Natural Vibrations Of Spherical Inhomogeneity In A Viscoelastic Medium

Zafar Boltai, Ismoil Safarov, Tulkin Razokov

Abstract: This article discusses the natural vibrations of inhomogeneous systems, i.e., spherical bodies located in a deformable viscoelastic medium. The main attention is paid to the study of low-contrast inhomogeneities in a viscoelastic medium. The problem of radial, torsional and spheroidal vibrations of a deformable spherical inclusion in an infinite viscoelastic medium is considered. The theory and methods of calculating the complex eigenfrequencies and vibration modes of a viscoelastic spherical inhomogeneity in a viscoelastic medium are constructed. Such vibrations are classified into radial, torsional and spheroidal. The frequency transcendental equation of spherical inhomogeneity in a viscoelastic medium is constructed. The general solution of the vector equation of motion of the three-dimensional theory of viscoelasticity in potentials of displacements of a spherical coordinate system is used. Using special functions of the mathematical physics of Bessel, Hankel and Legendre, the Mueller method and the Gauss method, an algorithm has been developed for solving the stated computer problems. Based on the constructed complex frequency equations with complex output parameters, numerical results are obtained. The considered problems were reduced to finding those complex natural frequencies at which the system of equations of motion of a spherical inclusion with shortened radiation conditions has a nonzero solution in the class of infinitely differentiable functions. Detailed numerical calculations of the complex eigenfrequencies of three modes of radial, torsional, and spheroidal oscillations of a spherical inhomogeneity in an infinite viscoelastic medium are performed. It is shown that the problem has a discrete complex spectrum. It was found that the eigenfrequencies of spheroidal oscillations of low-contrast inhomogeneity are divided into two groups: radial-like and torsion-like. It was revealed that at some values of viscoelastic density parameters, low-frequency eigenoscillations arise, which is essentially some aperiodic motion, since the imaginary part of the eigenfrequency is large. The results obtained make it possible to predict the scattering of viscoelastic (or seismic) waves in deformable media in the presence of inclusion.

Index Terms: cavity, equations of motion, frequency equations, inclusion, inhomogeneous system, viscoelastic medium, torsional vibrations, natural frequency, three-dimensional theory of viscoelasticity, axisymmetric vibrations, relaxation core, radial vibrations, spheroidal vibrations.

1 INTRODUCTION

Due to the increased interest in the development of new approaches to solving urgent modern problems related to non-destructive remote control of medical diagnostics materials, geo-acoustic problems, seismic acoustic sensing, seismic resistance assessment of underground engineering structures and some others, it becomes necessary to formulate and solve some model problems, the analysis of which will give a key to forecasting and effective management of the processes under consideration [1]. Any inhomogeneity (inclusion) located in an elastic or viscoelastic medium together with it possesses, like any mechanical system, a certain spectrum of natural frequencies. In a dynamic process, the inclusion and the medium work together, since they are interconnected, while the radiation of the waves leads to damping of the oscillations and the natural frequencies of the oscillations of this system (i.e., the inclusion and the medium) will be complex. Thus, the solution of problems associated with the identification of heterogeneities, with the determination of their size and physical characteristics, are very important and relevant. Typically, geophysicists use different approaches for these purposes, i.e., like gravity exploration, electromagnetic methods, the study of electrical conductivity, etc., compared with these methods, the seismic method is the most direct and, when interpreted, gives the least doubtful results. Since when the frequency of the incident wave is close to the real component of the natural frequency, the inhomogeneity will begin to radiate energy in the resonant mode.

