An Exploration Of The Generalized Cantor Set

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Abstract: In this paper, we study the prototype of fractal of the classical Cantor middle-third set which consists of points along a line segment, and possesses a number of fascinating properties. We discuss the construction and the self-similarity of the Cantor set. We also generalized the construction of this set and find its fractal dimension.

Keywords: Cantor set, Dimension, Fractal, Generalization, Self-similar.

1 INTRODUCTION

A fractal is an object or quantity that displays self-similarity. The object needs not exhibit exactly the same structure but the same type of structures must appear on all scales. Fractals were first discussed by Mandelbrot[1] in the 1970, but the idea was identified as early as 1925. Fractals have been investigated for their visual qualities as art, their relationship to explain natural processes, music, medicine, and in Mathematics. The Cantor set is a good example of an elementary fractal. The object first used to demonstrate fractal dimensions is actually the Cantor set. The process of generating this fractal is very simple. The set is generated by the iteration of a single operation on a line of unit length. In each iteration, the middle third from each lines segment of the previous set is simply removed. As the number of iterations increases, the number of separate line segments tends to infinity while the length of each segment approaches zero. Under magnification, its structure is essentially indistinguishable from the whole, making it self-similar[2]. We studied the dimension of the Cantor set that its magnification factor is three, or the fractal is self-similar if magnified three times. Then we noticed that the line segments decompose into two smaller units. We also studied the fractal dimensions of the generalized Cantor sets. We explore the generalization of the Cantor set with fractal dimension and demonstrate the diagram of self-similarity of this generalized Cantor set in three phases. Although we used the typical middle-thirds or ternary rule[3] in the construction of the Cantor set, we generalized this one-dimensional idea to any length other than $\frac{1}{3}$, excluding the degenerate cases of 0 and 1.

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2 BASIC DEFINITIONS

Definition 2.1: A set S is self-similar if it can be divided into N congruent subsets, each of which when magnified by a constant factor M yields the entire set S.

Definition 2.2: Let S be a compact set and $N(S,r)$ be the minimum number of balls of radius *r* needed to cover S. Then the fractal dimension[4] of S is defined as

$$\dim S = \lim_{r\to 0} \frac{\log N(S,r)}{\log(1/r)}.$$

In the line ($\mathbb{R}^1$), a ball is simply a closed interval.

3 CONSTRUCTION OF CANTOR SET

3.1. Cantor middle-1/3 set

To construct this set (denoted by C_3), we begin with the interval $[0,1]$ and remove the open set $\left(\frac{1}{3},\frac{2}{3}\right)$ from the closed interval $[0,1]$. The set of points that remain after this first step will be called K_1, that is, $K_1 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$. In the second step, we remove the middle thirds of the two segments of K_1, that is, remove $\left(\frac{1}{9},\frac{2}{9}\right) \cup \left(\frac{7}{9},\frac{8}{9}\right)$ and set $K_2 = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{3}{9}\right] \cup \left[\frac{6}{9},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$ be what remains after the first two steps. Delete the middle thirds of the four remaining segments of K_2 to get K_3. Repeating this process, the limiting set $C_3 = K_\infty$ is called the Cantor middle 1/3 set.

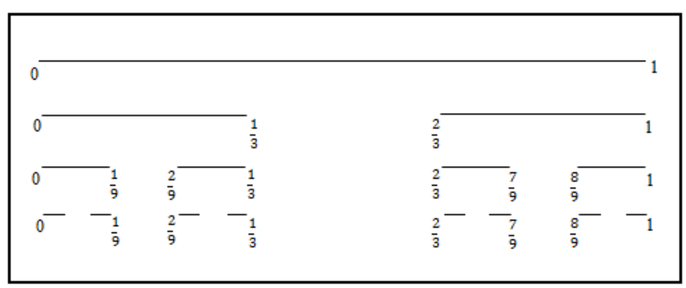

Figure 1. The Cantor set, produced by the iterated process of removing the middle third from the previous segments. The Cantor set has zero length, and non-integer dimension.

