Solution and stability of cubic functional equation in fuzzy normed spaces

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Abstract: In this present work, we introduce a new type of finite dimensional cubic functional equation of the form

\[
\phi \left( \sum_{a=1}^{l} an_a \right) = \sum_{1 \leq a < b < c \leq l} \phi (an_a + bn_b + cn_c) + (3 - l) \sum_{1 \leq a < b \leq l} \phi (an_a + bn_b) + \left( \frac{l^2 - 5l + 6}{2} \right) \sum_{a=0}^{l-1} \phi (n_{a+1})
\]

where \( l \geq 4 \) is an integer, and derive its general solution. The main purpose of this work is to investigate the Hyers-Ulam stability results for the above mentioned functional equation in fuzzy normed spaces by means of direct and fixed point approaches.


1. INTRODUCTION

Function \( f : X \to Y \) between real vector spaces is called a cubic Abic function if

\[
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \quad \forall x, y \in X.
\]

The functional equation (1.1) is called a cubic functional equation. Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [10,11,12,14].

In modelling applied problems only partial information may be known (or) there may be a degree of uncertainty in the parameters used in the model or some measurements may be imprecise. Due to such features, we are tempted to consider the study of functional equations in the fuzzy setting. For the last 40 years, fuzzy theory has become very active area of research and a lot of development has been made in the theory of fuzzy set to find the fuzzy analogues of the classical set theory. This branch finds a wide range of applications in the field of science and engineering.


In this present work, we introduce a new type of finite dimensional cubic functional equation of the form

\[
\phi \left( \sum_{a=1}^{l} an_a \right) = \sum_{1 \leq a < b \leq l} \phi (an_a + bn_b + cn_c) + (3 - l) \sum_{1 \leq a < b \leq l} \phi (an_a + bn_b) + \left( \frac{l^2 - 5l + 6}{2} \right) \sum_{a=0}^{l-1} \phi (n_{a+1})
\]

where \( l \geq 4 \) is an integer, and derive its general solution. The main purpose of this work is to investigate the Hyers-Ulam stability results for the above mentioned functional equation in fuzzy normed spaces by means of direct and fixed point approaches.

2 PRELIMINARIES

We recall some basic facts concerning fuzzy normed spaces and some preliminary results. We use the definition of fuzzy normed spaces given in [1].

Definition 2.1.[1] Let \( X \) be a real vector space. A function \( N : X \times R \to [0, 1] \) is said to be fuzzy norm on \( X \) if for all \( x, y \in X \) and \( s, t \in R \),

\[
\begin{align*}
(N_1) \quad N(x, s) = 0 & \text{ for } s \leq 0 \\
(N_2) \quad x = 0 & \text{ iff } N(x, s) = 1 \\
(N_3) \quad N(sx, t) = N \left( x, \frac{t}{s} \right) & \text{ if } s \neq 0 \\
(N_4) \quad N(x + y, s + t) & \geq \min \{N(x, s), N(y, t)\} \\
(N_5) \quad N(x, s) & \text{ is a non-decreasing function on } R \text{ and } \lim_{s \to 0} N(x, s) = 1.
\end{align*}
\]

For \( x \neq 0, N(x, s) \) is continuous on \( R \).
The pair \((X, N)\) is called fuzzy normed vector space.

**Example 2.2.**[15] Let \((X, \|\|)\) be a normed linear space and \(\alpha, \beta > 0\). Define \(N : X \times R \to [0,1]\) by

\[
N(x,t) = \frac{\alpha t}{\alpha t + \beta \|x\|} \quad t > 0, \quad x \in X;
\]

\[
0, \quad t \leq 0, \quad x \in X.
\]

It is easy to check that \(N\) is fuzzy norm on \(X\).