Therefore, for practical purposes, identifying possible resonance peaks on the spectral curve and establishing their relationship with the corresponding inhomogeneities, it is very important to determine the natural frequencies of oscillations of elastic inclusions in an infinite elastic medium [2,3] becomes an urgent task. Based on a numerical analysis, the authors of the monograph [4] suggested that extreme values of the scattering cross section are observed when the frequency of the incident wave is close to the resonant frequencies of the medium — scattering center system. This gap was substantially filled after the publication of a number of articles [5–7], in which, using the echo signal analysis, the resonance properties of liquid cylinders and spheres placed in an infinite elastic medium are studied [20, 21]. A similar approach can be used to study the tectono-physical phenomenon, i.e. earthquake source behavior. In the seismic field recorded by the seismic receiver, heterogeneity in the medium will leave a mark in the form of some distortions of the original seismic wave. In spectral analysis, these distortions should appear in the form of some resonance peaks in the spectral curve, as is the case when determining the scattering cross section. As you know, an ideal elastic body has no losses [8,9,10]. Even if the equation is linear with respect to stress and strain, the presence of time derivatives is always associated with dissipation. As a result, under alternating voltage, a hysteresis effect occurs. This means that in the frequency range in which the attenuation is noticeable, the deformation will lag behind the stress. Such a connection, firstly, leads to the interaction of the considered elastic wave with other waves (for example, with thermal vibrations) and as a result, the energy is redistributed between the waves. Secondly, the wave under consideration will generate higher harmonics, transmitting its energy to them. In both cases, the interaction depends on the strain amplitude. The nonlinear relationship between stress and strain in the presence of time derivatives also leads to damping, depending on the strain amplitude. Today, the study of heterogeneities is of great interest for predicting an

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important tectono-physical phenomenon, i.e. the behavior of the source of the impending earthquake. Now, among seismologists, the idea of the zone of upcoming seismic shocks is widely accepted as an area with elastic-density characteristics changing as a result of tectonic movements. From a mechanical point of view, this corresponds to the study of heterogeneity with longitudinal and transverse wave velocities slightly changed relative to the external elastic medium, as well as, possibly, with density. Any heterogeneity in the medium must possess, like any elastic mechanical system, some spectrum of natural frequencies. Since the oscillations of the inclusion and the medium are interconnected, there will be a damping of the oscillations due to the emission of elastic waves and, therefore, the natural frequencies will be complex [11]. From a physical point of view, attenuation in an ideal elastic medium is explained by the radiation of the energy of waves excited by natural vibrations due to diverging elastic waves. The interest in studying the eigenfrequencies of the “elastic inclusion - medium” system is also due to the following circumstance. When an inhomogeneity is detected by seismic waves either from weak earthquakes or from pulsed artificial sources such as pneumatic emitters, the scattering problem must be solved in an unsteady setting. In this case, the state of an inhomogeneous body is described by a linear unambiguous relationship between stress and deformation during the entire period of alternating stress. It follows that stress and strain are always in phase. The energy dissipation of the elastic wave will occur if the stress and strain are not connected by a single-valued dependence during the oscillation period. The absence of such an unambiguous relationship between stress and strain arises when time derivatives appear in the equation connecting these quantities [12]. As is known, in this case, for calculating the wave field, the stationary solution should be integrated over the frequency along with the spectrum of the given incident pulse. The resulting integral can, generally speaking, be calculated by any direct numerical method. In this case, preference should be given to the method of integration using the theory of residues in the form of an expansion at the poles of the integrand, because it is this method that can reveal a number of useful physical features of the diffraction process. We note that the poles of interest to us coincide with the roots of the eigenfrequency equation and, therefore, in order to be able to deal with the problems of unsteady diffraction of elastic waves in the future, a thorough study of the behavior of the roots of the frequency equations depending on the ratio of the elastic-density parameters of the medium and inclusion is necessary [13]. In [14], the problem is considered in a stationary setting, when the incident wave is a sinusoid infinite in space and time. In this case, a number of difficulties arise due to the fact that the eigenfunctions of the problem under study cannot be considered as a vector in a Hilbert space: they are not normalized due to the exponential growth of distance. To eliminate it, it is proposed to take into account that oscillations cannot exist for an infinitely large period of time, and, therefore, we come to the necessity of limiting at a small initial moment in time. In [15], the problem of “with an exponential catastrophe” is solved by developing special radiation conditions and boundary conditions. This article discusses the vibrations of spherical bodies in a deformable medium. The main attention in the work will be given to the study of low-contrast inhomogeneities. Along with this, the behavior of complex eigenfrequencies will be investigated depending on the geometric and physico-mechanical parameters of the inhomogeneous system. The physical nature of the considered heterogeneities is closely related to convective currents in the Earth’s interior, as well as to various areas of faults and fragmentations. Such inclusions are very common and, therefore, have a significant effect on the scattering of seismic waves in various media.