3.2. Fractal dimension of the Cantor middle-1/3 set

The set C_3 is contained in K_n for each n. Just as K_1 consists of 2 intervals of length $\frac{1}{3}$, and K_2 consists of 2^2 intervals of length $\frac{1}{3^2}$ and K_3 consists of 2^3 intervals of length $\frac{1}{3^3}$. In general, K_n

consists of 2^n intervals, each of length $\frac{1}{3^n}$. Further, we know that C contains the endpoints of all 2^n intervals, and that each pair of endpoints lie 3^{-n} apart. Therefore, the smallest number of 3^{-n}-boxes covering C_3 is $N(C_3, 3^{-n}) = 2^n$. Compute the dimension of the Cantor middle-1/3 set C_3 as

$$\dim(C_3) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln 3^n} = \lim_{n\to\infty} \frac{n\ln 2}{n\ln 3} = \frac{\ln 2}{\ln 3} = 0.6309.$$

3.3. Self-similarity of the Cantor middle-1/3 set

One of the most important properties of a fractal is known as self-similarity[5]. Roughly speaking, self-similarity means if we examine small portions of the set under a microscope, the image we see resembles our original set. To see this let us look closely at C_3. Note that C_3 is decomposed into two distinct subsets, the portion of C_3 in $[0, 1/3]$ and the portion in $[2/3, 1]$. If we examine each of these pieces, we see that they resemble the original Cantor set C_3. Indeed, each is obtained by removing middle-thirds of intervals. The only difference is the original interval is smaller by a factor of $1/3$. Thus, if we magnify each of these portions of C_3 by a factor of 3, we obtain the original set. More precisely, to magnify these portions of C_3, we use an affine transformation. Let $L(x) = 3x$. If we apply L to the portion of C_3 in $[0,1/3]$, we see that L maps this portion onto the entire Cantor set. Indeed, L maps $[1/9, 2/9]$ to $[1/3, 2/3]$, $[1/27, 2/27]$ to $[1/9, 2/9]$, and so forth (Fig. 2). Each of the gaps in the portion of C_3 in $[0,1/3]$ is taken by L to a gap in C_3. That is, the "microscope" we use to magnify $C_3 \cap [0,1/3]$ is just the affine transformation $L(x) = 3x$.

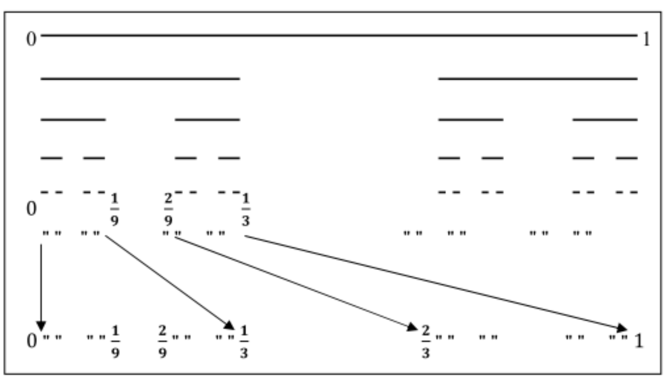

Figure 2. Self-similarity of the Cantor middle-thirds set

To magnify the other half of C_3, namely $C_3 \cap [2/3, 1]$, we use another affine transformation, $R(x) = 3x - 2$. Note that $R(2/3) = 0$ and $R(1) = 1$ so R takes $[2/3, 1]$ linearly onto $[0, 1]$. As with L, R takes gaps in $C_3 \cap [2/3, 1]$ to gaps in C_3, so R again magnifies a small portion of C_3 to give the entire set. Using more powerful "microscope", we may magnify arbitrarily small portions of C_3 to give the entire set. For example, the portion of C_3 in $[0, 1/3]$ itself decomposes into two self-similar pieces: one in $[0, 1/9]$ and one in $[2/9, 1/3]$. We may magnify the left portion via $L_2(x) = 9x$ to yield C_3 and the right portion via $R_2(x) = 9x - 2$. Note that R_2 maps $[2/9, 1/3]$ onto $[0, 1]$ linearly as required. Note also that at the nth stage of the construction of C_3, we have 2^n small copies of C_3, each of which may be magnified by a factor of 3^n to yield the entire Cantor set.

