Accelerated Genetic Algorithm Solutions Of Some Parametric Families Of Stochastic Differential Equations

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Absract: In this project, A new method for solving Stochastic Differential Equations (SDEs) deriving by Wiener process numerically will be construct and implement using Accelerated Genetic Algorithm (AGA). An SDE is a differential equation in which one or more of the terms and hence the solutions itself is a stochastic process. Solving stochastic differential equations requires going away from the recognizable deterministic setting of ordinary and partial differential equations into a world where the evolution of a quantity has an inherent random component and where the expected behavior of this quantity can be described in terms of probability distributions. We applied our method on the Ito formula which is equivalent to the SDE, to find approximation solution of the SDEs. Numerical experiments illustrate the behavior of the proposed method.

Index Terms: accelerated genetic algorithm, stochastic differential equations, Ito formula

1 INTRODUCTION

Stochastic differential equations (SDEs) arise when a random noise is introduced into ordinary differential equations (ODEs). Let us consider first an example to illustrate the need for simulated and to analyze the properties of solution of SDEs. Many processes in nature and technology are driven by (temperature, energy, velocity, concentration,…) changes. Such processes are called diffusion processes because the quantity considered (i.e. temperature) is distributed to an equilibrium state is established (i.e. until the differences that drive the process are minimized). There are many examples of diffusion processes with slight notational variations, are standard in many books with applications in different fields, [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12]. If X_t is a differentiable function defined for $t \geq 0$, $f(x, t)$ is a function of x and t, and the following relation is satisfied for all t , $0 \leq t \leq T$,

$$\frac{dX_t}{dt} = X' = f(X_t, t) \text{ and } X_0 = x_0 \; (1)$$

then X_t is a solution of the ODE with the initial condition x_0. The above equation can be written in other forms (by continuity of X'_t):

$$X_t = X_0 + \int_0^t f(X_s, s)ds$$

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Before we give a rigorous definition of SDEs, we show how they arise as a randomly perturbed ODEs and give a physical interpretation. The White noise process ξ_t is formally defined as the derivative of the Wiener process,

$$\xi_t \equiv \frac{dW_t}{dt} = W'(t) \; (2)$$

It does not exist as a function of t in the usual sense, since a Wiener process is nowhere differentiable. If g(x, t) is the intensity of the noise at point x at time t, then it is agreed that

$$\int_0^T g(X_t, t)\xi dt = \int_0^T g(X_t, t)W'(t)dt = \int_0^T g(X_t, t)dW_t$$

is Ito integral [4]. SDEs arise when the coefficients of ordinary equation "(1)" are perturbed by White noise. If X_t denotes the population density, then the population growth can be described by the ODE: $dX_t/dt = aX_t(1 - X_t)$. The growth is exponential with birth rate a, when this density is small, and slows down when the density increases. Random perturbation of the birth rate results in the equation:

$\frac{dX_t}{dt} = (a + \sigma \xi_t)X_t(1 - X_t)$, or the SDE:

$$dX_t = aX_t(1 - X_t)dt + \sigma X_t(1 - X_t)dW_t , X_0 = x_0 .$$

There are two widely used types of stochastic calculus, Stratonovich and Ito [9],[10], differing in respect of the stochastic integral used. Modeling issues typically dictate which version in appropriate, but once one has been chosen a corresponding equation of the other type with the same solutions can be determined. Thus it is possible to switch between the two stochastic calculus. Specifically, the processes $\{X_t , t \geq 0\}$ solution to the Ito SDE:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t \; (3)$$

where $\{W_t , t \geq 0\}$ is the standard Wiener process or standard Brownian motion, the drift $f(t,X_t)$ and diffusion $g(t,X_t)$ are known functions that are assumed to be sufficiently regular (Lipschitz, bounded growth), for existence and uniqueness of solution [13], has the same solutions as the Stratonovich SDE:

$$dX_t = \underline{f}(t, X_t)dt + g(t, X_t) \circ dW_t \quad (4)$$

with the modified drift coefficient which is defined by:

$$\underline{f}(t, X_t) = f(t, X_t) - \frac{1}{2} g(t, X_t) \frac{\partial g}{\partial x}(t, X_t)$$