2 METHODOLOGY

2.1 BASIC RELATIONSHIPS AND EQUATIONS

The basic equations of motion of heterogeneous systems, i.e. deformable (linear elastic or viscoelastic) spherical inclusions in infinite viscoelastic medium describing natural vibrations have the form of

\[
(\lambda_j + 2\beta_j \gamma) \text{grad} \phi_j - \mu_j \text{rot} \text{rot} u_j = \rho_j \frac{\partial^2 u_j}{\partial t^2}, \quad j=1,2
\]

(1)

where \(\lambda_j, \mu_j\) and \(\beta_j\) are the material constants of the inclusion. On the function of influence, the problem of inhomogeneity, changing parameters of the external elastic medium, as well as, possibly, with density. Any heterogeneity in the medium must possess, like any elastic mechanical system, some spectrum of natural frequencies. Since the oscillations of the inclusion and the medium are interconnected, there will be a damping of the oscillations due to the emission of elastic waves and, therefore, the natural frequencies will be complex [11]. From a physical point of view, attenuation in an ideal elastic medium is explained by the radiation of the energy of waves excited by natural vibrations due to diverging elastic waves. The interest in studying the eigenfrequencies of the “elastic inclusion - medium” system is also due to the following circumstance. When an inhomogeneity is detected by seismic waves either from weak earthquakes or from pulsed artificial sources such as pneumatic emitters, the scattering problem must be solved in an unsteady setting. In this case, the state of an inhomogeneous body is described by a linear unambiguous relationship between stress and deformation during the entire period of alternating stress. It follows that stress and strain are always in phase. The energy dissipation of the elastic wave will occur if the stress and strain are not connected by a single-valued dependence during the oscillation period. The absence of such an unambiguous relationship between stress and strain arises when time derivatives appear in the equation connecting these quantities [12]. As is known, in this case, for calculating the wave field, the stationary solution should be integrated over the frequency along with the spectrum of the given incident pulse. The resulting integral can, generally speaking, be calculated by any direct numerical method. In this case, preference should be given to the method of integration using the theory of residues in the form of an expansion at the poles of the integrand, because it is this method that can reveal a number of useful physical features of the diffraction process. We note that the poles of interest to us coincide with the roots of the eigenfrequency equation and, therefore, in order to be able to deal with the problems of unsteady diffraction of elastic waves in the future, a thorough study of the behavior of the roots of the frequency equations depending on the ratio of the elastic-density parameters of the medium and inclusion is necessary [13]. In [14], the problem is considered in a stationary setting, when the incident wave is a sinusoid infinite in space and time. In this case, a number of difficulties arise due to the fact that the eigenfunctions of the problem under study cannot be considered as a vector in a Hilbert space: they are not normalized due to the exponential growth of distance. To eliminate it, it is proposed to take into account that oscillations cannot exist for an infinitely large period of time, and, therefore, we come to the necessity of limiting at a small initial moment in time. In [15], the problem of “with an exponential catastrophe” is solved by developing special radiation conditions and boundary conditions. This article discusses the vibrations of spherical bodies in a deformable medium. The main attention in the work will be given to the study of low-contrast inhomogeneities. Along with this, the behavior of complex eigenfrequencies will be investigated depending on the geometric and physico-mechanical parameters of the inhomogeneous system. The physical nature of the considered heterogeneities is closely related to convective currents in the Earth’s interior, as well as to various areas of faults and fragmentations. Such inclusions are very common and, therefore, have a significant effect on the scattering of seismic waves in various media.

Further, applying the freezing procedure [18], we replace (2) with an approximate relation of the form

\[
\lambda_j \phi_j = \lambda_j \phi_j [1 - i\alpha_j^2(\omega_p) - i\beta_j^2(\omega_p)] \psi_j, \quad \phi_j = \mu_j [1 - i\alpha_j^2(\omega_p) - i\beta_j^2(\omega_p)] \phi_j \psi_j
\]

(3)

where \(\alpha_j, \beta_j, \lambda_j\) and \(\mu_j\) are the operator moduli of elasticity [16,17], \(\phi_j\) - arbitrary function of time; \(\beta_j\) - density, \(R_{\lambda_j}(t - \tau)\) and \(R_{\mu_j}(t - \tau)\) - relaxation nuclei and \(\lambda_j, \mu_j\) instant moduli of elasticity. For \(j = 1\), equations (1) and (2) relate to the sphere, and for \(j = 2\) to the external environment. We take the integral terms in (2) small, then the function \(\phi_j(t) = \phi_j(t) e^{-\omega_p t}\) where \(\phi_j(t)\) slowly changing function of time, \(\omega_p\) - real constant.