**Definition 2.3.** Let \((X, N)\) be a fuzzy normed space. A sequence \({x_n}_{n=1}^{\infty}\) in \(X\) is said to be convergent if there exists \(x \in X\) such that

\[
\lim_{n \to \infty} N(x_n - x, t) = 1 \quad \text{for all } t > 0.
\]

In this case, \(x\) is called the limit of the sequence \({x_n}_{n=1}^{\infty}\) and we denote it by

\[
N = \lim_{n \to \infty} x_n = x.
\]

It is easy to see that the limit of the convergent sequence \({x_n}_{n=1}^{\infty}\) in a fuzzy normed space \((X, N)\) is unique (see [15]).

**Definition 2.4.** A sequence \({x_n}_{n=1}^{\infty}\) in a fuzzy normed space \((X, N)\) is called a Cauchy sequence if for each \(\varepsilon > 0\) and each \(t > 0\) there exists an \(M \in \mathbb{N}\) such that for all \(n \geq M\) and all \(p > 0\), we have

\[
N(x_{n+p} - x, t) > 1 - \varepsilon.
\]

The property \((N_4)\) implies that every convergent sequence in a fuzzy normed space is a Cauchy sequence. A fuzzy normed space \((X, N)\) is called a fuzzy Banach space if each Cauchy sequence in \(X\) is convergent.

We say that a mapping \(f : X \to Y\) between fuzzy normed vector spaces \(X\) and \(Y\) is continuous at a point \(x_0 \in X\) if for each sequence \({x_n}\) converging to \(x_0 \in X\), then the sequence \({f(x_n)}\) converges to \(f(x_0)\). If \(f : X \to Y\) is continuous at each \(x \in X\), then \(f : X \to Y\) is said to be continuous on \(X\).

We will use the following fundamental result in fixed point theory.

**Theorem 2.5.**[5] Let \((X, d)\) be a generalized complete metric space and \(\Lambda : X \to X\) be a strictly contractive function with the Lipschitz constant \(L < 1\). Suppose that for a given element \(a \in X\) there exists a nonnegative integer \(k\) such that

\[
d(\Lambda^{k+1}a, \Lambda^k a) < \infty.
\]

Then

(i) The sequence \({\Lambda^n a}_{n=1}^{\infty}\) converges to a fixed point \(b \in X\) of \(\Lambda\);

(ii) \(b\) is the unique fixed point of \(\Lambda\) in the set

\[
Y = \{y \in X : d(\Lambda^k a, y) < \infty\};
\]

(iii) \(d(y, b) \leq \frac{1}{1-L} d(y, \Lambda y)\) for all \(y \in Y\).

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**3. GENERAL SOLUTION FOR THE FUNCTIONAL EQUATION (1.2)**

In this segment, we achieve the general solution of the functional equation (1.2).

**Theorem 3.1.** If a mapping \(\phi : A \to B\) satisfies the functional equation (1.2), then the function \(\phi : A \to B\) satisfies the functional equation (1.1).

**Proof.** Assume that \(\phi : A \to B\) satisfies the functional equation (1.2). Substituting \((n_1, n_2, \ldots, n_l)\) by \((0, 0, \ldots, 0)\) in (1.2), we receive \(\phi(0) = 0\). Replacing \((n_1, n_2, \ldots, n_l)\) by \((n, 0, 0, \ldots, 0)\) in (1.2), we get \(\phi(-n) = -\phi(n)\) for all \(n \in A\).

Hence \(\phi\) is odd function. Again replacing \((n_1, n_2, \ldots, n_l)\) by \(y \left(\frac{n}{2}, 0, \ldots, 0\right)\) in (1.2), we have

\[
\phi\left(2n\right) = 2^3 \phi(n)
\]

(3.1)

for all \(n \in A\). Now, letting \(n\) by \(2n\) in (3.1), we get

\[
\phi\left(4n\right) = 4^3 \phi(n)
\]

(3.2)

for all \(n \in A\). In general for any positive integer \(a\), we obtain

\[
\phi\left(an\right) = a^3 \phi(n)
\]

