Extension Of Lagrange Interpolation

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Abstract: In this paper is to present generalization of Lagrange interpolation polynomials in higher dimensions by using Gramer's formula .The aim of this paper is to construct a polynomials in space with error tends to zero.

Keywords: Lagrange interpolation ;multivariable interpolation.

1. Introduction:

More than two hundred years ,in 1796 in his Lecous elementaries sure les mathematics that the French mathematician J.L.Lagrange formulated the interpolation polynomial called after him .He fitted on n points of the space R^2 a polynomial of (n-1) degree and constructed it a suitable linear combination at basic polynomial $l_i(x)$ giving in the i^{th} point one, in the more point is zero. The Lagrangian interpolation polynomial and the Newtonian one are equivalent, but Lagrange interpolation has advantage ,its wanted polynomial can be written immediately, without solving a system of (n+1) equations, [4],[5]. Polynomial interpolation is a classical topic of numerical analysis which is useful in various area of applied mathematics. Polynomial are among the mathematical objects which can be handled most easily in practice, they can be represented by finite in formation and can be easily integrated and differentiated symbolically. Therefore, there is a wide area of applications for polynomial interpolation in several variables which range from surface reconstruction to cubature, finite elements and even optimization,[1] . Interpolation, a fundamental topic in numerical analysis, is the problem of constructing function which goes through a given set of data points.these data points are obtained by sampling of a function, the values of the f^n.can be used to construct an interpolation ,which must be agree with the interpolated function at the data points. Multivariate interpolation has application in computer graphics , numerical quadrature ,and numerical solution of differential . [2][3]

2. polynomial interpolation

Lagrange gave the following interpolation polynomial p(x) of degree n given at (n+1) points (x_i, y_i) $, i = 0,1,.., n$. such that

$$y = p(x) = \sum_{i=0}^{n} y_i l_i(x)$$

Where $l_i(x)$ are Lagrange basic polynomials defined by

$$l_i(x) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_i - x_j)} \cdots\cdots\cdots\cdots(1)$$

$$l_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ldots\ldots\ldots\ldots(2)$$

The benefit of Lagrange interpolation is that we can find and write down the interpolation immediately ,without computing the coefficients in the function.

3. Multivariable Lagrange interpolation:

Let $f = f(X_1, X_2,.., X_n)$ be an m-variable multinomial function of degree n, since there are n points , $p = \begin{pmatrix} n+m \\ n \end{pmatrix}$, it is a necessary condition that we have p distinct points $(X_1,..., X_{m,i}, f_i) \in R^{m+1}$ $,1 \leq i \leq p$, $f_i = f(X_1,..., X_{m,i})$ for f be uniquely defined

$$f(X_1,..., X_m) = \sum_{e_i 1 \leq n} \alpha_{e_i} X^{e_i} \ldots\ldots\ldots\ldots(3)$$

where α_{e_i} are the coefficients in f

$X^{e_i} = \prod_{j=1}^{m} X_j^{e_{ij}}$ hence we write f in the form $\sum_{i=1}^{p} f_i l_i(X)$,where $l_i(X)$ multinomial function in independent variables $X_1,..., X_m$,consider the linear equations $f_i = \sum_{e_j 1 \leq n} \alpha_{e_j} X_j^{e_j}$,where $1 \leq i \leq p$ we can construct the matrix

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$$M = [X_i^{e_i}]$$

$$M = \begin{pmatrix} X_1^{e_1} \ldots\ldots\ldots X_1^{e_p} \\ \vdots \\ X_i^{e_1} \ldots\ldots\ldots X_i^{e_p} \\ \vdots \\ X_p^{e_1} \ldots\ldots\ldots X_p^{e_p} \end{pmatrix} \ldots\ldots\ldots\ldots\ldots (4)$$

Provided that $\det(m) \neq 0$ (non singular move) and square matrix. Let $\Delta = \det(m)$, substitutions $X_j = X$ in M

$$M_j(X) = \begin{pmatrix} X_1^{e_1} \ldots\ldots\ldots X_1^{e_p} \\ \vdots \\ X_j^{e_i} \ldots\ldots\ldots X_j^{e_p} \\ \vdots \\ X_p^{e_1} \ldots\ldots\ldots X_p^{e_p} \end{pmatrix} \leftarrow j^{th} row \ldots\ldots\ldots (5)$$

$$\Delta_j(X) = \det(m_j(X))$$

Make another substitutions $X = X_i \; in \; M_j(X) \quad ,(i \neq j)$

$$(M_j)_i = \begin{pmatrix} X_1^{e_1} \ldots\ldots X_1^{e_p} \\ \vdots \\ X_i^{e_1} \ldots\ldots\ldots X_i^{e_p} \\ \vdots \\ X_i^{e_1} \ldots\ldots\ldots X_i^{e_p} \\ \vdots \\ X_p^{e_1} \ldots\ldots\ldots X_p^{e_p} \end{pmatrix} \begin{array}{l} \leftarrow i^{th} row \\ \hline \leftarrow j^{th} row \end{array}$$