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Further, it is necessary to study periodic processes in a continuous elastic medium with a spherical inclusion, differing in their elastic-density and rheological characteristics from the corresponding characteristics in an infinite medium, satisfying equations (1), (2), (3). To solve the problem under consideration, we take the dependence \(u\) from time to time in the form \(u = U(r, \theta, \phi) e^{i\omega t}\).

In this case, the spatial coordinate function \(U(r, \theta, \phi)\) can be represented as the amount of potential \(U_p = \text{grad} \phi\) and solenoidal \(U_s = \text{rot} \psi\) parts: \(U = U_p + U_s\), which satisfy the following equations

\[
(\Delta + k^2) U_p = 0; \quad (\Delta + k^2) U_s = 0, \quad \text{div}\ U_p = 0; \quad \text{div}\ U_s = 0,
\]

(4)

where \(k^2 = \omega^2 / \mu + \nu^2 / \rho\).
\[ u_{p1} = u_{p2}; \quad u_{q1} = u_{q2}; \quad u_{r1} = u_{r2}, \quad (5) \]

where index 1 refers to the inside of the sphere, and 2 refers to the external environment where the sphere is located. It is known [19] that the vector displacement field \( \mathbf{U} \) admits (in spherical coordinates) decomposition into three vector fields, each of which is determined by only one scalar function \( U_\alpha \), \( U_\beta \), and \( U_\gamma \)-transverse parts of the solution. Their expressions through scalar functions have the form

\[ \mathbf{U}_\alpha = \frac{1}{k_p} \text{grad} \psi_\alpha, \quad \mathbf{U}_\beta = \text{rot} (\mathbf{r} \psi_\beta), \quad \mathbf{U}_\gamma = \frac{1}{k_q} \text{rot rot} (\mathbf{r} \psi_\gamma), \]

so, each of the scalar functions \( \psi_i \) (\( i = 0, 1, 2 \)) satisfies the equation

\[ (\Delta + k_i^2)\psi_i = 0; \quad k_i = [k_p, i = 0; k_q, i = 1, 2, 3] \]

Here \( \Delta \) - second order operator in spherical coordinates.

The solution of the scalar Helmholtz equation (6) for each of the functions \( \psi_i (r, \theta, \phi) \) has the form

\[ \psi_i = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} h_l (k_i r) P_l^m (\cos \theta) \exp (im \phi); \]

\[ \psi_l = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} h_l (k_i r) P_l^m (\cos \theta) \exp (im \phi); \]

\[ \psi_2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} h_l (k_i r) P_l^m (\cos \theta) \exp (im \phi), \]

where \( h_l (z) \) - Bessel spherical function; \( l = 1, 2 \); \( l = 1 \) refers to a spherical body, and \( l = 2 \) - to the medium; \( P_l^m (\cos \theta) \) is the associated Legendre function of the first kind of the \( nth \) degree and \( mth \) order. In calculating the Legendre function \( n >> 1 \), the asymptotic formulas from \([3]\) were used

\[ P_{l-1/2} (\cos \theta) = (2/\sin \theta)^{1/2} \cos (n(\Delta - \pi/4) + \phi/\sin (n\theta-\pi/4) + \theta(1/n^2)), \]

For an external problem (oscillations of the medium \( l = 1 \)) we will take as \( h_l (z) \) Hankel function of the second kind.

\[ h_l (z) = \sqrt{\frac{\pi}{2}} H_{l+1/2}^2 (z) \]

which highlights at infinity \( (r \to \infty) \) diverging waves.

For the internal problem (switching oscillations, \( l = 2 \)) we will take as \( h_l (z) \) Bessel function of the first kind

\[ h_l (z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2} (z) = j_{n} (z) \]

which satisfies the condition of boundedness at zero. As a result, we obtain the following expressions for displacements \( \mathbf{U} \), which describes the oscillatory process in the medium under consideration with spherical inhomogeneities:

\[ u_{p1} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} D_l (k_p r) + \frac{B_{lm}}{k_q^2 r} h_l (k_q r) \right] \Phi_l \]

\[ u_{p2} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} h_l (k_p r) + \frac{B_{lm}}{k_q^2 r} D_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \phi} \]

\[ u_{p3} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} h_l (k_p r) + \frac{B_{lm}}{k_q^2 r} D_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \theta} \]

\[ u_{q1} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} h_l (k_p r) + \frac{B_{lm}}{k_q^2 r} D_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \rho} \]

\[ u_{q2} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} D_l (k_p r) + \frac{B_{lm}}{k_q^2 r} h_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \phi} \]

\[ u_{q3} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} h_l (k_p r) + \frac{B_{lm}}{k_q^2 r} D_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \theta} \]