2 ITO'S LEMMA

Let us now consider an arbitrary function of Brownian motion .[11],[13]

f(Wt) (5)

f = f(w) being a sufficiently many times continuously differentiable function of w ϵ R. We would like to know by how much $f(W_t)$ will change along a Brownian motion path, when going one infinitesimal time step into the future, from t to t + dt . we would like to compute

df (W_t) = f(W_t+dt) − f(Wt).

Note that this will in general be a stochastic quantity. The idea for analyzing "(5)" is to simply Taylor expand f around W_t : since W_t+dt = W_t + dW_t , we have that , by Taylor expand with x = W_t and h = dW_t ,

$$f(W_{t+dt}) = f(W_t + dW_t)$$

$$= f(W_t) + f'(W_t)dW_t + \frac{1}{2} f''(W_t)(dW_t)^2 + O(dW_t)^3$$

$$= f(W_t) + f'(W_t)dW_t + \tfrac{1}{2} f''(W_t)dt \quad (6)$$

where we used Ito's rules to replace $(dW_t)^2$ by dt and $(dW_t)^3$ by 0. Subtracting $f(W_t)$, we find the following :

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2} f''(W_t)dt \quad (7)$$

We now observe that "(7)" contains not only a dW_t but also a dt. This suggests that to have a sufficiently general and smoothly applicable formalism we go a bit beyond functions of Brownian motion only, and consider functions of both Brownian motion and time: f(W_t, t),with f = f(w, t) a sufficiently differentiable function of two variables, w and t. We follow the same strategy as before, but we now use the two-variable Taylor expansion, with n = 2 and (x_1, x_2) = (w, t). We easily find that

$$df(W_t, t) = \left(\tfrac{\partial f}{\partial t}(W_t, t) + \tfrac{1}{2} \tfrac{\partial^2 f}{\partial w^2}(W_t, t) \right) dt + \tfrac{\partial f}{\partial w}(W_t, t)dW_t \quad (8)$$

Theorem 1. *(Ito's for functions of Brownian motion and time). Let f = f(w, t) : R → R be continuously differentiable function. Then the infinitesimal change of f(W_t, t) along a Brownian motion path is given by. [1],[13]*

$$df(w,t) = \left\{ \tfrac{1}{2} \tfrac{\partial^2 f}{\partial w^2}(w,t) + \tfrac{\partial f}{\partial t}(w,t) \right\} dt + \tfrac{\partial f}{\partial w}(w,t) dW_t \quad (9)$$

where all derivatives of f in the right hand side are to be evaluated at the point (W_t, t).

3 ITO PROCESSES

For any stochastic process $(X_t)_{t\geq 0}$ we can consider its change over an infinitesimal time step into the future:

dX_t := X_t+dt − X_t, dt>0. (10)

An important special case are processes for which dX_t is an infinitesimal normal random variable conditionally upon the process up till t:

Definition

$(X_t)_{t\geq 0}$ is called an Itˆo process if dX_t is related to an underlying Brownian motion $(W_t)_{t\geq 0}$ by

$dX_t = a_t\, dt + b_t\, dW_t$, (11)

where $(a_t)_{t\geq 0}$ and $(b_t)_{t\geq 0}$ are two auxiliary stochastic processes having the important property that a_t and b_t only depend on the Brownian motion through its past values W_s, s ≤ t :

a_t , b_t = functions of $((W_s)_{s\leq t}$,t) (12)

Theorem 2. (Ito's for functions of Ito processes).Let f = f(x, t) be a sufficiently many times continuously differentiable function .Then

$$df(X_t, t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2$$

$$= \left(\tfrac{\partial f}{\partial t} + a_t \tfrac{\partial f}{\partial x} + b_t{}^2 \tfrac{1}{2} \tfrac{\partial^2 f}{\partial x^2} \right) dt + b_t \tfrac{\partial f}{\partial x} dW_t \quad (13)$$

where all derivatives on the right are to be evaluated at (X_t, t). This is the form of Ito's lemma which is the most useful for applications. [1],[11],[13]