(3.3)

for all \(n \in A\). Setting \((n_1, n_2, \ldots, n_l)\) by \(\left(x, -\frac{x}{2}, \frac{y}{3}, \frac{y}{4}, 0, \ldots, 0\right)\) in (1.2) and using (3.1), we receive

\[
3\phi(x+y) = -6\phi(x) + 3\phi(y) + \phi(2x+y) + \phi(x-y)
\]

(3.4)

for all \(x, y \in A\). Replacing \(y\) by \(-y\) in (3.4), we obtain

\[
3\phi(x-y) = -6\phi(x) - 3\phi(y) + \phi(2x-y) + \phi(x+y)
\]

(3.5)

for all \(x, y \in A\). Adding (3.4) and (3.5), we get our desired result (1.1).

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**4 STABILITY RESULTS FOR THE FUNCTIONAL EQUATION (1.2): DIRECT METHOD**

In the rest of this paper, we take \(A(B, P)\) and \((Z, Q)\) are linear space, fuzzy Banach space and fuzzy normed space, respectively. For notational convenience, we use the following abbreviation for a given mapping \(\phi : A \to B\) by
\[
D\phi(n_1, n_2, ..., n_l) = \phi\left(\sum_{a=1}^{l}an_a\right)
- \sum_{1 \leq a < b < c \leq l}^{\phi}(an_a + bn_b + cn_c)
- (3 - l)\sum_{1 \leq a < b \leq l}^{\phi}(an_a + bn_b)
- \left(\frac{l^2 - 5l + 6}{2}\right)\frac{l-1}{2}\sum_{a=0}^{l-1}(a + 1)^3\phi(n_{a+1})
\]
for all \(n_1, n_2, ..., n_l \in A\). In this segment, we examine a fuzzy version of the Hyers-Ulam stability for the functional equation (1.2) in fuzzy normed spaces by means of direct method.

Theorem 4.1. Let \(u \in \{-1, 1\}\) be fixed, also consider
\[
\chi : A^l \rightarrow Z
\]
be a mapping such that for some \(\zeta > 0\) with
\[
\left(\frac{\zeta}{2^3}\right) < 1
\]
\[
Q\left(\chi(2^nu, 2^nu, 0, ..., 0)\right) \geq Q\left(\zeta u\chi(n, n, 0, ..., 0)\right)\tag{4.1}
\]
including
\[
\lim_{m \rightarrow \infty} Q\left(\chi(2^mu_{n_1}, 2^mu_{n_2}, ..., 2^mu_{n_l}), 2^3u\varepsilon\right) = 1
\]
for all \(n_1, n_2, ..., n_l \in A\) and \(\varepsilon > 0\). Suppose an odd mapping
\(\phi : A \rightarrow B\) with \(\phi(0) = 0\) fulfills the inequality
\[
P\left(D\phi(n_1, n_2, ..., n_l)\right) \geq Q\left(\chi(n_1, n_2, ..., n_l)\right)\varepsilon\tag{4.2}
\]
for all \(n_1, n_2, ..., n_l \in A\) and \(\varepsilon > 0\). Then the limit
\[
C(n) = P - \lim_{m \rightarrow \infty} P\left(\phi(2^mu_{n})\right)
\]
exists for all \(n \in A\) and the mapping \(C : A \rightarrow B\) is a unique cu

\[
P\left(\phi(n) - C(n)\right) \geq Q\left(\chi(n, n, 0, ..., 0)\left(\frac{\varepsilon^2}{3} - \zeta^2\right)\right)\tag{4.3}
\]
for all \(n \in A\) and \(\varepsilon > 0\).