(7)

Here

\[ \Phi_l = n \text{rad} \theta \exp (im \phi), \quad D_l (z) = i n h_l (z) - \Phi_l, \quad D_l (z) = (n + 1) h_l (z) - \Phi_l, \]

Tension \( \sigma_{\theta \theta}, \sigma_{\rho \theta}, \sigma_{\phi \phi} \) are written out through displacements in spherical coordinates according to the following formulas:

\[ \sigma_{\theta \theta} = \mu \left( \frac{\partial u_{\theta}}{\partial \theta} + \frac{\partial u_{\phi}}{\partial \phi} \right), \]

\[ \sigma_{\rho \phi} = \mu \left( \frac{\partial u_{\rho}}{\partial \phi} - \frac{\partial u_{\phi}}{\partial \rho} \right). \]

(8)

Substituting (7) into (8), we obtain for determining the stresses inside and outside the heterogeneity of the expression:

\[ \sigma_{\rho \phi} = \frac{2 \pi}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} D_l (k_p r) + \frac{B_{lm}}{k_q^2 r} h_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \rho} \]

\[ \sigma_{\theta \theta} = \frac{2 \pi}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} h_l (k_p r) + \frac{B_{lm}}{k_q^2 r} D_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \theta} \]

\[ \sigma_{\phi \phi} = \frac{2 \pi}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{A_{lm}}{k_p^2 r} h_l (k_p r) + \frac{B_{lm}}{k_q^2 r} D_l (k_q r) \right] \frac{\partial \Phi_l}{\partial \phi} \]

(9)

Where

\[ D_l (z) = (n^2 - n - 1/z^2) \cdot h_l (z) + z h_{l+1} (z), \quad D_4 (z) = (n^2 - 1) \cdot h_l (z) - \Phi_l, \quad D_4 (z) = (n^2 - 1) \cdot h_l (z) + \Phi_l. \]

The coefficients \( A_{lm}, B_{lm} \) should be determined from the boundary conditions (6) on the surface of the sphere.

The results obtained (7), (8) allow us to study various natural vibrations of elastic and viscoelastic inhomogeneities located inside an infinite deformable medium, i.e. radial, torsional and spheroidal natural vibrations of the system under consideration.

2.2 RADIAL VIBRATIONS

We define the complex natural frequency of the radial vibrations of a spherical cavity in an unbounded viscoelastic medium for which \( c_n) \mathbf{c}_n \mathbf{c}_r = \mathbf{n} = \mathbf{u} = \mathbf{\Phi} \mathbf{r} \). The equation of oscillations (1) in the potentials (longitudinal waves) of displacements, taking into account the periodicity of oscillations, reduces to the Helmholtz equation.
The solution to equation (10) is sought in the form of a diverging spherical wave \( \phi = A e^{i \omega t} / r \).

Radial stress \( \sigma_{rr} = \rho \left[ c_0^2 - 2 \omega^2 \right] A \phi + 2 \omega^2 A \phi_{,t} \),
or, using equation (10), we obtain:

\[
\frac{1}{\rho^2} \sigma_{rr} = -\omega^2 \phi - 4c_0^2 \phi_{,t} / r.
\]

Using boundary conditions \( \sigma_{rr}(R) = 0 \) leads to the equation

\[
\left( k_p R \frac{c_p}{c_s} \right)^2 = 4(1 - i k_p R / \sqrt{\rho}) \phi - \frac{2c_\perp}{R} (1 - i \frac{c_p}{c_s})
\]

From here, when \( R = \delta \), then \( 0 = c_\perp(1 - i \sqrt{3}) \).