3.1 Discretized Brownian Motion.

A scalar standard Brownian motion, or standard Wiener process, over [t_0, T] is a random variable W(t) that depends continuously on t ϵ [t_0, T]. For t_0 ≤ s <t ≤ T . Let's take t_0 = 0 and divide the interval [0, T] into N steps such as: h = T/N. Lets also denote W_j = W(t_j) with t_j = jh [4].

$W_j = W_{j+1} + dW_j$ $W_0 = 0$ j = 1,2, … ,N

where each dW_j is an independent random variable of the form $\sqrt{h}N(0,1)$ The Fig.1 below displays the realizations of Wiener processes. [1],[11]

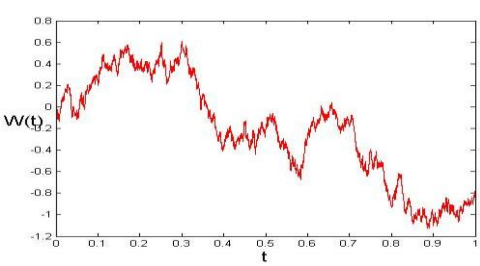

Fig 1: *1-dimensional Brownian path with* T = 1 and N = 5000.

4 GENETIC ALGORITHM AND GRAMMATICAL EVALUATION

Genetic algorithms are simulations of evolution process based on sexual and asexual reproduction, natural selection, mutation, and so on. However ,genetic algorithms are probabilistic optimization methods which are based on the principles of evolution. Grammatical evolution is an evolutionary algorithm that can produce code in any programming language[14] ,[15] , [16]. The algorithm requires as inputs the Backus–Naur Form (BNF) grammar definition of the target language and evaluate fitness function. We applied the following grammar to find the numerical solutions of stochastic differential equations with different models. Further details about grammatical evolution can be found in [17], [18], [19], [20],[21]. Chromosomes in grammatical evolution, are not expressed as parse trees, but as vectors of integers. Each integer denotes a production rule from the BNF grammar. The selection is performed in two steps:
1) We read an element from the chromosome (with value V).
2) We select the rule according to the scheme

Rule = V mod NR

where NR is the number of rules for the specific non-terminal symbol. In our method we used grammar as we can see in Fig 2. The numbers in parentheses denote the sequence number of the corresponding production rule to be used in the selection procedure described above.

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Fig 2: *The grammar of the proposed method*

Further details about grammatical evolution can be found in [19], [20],[21].

4.1 Technique of the algorithm

The algorithm has the following steps :

4.1.1 Initialization

In the initialization phase the values for mutation rate and selection rate are set.

4.1.2 Fitness evaluation

We express the PDE's in the following form:

$$f\left(x, y, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y), \frac{\partial^2 u}{\partial x^2}(x,y), \frac{\partial^2 u}{\partial y^2}(x,y)\right) = 0 \,,$$

$$x \in [x_0, x_1] \,, y \in [y_0, y_1]$$

The associated Dirichlet boundary conditions are expressed as:

$$u(x_0\,, y) = f_0(y)\,, u(x_1\,, y) = f_1(y)\,,$$

$$u(x\,, y_0) = g_0(y)\,, u(x\,, y_1) = g_1(y)$$

The steps for the fitness evaluation of the population are the following:

1. Choose N^2 equidistant points in the box $[x_0, x_1] \times [y_0, y_1]$, Nx equidistant points on the boundary at $x = x_0$ and at $x = x_1$, Ny equidistant points on the boundary at $y = y_0$ and at $y = y_1$
2. For every chromosome i
   1) Construct the corresponding model $M_i(x \;\; ,y)$, expressed in the above grammar.
   2) Calculate the quantity
      $E(M_i) =$
      $\sum_{j=0}^{N^2}(f(x_j, y_j, \frac{\partial}{\partial x} M_i(x_j, y_j), \frac{\partial}{\partial y} M_i(x_j, y_j), \frac{\partial^2}{\partial x^2} M_i(x_j, y_j), \frac{\partial^2}{\partial y^2} M_i(x_j, y_j))$
   3) Calculate an associated penalty Pi(Mi) . The penalty function P depends on the boundary conditions and it has the form:
      $P_1(M_i) = \sum_{j=1}^{N_x}(M_i(x_0, y_j) - f_0(y_j))^2$
      $P_2(M_i) = \sum_{j=1}^{N_x}(M_i(x_1, y_j) - f_1(y_j))^2$
      $P_3(M_i) = \sum_{j=1}^{N_y}(M_i(x_j, y_0) - g_0(x_j))^2$
      $P_4(M_i) = \sum_{j=1}^{N_y}(M_i(x_j, y_1) - g_1(x_j))^2$
   4) Calculate the fitness value of the chromosome as:
      $v_i = E(M_i) + P_1(M_i) + P_2(M_i) + P_3(M_i) + P_4(M_i)$

4.1.3 Genetic operators

The genetic operators that are applied to the genetic population are the initialization, the crossover and the mutation as shown in, [19] , [20]. The parents are selected via tournament selection.

4.1.4 Termination control

The genetic operators are applied to the population creating new generations, until a maximum number of generations or the best chromosome in the population has fitness better than a preset threshold.

4.2.Technical of the accelerated method

To make the method is faster to arrived the approximation solution of the stochastic differential equations by the following :

1- Insert the boundary conditions of the problem as a part of chromosomes in the our population of the problem, the algorithm gives the best approximation solution in a few generations.
2- Insert a part of exact solution of the stochastic differential equation if exist as a part of a chromosome in the population.
3- Insert the vector of numerical solution of the stochastic differential equation where obtained by the Euler-Maruyama method as a chromosome in the our population of the problem.

5 APPLICATIONS OF THE ALGORITHM

To find numerical solutions of stochastic differential equations we applied the accelerated genetic algorithm method on the Ito formula which equivalent to the stochastic differential equation "(15)". The crossover rate was set to 75% (that is, replication rate equal to 25%) and mutation rate was set to 1%. Each experiment to find the numerical solution was performed 50 times. The population size was set to 200 and the length of each chromosome to 50. The size of the population is a critical parameter. The generations number set to 250 iterations for each experiment. The function (randi) in Matlab (R2010b) used to generate the initial population.

6 NUMERICAL SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS (SDEs)

In this section we are using the following technique to finding an approximation solutions for any stochastic differential equation driven by Wiener process by applying the Ito formula to corresponding dynamics as follows: Consider a stochastic differential equation (SDE)

$$dX_t = a(t,X_t)dt + b(t,X_t)dB_t, \quad (14)$$
$$X_0 = x_0$$

If we are interested in finding the strong solution to this equation then we are searching for a function f : [0,1] × R → R such that $X_t = f(t,B_t)$. Because the changes of X_t is governed by t and B_t . The coefficients $a(t,X_t)$ and $b(t,X_t)$ only describe the effects changes in t and Bt respectively have on changes in Xt .Because the process $\{X_t\}$ has the dynamics as described in "(14)", there will be a corresponding dynamics for the process $f(t,B_t)$. The dynamics is obtained by applying the Ito Formula to $f(t,B_t)$. This gives

$$f(t,B_t) = f(0,0) + \int_0^t \left\{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,B_s) + \frac{\partial f}{\partial s}(s,B_s)\right\} ds + \int_0^t \frac{\partial f}{\partial s}(s,B_s)dB_s$$

If we compare this with the dynamics for X_t

$$X_t = x_0 + \int_0^t a(s,B_s)ds + \int_0^t b(s,B_s)dB_s$$

If we choose the function f so that it satisfies the following system of partial differential equations then we will have a candidate solution for the SDE "(14)":