Proof. Initially we consider \(u = 1\). Substituting \((n_1, n_2, ..., n_l)\) through \((n, n, 0, ..., 0)\) in (4.2), we reach
\[
P\left(\left(\frac{l^2 - 5l + 6}{2}\right)\frac{l-1}{2}\sum_{a=0}^{l-1}(a + 1)^3\phi(n_{a+1})\right)
\geq Q\left(\chi(n, n, 0, ..., 0)\right) \forall n \in A, \varepsilon > 0.
\]
Then we have
\[
P\left(\phi(2n), \frac{\varepsilon}{8(5l - 1 + 6)}\right)
\geq Q\left(\chi(n, n, 0, ..., 0)\right) \forall n \in A, \varepsilon > 0.
\](4.4)
Exchanging \(n\) through \(2^m n\) in (4.4), we acquire
\[
P\left(\phi(2^{m+1}n), \frac{\varepsilon}{8(5l - 1 + 6)}\right)
\geq Q\left(\chi(2^m n, 2^m n, 0, ..., 0)\right) \forall n \in A, \varepsilon > 0.
\]
Utilizing (4.1) and \((N_3)\) in the above inequality, we reach
\[
P\left(\phi(2^{m+1}n), \phi(2^m n), \frac{\varepsilon}{2^{m+1} \frac{\varepsilon}{8(5l - 1 + 6)}}\right)
\geq Q\left(\chi(n, n, 0, ..., 0)\right) \forall n \in A, \varepsilon > 0.
\]
Switching \(\varepsilon\) through \(\zeta^m \varepsilon\) in the last inequality, we acquire
\[
P\left(\phi(2^{m+1}n), \phi(2^m n) - \frac{\zeta^m}{2^{m+1}} \frac{\varepsilon}{(5l - 1 + 6)}\right)
\geq Q\left(\chi(n, n, 0, ..., 0)\right) \forall n \in A, \varepsilon > 0.
\]
From (4.5), we obtain
\[
P\left(\phi(2^m n), \phi(n), \frac{m-1}{2^m} \frac{\varepsilon}{2^{m+1} \frac{\varepsilon}{8(5l - 1 + 6)}}\right)
= P\left(\phi(2^m n), \phi(n), \frac{\sum_{a=0}^{m-1} \frac{\varepsilon}{2^{3(a+1)}}}{\frac{\varepsilon}{2^{m+1} \frac{\varepsilon}{8(5l - 1 + 6)}}}\right)
\geq 0 \sum_{a=0}^{m-1} \frac{\varepsilon}{2^{3(a+1)}} \frac{\varepsilon}{2^{m+1} \frac{\varepsilon}{8(5l - 1 + 6)}}\right)
\geq Q\left(\chi(n, n, 0, ..., 0)\right) \forall n \in A, \varepsilon > 0.
\]
for all \(n \in A, \varepsilon > 0\) and \(m \in \mathbb{N}\). Substituting \(n\) by \(2^s n\) in (4.6) and utilizing (1.1) with \((N_3)\), we acquire
\[
P\left(\phi(2^{m+s} n), \phi(2^s n), \frac{m-1}{2^m} \frac{\varepsilon}{2^{m+1} \frac{\varepsilon}{8(5l - 1 + 6)}}\right)
\geq Q\left(\chi(2^s n, 2^s n, 0, ..., 0)\right) \forall n \in A, \varepsilon > 0.
\[
\begin{align*}
&\geq Q\left(\chi(n,n,0,...,0), \frac{\varepsilon}{s^{a}}\right), \\
&\text{and so} \\
&P\left(\frac{\phi\left(2^{m+s}n\right)}{2^{3m(s+1)}} - \frac{\phi\left(2^{s}n\right)}{2^{3s}}, m+s-1, \sum_{a=0}^{m-1} \frac{\varepsilon}{a} \frac{3(a+1)}{2^{3(a+1)}} (1^2 - 5l + 6)\right) \\
&\geq Q\left(\chi(n,n,0,...,0), \varepsilon\right)
\end{align*}
\]
for all \( n \in A, \varepsilon > 0 \) and all integer \( s, m \geq 0 \). Exchanging \( \varepsilon \) through \( \sum_{a=s}^{m+s-1} \frac{\varepsilon}{a} \frac{3(a+1)}{2^{3(a+1)}} (1^2 - 5l + 6) \) in the last inequality, we obtain

