end{array} \right. \quad \ldots \; (12) \]

Second: when the white noise \( a_t \sim \text{poisson with } (\lambda) \):

In this case we have the (pmf) as follows:

\[ f(a_t) = \left\{ \begin{array}{ll} \lambda^a \exp(-\lambda) / a! & a_t = 0, 1, 2, \ldots \text{ and } \lambda > 0 \\ 0 & \text{o.w.} \end{array} \right. \]

So that the characteristic function of white noise \( (a_t) \) is:

\[ \psi_a(s) = \exp \left\{ \lambda \exp(i s) - 1 \right\} \]

Or \[ \psi_z(s) = \psi_a(s) \prod_{j=0}^{\infty} \psi_a(s) \]

Therefore; the characteristic function of \( (Z_t = \text{ARMA}(1,1)) \) with white noise \( (a_t \sim \text{poi } (\lambda)) \) is:

\[ \psi_z(s) = \exp \left\{ \exp(i s) - 1 \right\} \sum_{j=1}^{\infty} \exp \left\{ i s \delta^{j-1}(\emptyset - \theta) - 1 \right\} \quad \ldots \; (13) \]
6. The moments of time series \( \{Z_t\} \) when \( a_t \sim \text{pois}(\lambda) \):

According to above formula (13) and by using the characteristic function of \( \{Z_t\} \) we have:

\[
\psi_{Z_t}(s) = \exp \left( \frac{1}{2} \sigma^2 \right) = \exp \left( \frac{1}{2} \sigma^2 \right) \quad \text{for } s = 0
\]

By the same way we derive the characteristic function of \( \{\text{second}, \text{third}, \text{and forth.} \) to find another moment of time series \( \{\} \) when white noise \( a_t \sim \text{pois}(\lambda) \), so, the mean and variance of \( \{\) is:

\[
\mu = \text{E}(\{Z_t\}) = \lambda \quad \text{and } \quad \text{Var}(\{Z_t\}) = \lambda \sum_{j=0}^{\infty} \psi_j
\]

7-Skewness and Kurtosis coefficients of \( \{Z_t\} \) when \( \{a_t \sim N(0, \sigma^2_a)\} \):

The skewness (Sk) is a measure of symmetry and all symmetric distributions have zero skewness, so

\[
\text{Sk} = \frac{\mu_3}{\sigma^3} = \frac{\text{E}(Z_t-\mu)^3}{(\text{E}(Z_t-\mu)^2)^{3/2}} = \frac{\text{E}(Z_t^3) - 3\text{E}(Z_t^2)\text{E}(Z_t) + 2(\text{E}(Z_t))^3}{(\text{E}(Z_t^2) - (\text{E}(Z_t))^2)^{3/2}} \quad \text{(18)}
\]

But the kurtosis (Ku) is a measure peaknessed of the (pdf) or the pf (pmf) (Usually, compared to the normal distribution), for the normal distribution, this value equal to zero; so

\[
\text{Ku} = \frac{\mu_4}{\sigma^4} = \frac{\text{E}(Z_t-\mu)^4}{(\text{E}(Z_t-\mu)^2)^{2}} = \frac{\text{E}(Z_t^4) - 4\text{E}(Z_t^3)\text{E}(Z_t) + 6\text{E}(Z_t^2)(\text{E}(Z_t))^2 - 3(\text{E}(Z_t))^4}{(\text{E}(Z_t^2) - (\text{E}(Z_t))^2)^2} \quad \text{(19)}
\]

By using the previous four moments of time series \( \{Z_t\} \) with formula (18) and (19) we find the skewness and kurtosis of \( \{Z_t\} \) as follows:

\[
\text{Sk}(Z_t) = 0 \quad \text{and } \quad \text{Ku}(Z_t) = 3
\]

8- Skewness and Kurtosis coefficients of \( \{Z_t\} \) with \( \{a_t \sim \text{pois}(\lambda)\} \)

By the same way we can find the skewness and kurtosis of \( \{Z_t\} \) as follows:

\[
\text{Sk}(Z_t) = \frac{\text{E}(Z_t^3) - 3\text{E}(Z_t^2)\text{E}(Z_t) + 2(\text{E}(Z_t))^3}{(\text{E}(Z_t^2) - (\text{E}(Z_t))^2)^{3/2}} \quad \text{(22)}
\]

And

\[
\text{Ku}(Z_t) = 3 + \frac{1}{\lambda (1 + (\sigma/\lambda) \cdot (\text{E}(Z_t^4) - 4\text{E}(Z_t^3)\text{E}(Z_t) + 6\text{E}(Z_t^2)(\text{E}(Z_t))^2 - 3(\text{E}(Z_t))^4))} \quad \text{(23)}
\]
Table (1-B): The kurtosis coefficients of ARMA (1, 1) when \( a_i \sim N (0, 1) \) and \( \text{poi} (\lambda) \)