As you know, the material part \( \omega \) gives its own oscillation frequency, and the imaginary part is the attenuation coefficient. It can be seen that for the Poisson medium, the wave intensely attenuates. In an incompressible environment \( (c_s \to \infty) \) attenuation would naturally be absent. With an increase in the radius of the hole, the corresponding frequencies and attenuation coefficients decrease with a hyperbolic law. When studying the radial vibrations of a spherical inclusion in a viscoelastic medium (7) and (9), it is necessary to take

\[
\Delta = \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial t^2} - k_p^2 \phi = 0
\]

Using the boundary conditions \( \sigma_{rr}(R) = 0 \) leads to the equation

\[
\left( k_p R \frac{c_p}{c_s} \right)^2 = 4(1 - i k_p R / \sqrt{\rho}) \phi - \frac{2c_\perp}{R} (1 - i \frac{c_p}{c_s})
\]

Further, equating the determinant of the system to zero, we obtain a transcendental equation for the natural frequencies of torsional vibrations of a spherical inclusion:

\[
\left( n - 1 - G_\delta(z_\omega) \right) - z_\omega (n - 1 - G_\delta(z_\omega)) = 0
\]

If \( z_\omega \to 0 \), then we naturally come to the real frequency equation of torsional vibrations of the ball \( n = 1 - G_\delta(z_\omega) \).

This equation determines the actual discrete spectrum natural frequencies of a spherical cavity in which there is no radiation. At \( z_\omega \to \infty \) we obtain a complex equation for determining the natural frequencies (the imaginary part of which is the damping coefficient of torsional vibrations of a spherical cavity in an infinite medium).

2.4 SPHEROIDAL VIBRATIONS

In this case, the eigenfrequencies of the spheroidal vibrations of the spherical cavity in an unbounded viscoelastic medium are determined.

When the radial component of the displacement vector vanishes \( (u_r = 0) \), as well as dilatations \( (d_{tt}) \), Eqs. (4) characterize torsional vibrations of an inhomogeneous inclusion in an infinite medium. Moreover, in the general solution (7), they correspond to the part including the coefficients Cmn. Substituting this part into the boundary conditions (5) leads to the following system of equations for determining the coefficients Cmn1, Cmn2:

\[
C_{mn1} h_{n,k}(k_* R) - C_{mn2} i_{n,k}(k_* R) = 0
\]

Further, equating the determinant of the system to zero, we obtain a transcendental equation for determining the natural frequencies of spheroidal vibrations of a spherical inclusion:

\[
\left( n - 1 - G_\delta(z_\omega) \right) - z_\omega (n - 1 - G_\delta(z_\omega)) = 0
\]

If \( z_\omega \to 0 \), then we naturally come to the real frequency equation of torsional vibrations of the ball \( n = 1 - G_\delta(z_\omega) \).

This equation determines the actual discrete spectrum natural frequencies of a spherical cavity in which there is no radiation. At \( z_\omega \to \infty \) we obtain a complex equation for determining the natural frequencies (the imaginary part of which is the damping coefficient of torsional vibrations of a spherical cavity in an infinite medium).
where the elements cij (i = 1, 2, 3, 4; j = 1, 2, 3, 4) are defined as:

\[
\begin{align*}
    c_{11} &= n - G_{1}(z_{j}z_{j}), \\
    c_{12} &= (n + 1), \\
    c_{13} &= n - G_{1}(z_{j}z_{j}), \\
    c_{14} &= (n + 1), \\
    c_{21} &= n - G_{1}(z_{j}z_{j}), \\
    c_{22} &= (n + 1), \\
    c_{23} &= n - G_{1}(z_{j}z_{j}), \\
    c_{24} &= (n + 1), \\
    c_{31} &= n - G_{1}(z_{j}z_{j}), \\
    c_{32} &= n - G_{1}(z_{j}z_{j}), \\
    c_{33} &= n - G_{1}(z_{j}z_{j}), \\
    c_{34} &= (n + 1), \\
    c_{41} &= n - G_{1}(z_{j}z_{j}), \\
    c_{42} &= n - G_{1}(z_{j}z_{j}), \\
    c_{43} &= n - G_{1}(z_{j}z_{j}), \\
    c_{44} &= (n + 1).
\end{align*}
\]

(15)

An analytical study of the complex eigenvalues of the transcendental equation (16) depending on the parameter \( z_{sh} \), for various n, in the general case, is impossible. The characteristic equation (16), in the present work, was solved by the Mueller method, the elastic problems were solved as the initial approximation (R\(\lambda\) = R\(\lambda\) = 0).