4 GENERALIZED CANTOR MIDDLE-ω SETS $(0 < \omega < 1)$

4.1. Cantor middle-1/5 set

To build this set (denoted by C_5) we can follow the same procedure as construction of the middle-third Cantor's set. First we delete the open interval covering its middle fifth from the unit interval $I = [0,1]$. That is, we remove the open interval $(\frac{2}{5}, \frac{3}{5})$. The set of points that remain after this step will be called K_1. That is, $K_1 = \left[0, \frac{2}{5}\right] \cup \left[\frac{3}{5}, 1\right]$. In the second step, we remove the middle fifth portion of each of the 2 closed intervals of K_1 and set

$$K_2 = \left[0, \frac{4}{25}\right] \cup \left[\frac{6}{25}, \frac{2}{5}\right] \cup \left[\frac{3}{5}, \frac{19}{25}\right] \cup \left[\frac{21}{25}, 1\right].$$

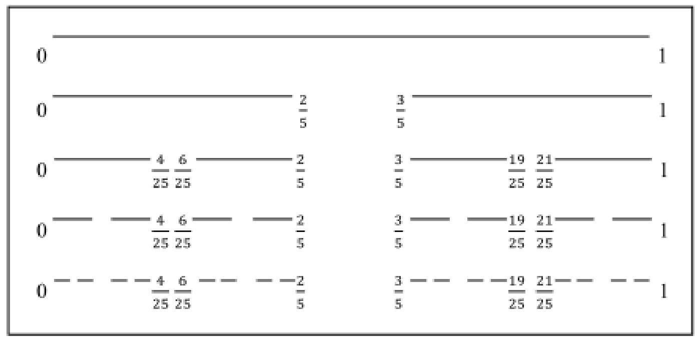

Figure 3. Construction of the Cantor middle-$1/5$ set.

Again, we remove the middle fifth portion of each of the 2^2 closed intervals of K_2 to get K_3. Repeating this process, the limiting set $C_5 = K_\infty$ is called the Cantor middle-$1/5$ set. The set C_5 is the set of points that belongs to all of the K_n.

4.2. Fractal dimension of the Cantor middle-1/5 set

The set C_5 is contained in K_n for each n. Just as K_1 consists of 2 intervals of length 2/5, and K_2 consists of 2^2 intervals of length $\frac{2^2}{5^2}$ and K_3 consists of 2^3 intervals of length $\frac{2^3}{5^3}$. In general, K_n consists of 2^n intervals, each of length $\left(\frac{2}{5}\right)^n$. Further, we know that C_5 contains the endpoints of all 2^n intervals, and that each pair of endpoints lie $\left(\frac{2}{5}\right)^n$ apart. Therefore, the smallest number of $\left(\frac{2}{5}\right)^n$-boxes covering C_5 is $N(C_5, 2^n 5^{-n}) = 2^n$. We compute the fractal dimension of the Cantor middle-1/5 set C_5 as

$$\dim(C_5) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln(5/2)^n} = \frac{\ln 2}{\ln 5 - \ln 2}.$$

4.3. Cantor middle-1/7 set

To build this set (denoted by C_7) we first delete the open interval covering its middle 7th from the unit interval $I = [0,1]$. That is, we remove the open interval $(\frac{3}{7}, \frac{4}{7})$. The set of points that remain after this step will be called K_1, that is, $K_1 = \left[0, \frac{3}{7}\right] \cup \left[\frac{4}{7}, 1\right]$. In the second step, we remove the middle 7th portion of each of the 2 closed intervals of K_1 and set

$$K_2 = \left[0, \frac{9}{49}\right] \cup \left[\frac{12}{49}, \frac{21}{49}\right] \cup \left[\frac{4}{7}, \frac{37}{49}\right] \cup \left[\frac{40}{49}, 1\right].$$

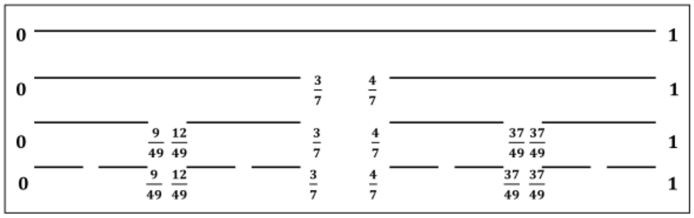

Figure 4. Construction of the middle-*1/7* Cantor set.

Again, we remove the middle 7th portion of each of the 2^2 closed intervals of K_2 to get K_3. Repeating this process, the limiting set $C_7 = \delta_\infty$ is called the Cantor middle-1/7 set. The set C_7 is the set of points that belongs to all of the K_n.