<table>
<thead>
<tr>
<th>n</th>
<th>(( \theta ) )</th>
<th>( K.U. (Z_i) )</th>
<th>T.N</th>
<th>T.P.</th>
<th>E.N .</th>
<th>E.P.</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>(-0.8, 0.9)</td>
<td>3.95</td>
<td>3.60</td>
<td>3.0</td>
<td>3.98</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>(-0.8, 0.9)</td>
<td>3.95</td>
<td>3.60</td>
<td>3.0</td>
<td>3.94</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>(-0.8, 0.9)</td>
<td>3.95</td>
<td>3.60</td>
<td>3.0</td>
<td>3.95</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>(0.3, 0.1)</td>
<td>3.74</td>
<td>3.64</td>
<td>3.0</td>
<td>3.75</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>(0.3, 0.1)</td>
<td>3.74</td>
<td>3.64</td>
<td>3.0</td>
<td>3.75</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>(0.3, 0.1)</td>
<td>3.74</td>
<td>3.64</td>
<td>3.0</td>
<td>3.77</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>(0.7, 0.5)</td>
<td>3.86</td>
<td>3.90</td>
<td>3.0</td>
<td>3.81</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>(0.7, 0.5)</td>
<td>3.86</td>
<td>3.90</td>
<td>3.0</td>
<td>3.80</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>(0.7, 0.5)</td>
<td>3.86</td>
<td>3.90</td>
<td>3.0</td>
<td>3.86</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>(0.9, 0.9)</td>
<td>3.99</td>
<td>4.00</td>
<td>3.0</td>
<td>3.99</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>(0.9, 0.9)</td>
<td>3.99</td>
<td>4.00</td>
<td>3.0</td>
<td>3.99</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>(0.9, 0.9)</td>
<td>3.99</td>
<td>4.00</td>
<td>3.0</td>
<td>4.55</td>
<td></td>
</tr>
</tbody>
</table>

But when \( (\theta = 20) \) and \( (n = 200) \) we get the following tables:

Table (2-A): The skewness coefficients of ARMA (1, 1) when\( a_i \sim \text{poi} (\lambda = 20) \)

<table>
<thead>
<tr>
<th>(( \theta ) )</th>
<th>Sk. (( Z_i ))</th>
<th>Sk.(( a_i ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.9, 0.9)</td>
<td>0.22361</td>
<td>0.22361</td>
</tr>
<tr>
<td>(-0.3, 0.1)</td>
<td>0.15986</td>
<td>0.15829</td>
</tr>
</tbody>
</table>

Table (2-B): The kurtosis coefficients of ARMA (1, 1) when\( a_i \sim \text{poi} (\lambda = 20) \)

<table>
<thead>
<tr>
<th>(( \theta ) )</th>
<th>Ku. (( Z_i ))</th>
<th>Ku.(( a_i ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.9, 0.9)</td>
<td>3.05</td>
<td>3.05</td>
</tr>
<tr>
<td>(-0.3, 0.1)</td>
<td>3.03649</td>
<td>3.03756</td>
</tr>
</tbody>
</table>

Table (3): Frequency distribution for ARMA (1, 1) with different methods

<table>
<thead>
<tr>
<th>n</th>
<th>(( \theta ) )</th>
<th>ESACF</th>
<th>C- table</th>
<th>ESPACF</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>(-0.8, 0.8)</td>
<td>481</td>
<td>478</td>
<td>479</td>
</tr>
<tr>
<td>100</td>
<td>(-0.8, 0.8)</td>
<td>492</td>
<td>481</td>
<td>488</td>
</tr>
<tr>
<td>200</td>
<td>(-0.8, 0.8)</td>
<td>497</td>
<td>486</td>
<td>495</td>
</tr>
</tbody>
</table>

So from above table, the results of method (ESPACF) and method (ESACF) gives more iterations and much better than (C-table) method.

C: Comparing between the three methods for efficiency ARMA (1, 1) model by using (percentage Error) measurement, we get the following table:

Table (4): The percentage Error ratio for different methods

<table>
<thead>
<tr>
<th>N</th>
<th>(( \theta ) )</th>
<th>ESACF</th>
<th>C- table</th>
<th>ESPACF</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>(-0.8, -0.8)</td>
<td>0.038</td>
<td>0.044</td>
<td>0.042</td>
</tr>
<tr>
<td>100</td>
<td>(-0.8, -0.8)</td>
<td>0.016</td>
<td>0.018</td>
<td>0.024</td>
</tr>
<tr>
<td>200</td>
<td>(-0.8, -0.8)</td>
<td>0.006</td>
<td>0.008</td>
<td>0.01</td>
</tr>
</tbody>
</table>

10 Conclusion

1- We find the results of method (ESPACF) and method (ESACF) gives more iterations and much better than (C-table) method

2- Find that Error ratio decreasing with increasing sample size and method (ESACF) is better than the methods (ESPACF) and (C-table) respectively.
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