\[
\sum_{a=s}^{m+s-1} \frac{\varepsilon}{a} \frac{3(a+1)}{2^{3(a+1)}} (1^2 - 5l + 6)
\]

for all \( n \in A, \varepsilon > 0 \) and all integer \( s, m \geq 0 \). Since

\[
\sum_{a=0}^{\infty} \frac{\varepsilon}{8(1^2 - 5l + 6)} < \infty,
\]

it follows from (4.7) and \((N_1)\) that

\[
\begin{align*}
&\left\{\phi\left(2^{m}n\right)\right\}_{m=1}^{\infty} \\
&\text{is a Cauchy sequence in } (B,P) \text{ for each} \\
&n \in A. \text{ Since } (B,P) \text{ is a fuzzy Banach space, this sequence converges to some point } C(n) \in B \text{ for each } n \in A. \text{ So one can define the mapping } C : A \to B \text{ by} \\
&C(n) := P - \lim_{m \to \infty} \frac{\phi\left(2^{m}n\right)}{2^{3m}}, \forall n \in A.
\end{align*}
\]

Since \( \phi \) is odd, C is odd. Letting \( s = 0 \) in (4.7), we obtain

\[
P\left(\frac{\phi\left(2^{m}n\right)}{2^{3m}} - \phi(n), \varepsilon\right)
\]

\[
\geq Q\left(\chi(n,n,0,...,0), \frac{\varepsilon}{m-1} \frac{3(a+1)}{2^{3(a+1)}} (1^2 - 5l + 6)\right)
\]

for all \( n \in A, \varepsilon > 0 \) and all integer \( m \geq 1 \). Then

\[
P(\phi(n) - C(n), \varepsilon + \alpha)
\]

\[
\geq \min \left\{ P\left(\frac{\phi\left(2^{m}n\right)}{2^{3m}} - \phi(n), \varepsilon\right), P\left(\frac{\phi\left(2^{m}n\right)}{2^{3m}} - C(n), \alpha\right)\right\}
\]

for all \( n \in A, \varepsilon, \alpha > 0 \) and all integer \( m \geq 1 \). Hence taking the limit as \( m \to \infty \) in the last inequality and using \((N_6)\), we get

\[
P(\phi(n) - C(n), \varepsilon + \alpha)
\]

\[
\geq Q\left(\chi(n,n,0,...,0), \frac{\varepsilon}{2^{3m}(1^2 - 5l + 6)}\right)
\]

\[
\forall n \in A, \varepsilon, \alpha > 0.
\]

Taking the limit as \( \varepsilon \to 0 \), we get (4.3). Now, we assert that C is cubic. It is clear that

\[
P(DC(n_1,n_2,...,n_l), 2\varepsilon)
\]

\[
\geq \min \left\{ P\left(DC(n_1,n_2,...,n_l) - \frac{1}{2^{3m}} D\phi(2^{m}n_1,2^{m}n_2,...,2^{m}n_l), \varepsilon\right), P\left(\frac{1}{2^{3m}} D\phi(2^{m}n_1,2^{m}n_2,...,2^{m}n_l), \varepsilon\right)\right\}
\]

By (4.2)

\[
\geq \min \left\{ P\left(DC(n_1,n_2,...,n_l) - \frac{1}{2^{3m}} D\phi(2^{m}n_1,2^{m}n_2,...,2^{m}n_l), \varepsilon\right), P\left(\frac{1}{2^{3m}} D\phi(2^{m}n_1,2^{m}n_2,...,2^{m}n_l), \varepsilon\right)\right\}
\]

\[
Q\left(\chi(2^{m}n_1,2^{m}n_2,...,2^{m}n_l), \varepsilon\right), \forall n \in A, \varepsilon > 0.
\]