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = a\big(s,f(s,x)\big)$$
$$\frac{\partial f}{\partial x}(s,x) = b\big(s,f(s,x)\big) \quad (15)$$
$$f(0,0) = x_0$$

The above technique is useful for linear and nonlinear coefficient functions a(s, x) and b(s, x). Now, we are applying this technique for finding the numerical solution to the stochastic differential equations using an accelerated genetic algorithm, as follows:

1- Find the approximation solutions f(s,x) for the system of partial differential equations "(15)".
2- Replace the variables s with t and x with Bt to obtain the numerical solution $f(t,B_t)$ of the original stochastic differential equation "(14)".

6.1 The fundamental stock price model .

A simple model for stock prices St is obtained by assuming that the return $(S_t{+}dt - S_t)/S_t$ over an infinitesimal period [t, t+dt] is normally distributed, with mean and variance both proportional to the length dt of the time interval :

$$\frac{dS_t}{dt} \sim N(\mu dt, \sigma^2 dt), \mu, \sigma^2 \text{ constant}$$

Now since $dW_t \sim N(0, dt)$, it follows that $\mu dt + \sigma\, dW_t \sim N(\mu dt,\sigma^2 dt)^3$. We can therefore model our stock price (the changes in the stock price) by

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (16)$$

This is an SDE for the stock price St. Given an initial value S_0, its solution is given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \quad (17)$$

This can be checked by applying Ito's lemma" (9)". To find numerical solution of "(16)" by our method we applied Ito formula "(15)" for f(s,x) then :

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = \mu f(s,x)$$

$$\frac{\partial f}{\partial x}(s,x) = \sigma f(s,x)$$

$$f(0,0) = x_0$$

If μ = σ = 2 , we find the following as a candidate solution of this system

$Gp2 = f(s,x) = \exp(2x)$, then
$S_t = \exp(2W_t)$ at generation 2.
$Gp10 = f(s,x) = \exp(2s + x)$, then
$S_t = \exp(2t + W_t)$ at generation 10
$Gp12 = f(s,x) = \exp(s + x)$, then
$S_t = \exp(t + W_t)$ at generation 12

All the above solutions were compared with the exact solution as in Fig 3 S_0=1. The process S_t is called geometric Brownian motion with drift μ and volatility σ^2.

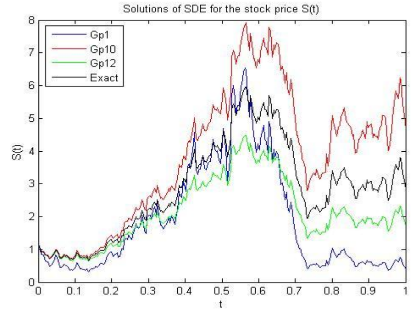


Fig 3: Comparison for Solutions of SDE for the stock price S_t

6.2 The Ornstein – Uehlenbeck process (OU)

The Ornstein – Uehlenbeck SDE is given by :

$$dX_t = \alpha(\theta - X_t)dt + \sigma\, dW_t \ (18)$$

To solve this SDE, set $Y_t = X_t - \theta$. Then $dY_t = dX_t$ and so

$$dY_t = -\alpha Y_t\, dt + \sigma\, dW_t.$$

To get rid of the $-\alpha Y_t dt$, multiply Y_t by $\exp(\alpha t)$; then

$$d(\exp(\alpha t)Y_t) = \exp(\alpha t)\, dY_t + \alpha \exp(\alpha t) Y_t\, dt$$

$$= \sigma \exp(\alpha t)\, dW_t$$

where we used the preceding equation for dY_t. Formally integrating from 0 to t, we would find that

$$\exp(\alpha t)Y_t = Y_t + \int_0^t \sigma \exp(\alpha t) dW_s$$

or, remembering what Y_t stands for,

$$X_t = \theta(1 - \exp(-\alpha t)) + X_0 \exp(-\alpha t) + \sigma \int_0^t \exp\big(\alpha(s - t)\big)\, dW_s \ (18a)$$