That is mean $\det(M_j)_i = 0$

$$l_i(X) = \frac{\Delta_i(x)}{\Delta} \quad , X = (x, y)$$

$$f(x,y) = \sum_{i=1}^{p} f_i l_i(X) = \sum_{i=1}^{p} f_i \frac{\Delta_i(X)}{\Delta}$$

$$z(x,y) \cong Z_1 \frac{\Delta_1(X)}{\Delta} + Z_2 \frac{\Delta_2(X)}{\Delta} + \ldots + Z_p \frac{\Delta_p(X)}{\Delta} \ldots (6)$$

$$X = (x, y)$$

$$Z \cong Z_1 l_1(x,y) + Z_2 l_2(x,y) + \ldots + Z_p l_p(x,y)$$

Experiment 1:
Suppose (1,0,0),(0,1,0) and (0,0,1) are there points are given and lie on z=f(x,y) . to construct polynomial of degree n=1, m=2 with

$$p = \begin{pmatrix} n+m \\ n \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3$$

Hence $z_i = \alpha_1 x_i + \alpha_2 y_i + \alpha_3 \quad ,1 \leq i \leq 3 \, , \alpha_1, \alpha_2, \alpha_3$ are coefficients

$$\begin{aligned} 0 &= \alpha_1 + \alpha_3 \\ 0 &= \quad + \alpha_2 + \alpha_3 \\ 1 &= \qquad\quad \alpha_3 \end{aligned}$$

From (4) we have:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; \det(M) = \Delta = 1$$

From (5) we get:

$$M_1 = \begin{pmatrix} X & Y & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\det(M_1) = \Delta_1 = x$$

$$\det(M_2) = \Delta_2 = y$$

$$\det(M_3) = \Delta_3 = 1 - y - x$$

By (6) we get

$$z = z_1 \frac{\Delta_1}{\Delta} + z_2 \frac{\Delta_2}{\Delta} + z_3 \frac{\Delta_3}{\Delta}$$

$$= (1 - y - x)$$

$$then$$

$$f(x,y) = 1 - y - x$$

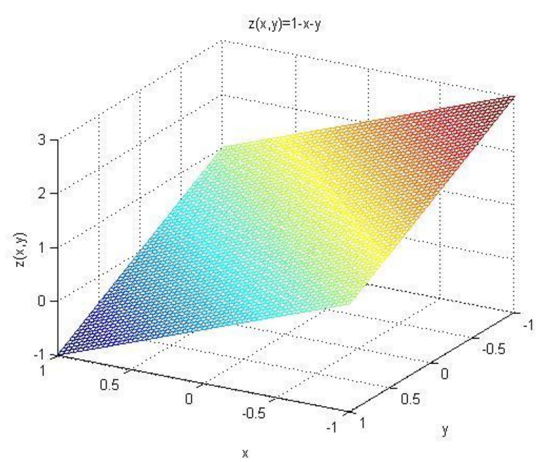

***Fig.1** Experiment (1)*

2. if n=2 , and m=2

$$z_i = f(x_i, y_i) \qquad 1 \le i \le 6$$
$$p = \binom{n+m}{n} = \binom{2+2}{2} = 6$$

Experiment 2:

Suppose the initial points of surface are (0,0,-1), (1,1,1),(1,0,-1),(2,1,5), ,(-2,-1,3),(3,2,9). Hence

$$\Delta = \det(M) = -4$$
$$\Delta_1 = \det(M_1) = 8x^2 - 20xy - 4x + 12y^2 + 8y - 4$$
$$\Delta_2 = \det(M_2) = -4x^2 + 12xy + 4x - 8y^2 - 8y$$
$$\Delta_3 = -2x^2 + 8xy - 2x - 8y^2 + 4y$$
$$\Delta_4 = -2x^2 + 2x + 6y^2 - 6y$$
$$\Delta_5 = -2x^2 + 4xy + 2x - 2y^2 - 2y$$
$$\Delta_6 = 4y - 2x - 4xy + 2x^2$$
$$and$$
$$z = 2x^2 - 2x - 3y^2 + 5y - 1$$

3. if n=3 and m=2 , p=10

$$z_i = f(x_i, y_i) \qquad 1 \le i \le 10$$

Experiment 3:

Suppose we are given set of point as:

$p_1(0,0,1)$, $p_2(1,0,2)$, $p_3(-1,1,1)$, $p_4(2,-1,8)$, $p_5(2,2,11)$, $p_6(-1,-1,-1)$, $p_7(3,1,29)$, $p_8(0,-2,-1)$, $p_9(-2,1,-6)$, $p_{10}(-2,-2,-9)$

First ,we construct 11 matrices of size 10 x10 , and by MATLAB program , we calculate all determinate of them for example,