Since

\[
\lim_{m \to \infty} P\left(DC(n_1,n_2,...,n_l) - \frac{1}{2^{3m}} D\phi(2^{m}n_1,2^{m}n_2,...,2^{m}n_l), \varepsilon\right) = 1,
\]

\[
\lim_{m \to \infty} C\left(\chi(2^{m}n_1,2^{m}n_2,...,2^{m}n_l), 2^{3m}\varepsilon\right) = 1.
\]

We infer \( P(DC(n_1,n_2,...,n_l), 2\varepsilon) = 1 \) for all \( n_1,n_2,...,n_l \in A \) and all \( \varepsilon > 0 \). Then \((N_2)\) implies \( DC(n_1,n_2,...,n_l) = 0 \) for all \( n_1,n_2,...,n_l \in A \). Therefore C : A \to B is cubic by Theorem 3.1. To show the uniqueness of C, let \( D : A \to B \) be another cubic mapping fulfilling (4.3). Since \( C(2^{m}n) = 2^{3m}C(n) \) and

\[
D(2^{m}n) = 2^{3m}D(n)
\]

for all \( n \in A \) and \( m \in \mathbb{N} \), it follows from (4.3) that

\[
P(C(n) - D(n), \varepsilon) = P\left(\frac{C(2^{m}n) - D(2^{m}n)}{2^{3m}}, \varepsilon\right)
\]

\[
\geq \min \left\{ P\left(\frac{C(2^{m}n)}{2^{3m}} - \frac{D(2^{m}n)}{2^{3m}}, \frac{\varepsilon}{2}\right), P\left(\frac{D(2^{m}n)}{2^{3m}} - \frac{D(2^{m}n)}{2^{3m}}, \frac{\varepsilon}{2}\right)\right\}
\]

\[
\text{for all } n \in A, \varepsilon > 0 \text{ and all integer } m \geq 1. \text{ Then}
\]
\[ P\left( \phi(n) - C(n), \varepsilon \right) \geq Q\left( \frac{L^{1-a}}{1-L} \sigma(n), \varepsilon \right), \quad \forall n \in A, \varepsilon > 0. \]  

\[ (5.4) \]

Proof. Let \( \varsigma \) be the generalized metric on \( \mathcal{Y} \):
\[ \varsigma(v, w) = \inf \left\{ r \in (0, \infty) : P(v) - w(n), \varepsilon \right\} \geq Q(r \sigma(n), \varepsilon), \quad n \in A, \varepsilon > 0 \}, \]

and we take, as usual, \( \inf \phi = +\infty. \) A similar argument provided in [9, Lemma 2.1] shows that \( \mathcal{Y}, \varsigma \) is a complete generalized metric space. Define \( \Phi_a : \mathcal{Y} \to \mathcal{Y} \) by
\[ \Phi_a(v) = \frac{1}{\gamma_a} \psi(a)n \] for all \( n \in A. \) Let \( v, w \in \mathcal{Y} \) be given such that \( \varsigma(v, w) \leq \alpha. \) Then
\[ P(v) - w(n), \varepsilon \geq Q(\alpha \sigma(n), \varepsilon), \quad \forall n \in A, \varepsilon > 0, \]

where
\[ P(\Phi_a(v) - \Phi_a(w), \varepsilon) \geq Q\left( \frac{\alpha}{\gamma_a} \sigma(a)n, \varepsilon \right), \quad \forall n \in A, \varepsilon > 0. \]

It follows from that (5.3) that
\[ P(\Phi_a(v) - \Phi_a(w), \varepsilon) \geq Q(\alpha L \sigma(n), \varepsilon), \quad \forall n \in A, \varepsilon > 0. \]