Note that here the solution up to time t depends on all Ws for times s ≤ t. To find numerical solution of "(18)" by our method we applied Ito formula "(15)" for f(s,x) then :

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = -5f(s,x)$$

$$\frac{\partial f}{\partial x}(s,x) = 3$$

$$f(0,0) = x_0$$

When $\alpha = 5, \theta = 0, \sigma = 3, x_0 = 10$,we find the approximation solutions of OU equation by using accelerated genetic algorithm at generations 50 , 18 , 32 respectively are:

$$Gp50 = f(s,x) = \exp(-5s)\,(10 + 3x)\,, \text{then}$$

$$X_t = \exp(-5t)(10 + 3W_t)$$

$$Gp18 = f(s,x) = \exp(-3s)(10 + x)\,, \text{then}$$

$$X_t = \exp(-3t)(10 + W_t)$$

$$Gp32 = f(s,x) = \exp(-4s)\left(10 - 2\sqrt{x^2}\right), \text{then}$$

$$X_t = \exp(-4t)(10 - 2\sqrt{W_t^2})$$

These solutions where shown in Fig 4 blow. When we are comparison the path of the above solutions with the Simulated path of the Ornstein-Uhlenbeck process as in [1], we found that these solution are a good approximations of path of the Ornstein-Uhlenbeck process.

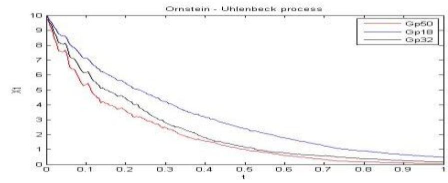


***Fig 4:** Numerical solutions of Ornstein –Uhlenbeck process*

6.3 The Cox-Ingersoll-Ross model (CIR)

Another interesting family of parametric models is that of the Cox-Ingersoll-Ross process. The CIR process is the solution to the stochastic differential Equation

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3\sqrt{X_t}dW_t\,, X_0 = x_0 > 0, (19)$$

sometimes parameterized as

$$dX_t = \theta(\beta - X_t)dt + \sigma\sqrt{X_t}dW_t\,, X_0 = x_0 > 0, (20)$$

where $\theta_1, \theta_2, \theta_3 \epsilon R_+$. if $2\theta_1 > \theta_3^2$, the process is strictly positive otherwise it is nonnegative, which means that it can reach the state 0. The stochastic differential equation "(19)" has the explicit solution :

$$X_t = \left(X_0 - \frac{\theta_1}{\theta_2}\right)\exp(-\theta_2 t) + \theta_3 \exp(-\theta_2 t)\int_0^t \exp(\theta_2 u)\sqrt{X_u}dW_u$$

The Stochastic Chain Rule

Let us suppose that we make some kind of change of variables, for example by letting $V(X) = \sqrt{X}$ We can use Ito's lemma to derive a corresponding SDE for V (X) and solve it. Then, if Ito's lemma is correct, squaring our solution for V (X) should give us the same result as as solving "(20)" directly. Now,

$$V(X) = \sqrt{X} \Rightarrow \frac{dV}{dX} = \frac{1}{2}\frac{1}{\sqrt{X}} \Rightarrow \frac{d^2V}{dX^2} = \frac{-1}{4}\frac{1}{X\sqrt{X}}$$

where $f(X) = \theta(\beta - X_t)$ and $g(X) = \sigma\sqrt{X_t}$ and using Ito's lemma "(13)" gives:

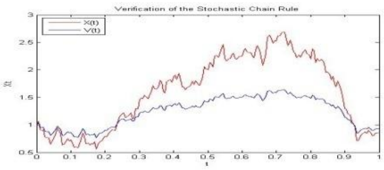
$$dV = \left(\frac{1}{2}\theta(\beta - X)\frac{1}{\sqrt{X}} - \frac{1}{8}\sigma^2 X\frac{1}{X\sqrt{X}}\right)dt + \frac{1}{2}\sigma\frac{\sqrt{X}}{\sqrt{X}}dW$$

$$dV = \left\{\left(\frac{1}{2}\theta\beta - \frac{1}{8}\sigma^2\right)\frac{1}{V} - \frac{1}{2}\theta V\right\}dt + \frac{1}{2}\sigma dW$$

$$dV = \left(\frac{4\theta\beta - \sigma^2}{8V} - \frac{\theta}{2}V\right)dt + \frac{\sigma}{2}dW \ (21)$$

$$V_0 = \sqrt{X_0}$$

The solutions of "(20)" & "(21)" by Euler-Maruyama method where $\theta = 1$, $\beta = 2$, $\sigma = 1$ as shown in Fig 5



(a)

(b)

Fig 5: *Verifying the Stochastic Chain Rule, X(t) vs V(t) and (b) X(t) vs V^2(t)*

By using the above values of parameters

$$dX_t = (2 - X_t)dt + \sqrt{X_t}dW_t \ (22)$$

$X_0 = x_0 > 0$, and

$$dV = \left(\frac{7}{8V} - \frac{V}{2}\right)dt + \frac{1}{2}dW_t \ (23)$$

$$V_0 = \sqrt{X_0}$$

To find numerical solution of "(22)" by our method we applied Ito formula "(15)" for f(s,x) to find :

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,x) + \frac{\partial f}{\partial s}(s,x) = 2 - f(s,x)$$

$$\frac{\partial f}{\partial x}(s,x) = \sqrt{f(s,x)} \ (24)$$

$$f(0,0) = x_0$$

Then we applied accelerated genetic algorithm on the system of PDEs "(24)" ,find the following candidate solutions at generations 5 , 12 , 30 , 26 respectively are :

$Gp5 = f(s,x) = x^2 + \sqrt{\frac{s}{2} + 1}$, then

$$X_t = W_t^2 + \sqrt{\frac{t}{2} + 1}$$

$Gp12 = f(s,x) = x^2 + 1$, then

$$X_t = W_t^2 + 1$$

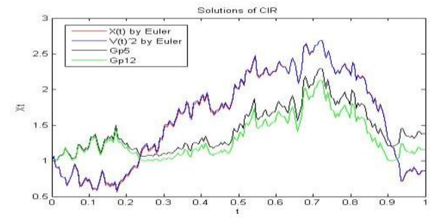
$Gp30 = f(s,x) = x + \frac{1}{2}s + 1$, then

$$X_t = W + \frac{1}{2}t + 1$$

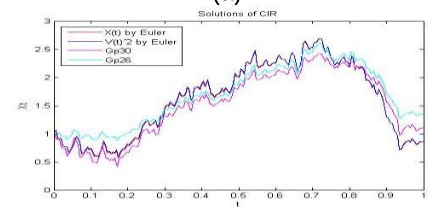
$Gp26 = f(s,x) = \exp(\frac{1}{2}(x + \sqrt{s}))$, then

$$X_t = \exp(\frac{1}{2}(W_t + \sqrt{t}))$$

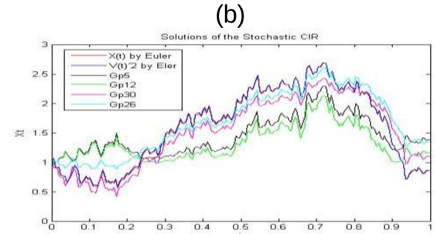
And compared of these solutions with the solution obtained by Euler - Maruyama method as shown in Fig 6 blow . From these comparison of paths ,we found that the above solutions are a good approximations for the CIR model .