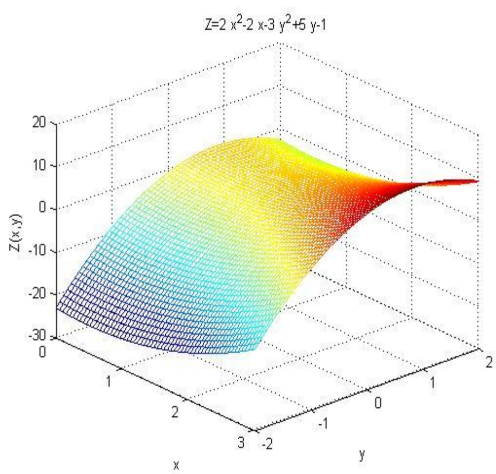

Fig.2 *graph of experiment (2)*

Det (M)=

0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	1	0	1	0	1
-1	1	-1	1	1	1	-1	-1	1	1
8	-4	2	-1	1	4	-2	2	-1	1
8	8	8	8	4	4	4	2	2	1
-1	-1	-1	-1	-1	1	1	-1	-1	1
27	9	3	1	1	9	3	3	1	1
0	0	0	-8	4	0	0	0	-2	1
-8	-2	-2	1	1	4	-2	-2	1	1
-8	8	-8	-8	4	4	4	-2	-2	1

= 87552

Det (M1)=

x^3	x^2*y	x*y^2	y^3	y^2	x^2	x*y	x	y	1
1	0	0	0	0	1	0	1	0	1
-1	1	-1	1	1	1	-1	-1	1	1
8	-4	2	-1	1	4	-2	2	-1	1
8	8	8	8	4	4	4	2	2	1
-1	-1	-1	-1	-1	1	1	-1	-1	1
27	9	3	1	1	9	3	3	1	1
0	0	0	-8	4	0	0	0	-2	1
-8	-2	-2	1	1	4	-2	-2	1	1
-8	8	-8	-8	4	4	4	-2	-2	1

$=-171648x^3+812160x^2y+162432x^2+499968xy^2-1169280xy-78336x+35712y^3-639360y^2-1377792y+87552$,

and so on

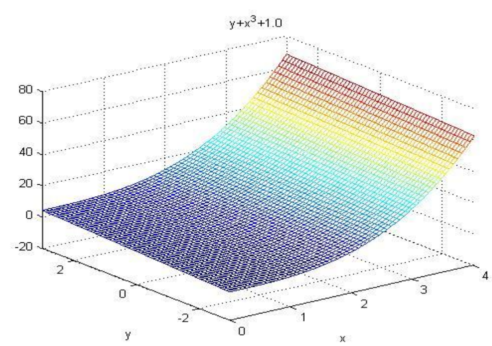

Fig.3 *Experiment (3)*

$$\Delta = \det(M) = 8.7552\ e + 004 = 87552$$

$$\Delta_1 = \det(M_1) = -171648x^3 + 812160x^2y + 162432x^2 + 499968xy^2 - 1169280xy - 78336x + 35712y^3 - 639360y^2 - 1377792y + 87552$$

$$\Delta_2 = \det(M_2) = 78336x^3 - 414720x^2y - 82944x^2 - 253440xy^2 + 608256xy + 92160x - 36864y^3 + 304128y^2 + 755712y$$

$$\Delta_3 = \det(M_3) = 73728x^3 - 293760x^2y - 80640x^2 - 20506x + 408960xy + 6912x + 10368y^3 + 259200y^2 + 476928y$$

$$\Delta_4 = \det(M_4) = -8448x^3 + 57600x^2y + 11520x^2 + 23040xy^2 - 99072xy - 3072x + 24576y^3 - 27648y^2 - 153600y$$

$$\Delta_5 = \det(M_5) = 2112x^3 - 14400x^2y - 2880x^2 - 5760xy^2 + 24768xy + 768x + 4800y^3 + 17856y^2 + 16512y$$

$$\Delta_6 = \det(M_6) = 52992x^3 - 241920x^2y - 48384x^2 - 140544xy^2 + 354816xy - 4608x - 6912y^3 + 177408y^2 + 382464y$$

$$\Delta_7 = \det(M_7) = -3840x^3 + 24192x^2y + 9216x^2 + 18432xy^2 - 31104xy - 5376x - 8064y^3 - 26496y^2 - 20736y$$

$$\Delta_8 = \det(M_8) = 13056x^3 - 69120x^2y - 13824x^2 - 27648xy^2 + 101376xy + 768x - 20736y^3 + 50688y^2 + 140544y$$

$$\Delta_9 = \det(M_9) = -25344x^3 + 85248x^2y + 34560x^2 + 69120xy^2 - 122112xy - 9216x - 13824y^3 - 82944y^2 - 110592y$$

$$\Delta_{10} = \det(M_{10}) = -10944x^3 + 54720x^2y + 10944x^2 + 21888xy^2 - 76608xy + 10944y^3 - 32832y^2 - 109440y$$

$$z = (\det(M_1) + 2\det(M_2) + \det(M_3) + 8\det(M_4) + 11\det(M_5) - \det(M_6) + 29\det(M_7) - \det(M_8) - 6\det(M_9) - 9\det(M_{10}) / \det(M))$$

$$z = x^3 + y + 1$$

4. Conclusion:

We tried to build polynomials of different degrees in R^3 and used MATLAB program from several different angles.

References

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