Hence, we have \( \varsigma(\Phi_a(v), \Phi_a(w)) \leq \alpha L \). This shows
\[ \varsigma(\Phi_a(v), \Phi_a(w)) \leq \varsigma(v, w), \quad i.e., \Phi_a \text{ is strictly contrapplicative mapping on } \mathcal{Y} \text{ with the Lipschitz constant } L. \]

Substituting \( n, n_2, \ldots, n_l \) by \( (n, n, 0, \ldots, 0) \) in (5.2) and utilizing \( (N_3) \), we get
\[ P \left( \frac{(2n)}{2^3} - \phi(n), \varepsilon \right) \geq Q \left( \frac{\chi(n, n, 0, \ldots, 0)}{2^3 (l^2 - 5l + 6)}, \varepsilon \right), \quad \forall n \in A, \varepsilon > 0. \]  

\[ (5.5) \]

Using (5.3) when \( \alpha = 0 \), it follows from (5.5) that
\[ P \left( \frac{(2n)}{2^3} - \phi(n), \varepsilon \right) \geq Q(L \sigma(n), \varepsilon), \quad \forall n \in A, \varepsilon > 0. \]

Therefore
\[ \varsigma(\Phi_0 \phi, \phi) \leq L = L^{1-a}. \]  

\[ (5.6) \]

Exchanging \( n \) through \( \frac{n}{2} \) in (5.5), we obtain
\[ P \left( \frac{(2n)}{2^3} - \phi(n), \varepsilon \right) \geq Q \left( \frac{(n)}{2^3} \sigma(n), \varepsilon \right), \quad \forall n \in A, \varepsilon > 0. \]

Therefore
\[ \varsigma(\Phi_1 \phi, \phi) \leq L = L^{1-a}. \]  

\[ (5.7) \]

Then from (5.6) and (5.7), we conclude
\[ \varsigma(\Phi_0 \phi, \phi) \leq L^{1-a} < +\infty. \]

Now from the fixed point alternative Theorem 2.5,
it follows that there exists a fixed point C of $\Phi_a$ in $\mathcal{Y}$ such that

(i) $\Phi_a C = C$ and $\lim_{m \to \infty} \zeta (\Phi_a^m \phi, C) = 0$;

(ii) $C$ is the unique fixed point of $\Phi$ in the set

$\zeta = \{v \in \mathcal{Y} : d (\phi, v) < \infty\}$;

(iii) $\zeta (\phi, C) \leq \frac{1}{1-L} \zeta (\phi, \Phi_a \phi)$.

Letting $\zeta (\Phi_a^m \phi, C) = \alpha_m$, we get

$P (\Phi_a^m \phi (n) - C(n), \varepsilon) \geq Q (\alpha_m \sigma (n), \varepsilon)$ for all $n \in A$ and all $\varepsilon > 0$. Since $\lim_{m \to \infty} \alpha_m = 0$, we infer

$C(n) = P - \lim_{m \to \infty} \frac{\phi (\psi_a^m n)}{\psi_a^m}, \quad \forall n \in A.$

Switching $(n_1, n_2, \ldots, n_l)$ by $(\psi_a^m n_1, \psi_a^m n_2, \ldots, \psi_a^m n_l)$ in (5.2), we obtain

$P \left( \frac{1}{\psi_a^m} D \phi (\psi_a^m n_1, \psi_a^m n_2, \ldots, \psi_a^m n_l), \varepsilon \right) \geq Q (\chi (\psi_a^m n_1, \psi_a^m n_2, \ldots, \psi_a^m n_l), \psi_a^m \varepsilon),$

for all $\varepsilon > 0$ and all $n_1, n_2, \ldots, n_l \in A$. Using the same argument as in the proof of Theorem 4.1, we can prove the function $C : A \to B$ is cubic. Since $\zeta (\phi, \Phi_a) \leq L^{-a}$, it follows from (iii)