(a)



(b)



(c)

Fig 6*: Comparison for solutions of CIR model with those obtained by Euler method, (a)X(t)and V(t)with Gp5 and Gp12, (b) X(t)and V(t)with Gp30 and Gp26, (c)all solutions.*

7 CONCLUSION

In this project , we have discussed a new technique for solving stochastic differential equations driven by Wiener process by using an accelerated genetic algorithm. Depending on Ito formula. By defining Brownian motion in a formal way. We also performed Euler – Maruyama method for compared these solutions obtained by our method. Finally, we established the Stochastic chain rule by experimental way to showing that the solution of one SDE and the solution of different SDE that was derived from the first using the Stochastic Chain rule, were in comparative agreement. There are many possible extensions to this project. The study of Stochastic differential equations is a moderately youthful field and consequently, there is much work still to be done. i.e. , one could using the method in an attempt to solving SDEs in more than one dimension, and stochastic partial differential equations.

REFERENCES

[1] S. M. Iacus , "Simulation and Inference for Stochastic Differential Equations , " Springer Science +Business Media, LLC , (2008).

[2] T.T. Soong, "Random differential equations in science and engineering," Academic Press, New York. L c Number: A274.23.S58., (1973).

[3] D. Henderson, and P. Plaschko , "Stochastic differential equations in science and engineering," World Scientific,(2006).

[4] K. Ito, "Stochastic integral," Tokyo, Proc. Jap. Acad , 20, pp. 519-529, (1944).

[5] E. Allen, "Modeling with Ito stochastic differential equations," Springer-Verlag.(2007).

[6] F.C. Klebaner , "Introduction to Stochastic Calculus with Applications," Imperial College Press, (2005).

[7] Y . Saito, T. Mitsui, "Simulation of Stochastic Differential Equations," The Annals of the Institute of Statistical Mathematics, 3, 419-432. (1993).

[8] P. E. Kloeden ,"A survey of numerical methods for stochastic differential Equations," Stochastic Hydrology and Hydraulics, 3, 155-178, (1989).

[9] P.E. Kloeden, E. Platen, "Relations between multiple Ito and stratonovich Integrals," Stochastic Analysis and Applications, 9(3), 311-321, (1991a).

[10] P.E. Kloeden, , E. Platen, "Stratonovich and Ito stochastic Taylor expansions," Mathematics Nachrichten , 151, 33-50, (1991b).

[11] P.E. Kloeden, E.Platen, "Numerical Solution of Stochastic Differential Equations," Springer- Verlag, New York, (1995).

[12] P.E. Kloeden, E. Platen, and H. Schurz, "Numerical solution of SDE through computer experiments," Springer-Verlag, New York, (1994).

[13] B. Oksendal, "Stochastic Differential Equations: An Introduction with Applications," 5th edition. Springer-Verlag, Berlin, (2000).

[14] G. Tsoulos .I.E, "Solving differential equations with genetic programming ," P.O. Box 1186 , Ioannina 45110, (2003).

[15] E.A. Hussain and Y.M. Alrajhi , "Solution of partial differential equations using accelerated genetic algorithm," Int. J. of Mathematics and Statistics Studies,Vol.2, No.1, pp.55-69, March (2014).

[16] P. Naur , "evised report on the algorithmic language ALGOL,"(1963).

[17] M. O'Neill and C. Ryan, "Under the hood of grammatical evolution,"(1999).

[18] M. O'Neill and C. Ryan, "Grammatical Evolution: Evolutionary Automatic Programming in a Arbitrary Language," Kluwer Academic Publishers, (2003).

[19] M. O'Neill and C. Ryan, "Grammatical Evolution, " IEEE Trans. Evolutionary Computation, Vol. 5, pp. 349-358, (2001).

[20] A. Sydow, "GMD-FIRST Genetic Algorithms and Genetic Programming Modern Concepts and Practical Applications , " Berlin, Germany, (2009)

[21] D.E. Goldberg, "Genetic algorithms in search, Optimization and Machine Learning, "Addison Wesley, (1989).

[22] E.A. Hussain and Y. M. Alrajhi , "Accelerated Genetic Algorithm Solution of Linear Black – Scholes Equation" , IJMCAR. (2014).