3 NUMERICAL RESULTS AND DISCUSSIONS

From the point of view of mechanics, the above problems are posed for non-conservative systems, i.e. for an infinite viscoelastic medium with an inhomogeneous inclusion. Such systems, if they have their own frequencies, they will be complex, i.e. \( \omega = \omega_{r} - i\omega_{i} \). The real part \( \omega_{r} \) complex parameter \( \omega_{i} \) in its physical essence is the frequency of free damped oscillations of the system, and the imaginary part \( \omega_{i} \) carries information about the rate of attenuation of oscillations and (up to a sign) is equal to the damping coefficient. The damping coefficient is a quantitative characteristic of the rate of damping of oscillations and determines the dissipative properties of the system as a whole. Therefore, in the future it will be necessary to determine the numerical values of the complex eigenfrequencies by solving the above-obtained transcendental equations (12) - (15).

Transcendental equations are solved numerically - using the Muller method [20, 21], as well as using the MAPLE-18 software package. The viscoelastic properties of the material are described using three parametric relaxation nuclei [22]:

\[
R_{j1}(t) = A_{j} e^{-\beta_{j}t} t^{1-\gamma_{j}} (j = 1,2)
\]

In specific calculations, the following values of the parameters of the material unlimited in a viscoelastic medium are accepted [22,23,24]:

\[
\beta = 0.02, \quad z_{sh} = 0.9, \quad z_{exp} = 0.95, \quad A_{1} = 0.01; \quad \beta = 0.05; \quad \alpha = 0.1
\]

Figure 1 shows the calculation results obtained by solving equation (12) for radial vibrations as a function of the dimensionless phase velocity of radial vibrations as a function of the ratio of the shear wave velocities of the medium and the ratio of the shear wave velocities of the medium and the inclusion. For radial oscillations, the frequencies are determined by solving the frequency equations (12) and (13) with complex parameters \( z_{j}^{(3)} \) [25,26]. Moreover, the presence of trigonometric functions for each number of oscillations \( n \) leads to the appearance of a countable set of complex frequencies, which are called overtones [27,28,29].

In fig. Figure 2 shows the change in the imaginary part of the complex phase velocity (damping velocity) from the ratio of shear wave velocities \( \beta = c_{s2}/c_{s1} \) at various values of \( z_{exp} \). An analysis of the results shows that in the region \( 0 \leq \beta \leq 1 \), damping speed at \( z_{exp} \geq 1 \) decreases moderately and then, at \( \beta \geq 1.1 \) increases dramatically. This means that free waves at \( \beta \geq 1.1 \) quickly decay. When calculating the stresses and displacements according to the formula (9), the summation was performed until the ratio of the current term to the current particular sum becomes less than 10-10. To ensure the convergence of
series (9), a numerical experiment was carried out each time [30,31].

Since in this case there is no dependence on the longitudinal velocities, there is a stronger sensitivity to a change in the density of an infinite medium with respect to heterogeneity [32,33]. From fig. Figure 3 shows that the first mode of the natural frequency suffers a discontinuity (the mechanical characteristics of the inclusion and the medium are close to each other). The imaginary roots of the frequency equation of torsional vibrations will be a finite number, and with radial vibrations - a countable set [34,35]. An analysis of the results of Fig. 4 shows that when the mechanical characteristics are close to each other (β = 1) imaginary parts of the complex frequency reach a maximum. This means that the corresponding torsional vibrations decay intensively.

4 CONCLUSION

1. A theory and methods have been constructed for assessing the dynamics of an inhomogeneous infinite viscoelastic medium with a spherical inclusion, which can make it possible to predict the scattering of seismic waves in various media in the presence of inclusions.

2. The study of low-contrast oscillations of inhomogeneities, reducing the problems considered to radial, torsional and spheroidal vibrations of a deformable spherical inclusion in an infinite viscoelastic medium.

3. The problems considered were reduced to the problem of finding the complex eigenfrequencies of the considered inhomogeneous systems (ie, inclusions in an infinite medium) \( \Omega_R + i\Omega_I \).

4. It was found that at some values of the viscoelastic - density parameters of inclusion in an infinite medium, low-frequency eigenoscillations arise. These oscillations are essentially some aperiodic motion, since the imaginary part of the natural frequency is large.

5. The obtained numerical results differ from the known results of V.A. Dubrovsky [6.7] to 10-15%.

5 ACKNOWLEDGMENT

The work was done with the financial support of the FFI of Uzbekistan (project No. F-4-14).

6 REFERENCES