$\zeta (\phi, C) \leq \frac{L^{-a}}{1-L}$ which means (5.4). To prove the uniqueness of $C$, let $D : A \to B$ be another cubic mapping fulfilling (5.4). Since $C (2^m n) = 2^m C(n)$ and $D (2^m n) = 2^m D(n)$ for all $n \in P$ and all $m \in \mathbb{N}$, we have

$P (C(n) - D(n), \varepsilon) = P \left( C (2^m n) - D (2^m n), \frac{2^m \varepsilon}{2^m} \right),$

$\geq \min \left\{ P \left( \frac{C (2^m n)}{2^m} - \frac{\phi (2^m n)}{2^m}, \frac{\varepsilon}{2} \right), P \left( \frac{\phi (2^m n)}{2^m} - \frac{D (2^m n)}{2^m}, \frac{\varepsilon}{2} \right) \right\}$

$\geq Q \left( \frac{L^{-a}}{1-L} \sigma (2^m n), \frac{2^m \varepsilon}{2} \right).$

By (5.1), we have

$\lim_{m \to \infty} Q \left( \frac{L^{-a}}{1-L} \sigma (2^m n), \frac{2^m \varepsilon}{2} \right) = 1.$

Consequently, $P (C(n) - D(n), \varepsilon) = 1$ for all $n \in A$ and all $\varepsilon > 0$. So $C(n) = D(n)$ for all $n \in A$, which ends the proof.

The upcoming corollaries are instantaneous outcome of Theorem 4.1 and 5.1, regarding the stability for the equation (1.2). In the following results, we assume that $A$, $(B, \mathcal{P})$ and $(\mathcal{L}, \mathcal{Q})$ are a linear space, a fuzzy Banach space and a fuzzy normed space, respectively.

Corollary 5.2. Suppose an odd function $\phi : A \to B$ fulfills $\phi (0) = 0$ and the inequality

$P \left( D \phi (n_1, n_2, \ldots, n_l), \varepsilon \right) \geq Q \left( \varepsilon + \theta \sum_{a=1}^{L} \|n_a\|^q, \varepsilon \right),$ 

for all $n_1, n_2, \ldots, n_l \in A$ and all $\varepsilon > 0$, where $\theta, \gamma, q$ are real co constants with $\gamma \in (0, 3)$. Then there exists a unique cubic mapping $C : A \to B$ such that

$P \left( \phi (n) - C(n), \varepsilon \right) \geq Q \left( \theta \varepsilon, \varepsilon \right), \quad \forall n \in A, \varepsilon > 0.$

Corollary 5.3. Suppose an odd function $\phi : A \to B$ fulfills $\phi (0) = 0$ and the inequality

$P \left( D \phi (n_1, n_2, \ldots, n_l), \varepsilon \right) \geq Q \left( \gamma \sum_{a=1}^{L} \|n_a\|^{pq} + \theta \sum_{a=1}^{L} \|n_a\|^q, \varepsilon \right),$ 

for all $n_1, n_2, \ldots, n_l \in A$ and all $\varepsilon > 0$, where $\alpha, \theta, p$ and $q$ are real co constants with $\gamma \in (0, 3)$. Then there exists a unique cubic mapping $C : A \to B$ such that

$P \left( \phi (n) - C(n), \varepsilon \right) \geq Q \left( \theta \sum_{a=1}^{L} \|n_a\|^q, \varepsilon \right), \quad \forall n \in A, \varepsilon > 0.$

Corollary 5.4. Suppose an odd function $\phi : A \to B$ fulfills $\phi (0) = 0$ and the inequality

$P \left( D \phi (n_1, n_2, \ldots, n_l), \varepsilon \right) \geq Q \left( \theta \sum_{a=1}^{L} \|n_a\|^q, \varepsilon \right),$ 

for all $n_1, n_2, \ldots, n_l \in A$ and all $\varepsilon > 0$, where $\alpha$ and $\theta$ are real co constants with $0 < \gamma \neq 3$. Then $\phi$ is cubic.

REFERENCES