4.4. Fractal dimension of the Cantor middle-1/7 set

In this case, K_1 consists of 2 intervals of length 3/7, and K_2 consists of 2^2intervals of length $\frac{3^2}{7^2}$ and K_3 consists of 2^3 intervals of length $\frac{3^3}{7^3}$. In general, K_n consists of 2^n intervals, each of length $\left(\frac{3}{7}\right)^n$. Further, we know that C_7 contains the endpoints of all 2^n intervals, and that each pair of endpoints lie $\left(\frac{3}{7}\right)^n$apart. Therefore, the smallest number of $\left(\frac{3}{7}\right)^n$-boxes covering C_5 is $N(3^n 7^{-n}) = 2^n$. We compute the fractal dimension of the Cantor middle-1/7 set C_7 as

$$\dim(C_7) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln(7/3)^n} = \frac{\ln 2}{\ln 7 - \ln 3}.$$

5 GENERALIZATION

In similar way, we can construct the middle-$(2m-1)$th Cantor's set, C_{2m-1} where $m \geq 2$ and then compute the fractal dimension of the Cantor middle-$1/(2m-1)$ setC_{2m-1}.
In this case, K_n consists of 2^n intervals, each of length $\frac{(m-1)^n}{(2m-1)^n}$ and C_{2m-1} contains the endpoints of all 2^n intervals, and each pair of endpoints lie $\frac{(m-1)^n}{(2m-1)^n}$ apart. Therefore, the smallest number of $\frac{(m-1)^n}{(2m-1)^n}$ -boxes covering C_{2m-1} is $N(C_{2m-1},(m-1)^n(2m-1)^{-n}) = 2^n$. We compute the fractal dimension of the Cantormiddle-$1/(2m-1)$ set C_{2m-1}as

$$\dim(C_{2m-1}) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln((2m-1)^n/(m-1)^n)} = \frac{\ln 2}{\ln(2m-1) - \ln(m-1)},$$

where $m \geq 2$.

Comment: We can generalize the Cantor's set by setting $\omega = \frac{1}{2m-1}$ as the Cantor middle-ω set and the fractal dimension of the Cantor middle-ω set is $\frac{\ln 2}{\ln 2 - \ln(1-\omega)}$, where $\omega = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots \dots \dots$.

Lemma 5.1[6]**:** If K_n is as defined above in constructionof the Cantor middle-ω set, then there are 2^n closed intervals in K_n and the length of each closed interval is $\left(\frac{1-\omega}{2}\right)^n$. Also, the combined length of the intervals in K_n is $(1-\omega)^n$, which approaches 0 as n approaches ∞. We are now ready to say that Cantor middle-ω sets are appropriately named.

Proposition 5.2. The Cantor middle-ω set is a Cantor set, where $0 < \omega < 1$.

Proof: The proof can be found in [7].

6 SPECIAL CASES OF GENERALIZED CANTOR MIDDLE-*ω* SET

6.1. Removing the alternative segments (obviously the number of segments is odd).

(a) When $\omega = 2/5$, construction of the Cantor middle-2/5 set:
In this case, we delete the middle second and fourth of 5 portions of the unit interval $I = [0,1]$. Then we have $K_1 = \left[0, \frac{1}{5}\right] \cup \left[\frac{2}{5}, \frac{3}{5}\right] \cup [\frac{4}{5}, 1]$ and

$$K_2 = \left[0, \frac{1}{25}\right] \cup \left[\frac{2}{25}, \frac{3}{25}\right] \cup \left[\frac{4}{25}, \frac{1}{5}\right] \cup \left[\frac{2}{5}, \frac{11}{25}\right] \cup \left[\frac{12}{25}, \frac{13}{25}\right] \cup \left[\frac{14}{25}, \frac{3}{5}\right] \cup \left[\frac{4}{5}, \frac{21}{25}\right] \cup \left[\frac{22}{25}, \frac{23}{25}\right] \cup [\frac{24}{25}, 1].$$

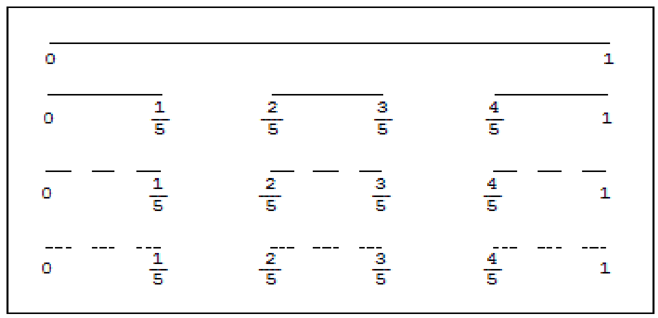

Figure 5. Construction of the middle-*2/5* Cantor set.

Repeating this process, the limiting set $C_5 = K_\infty$ can be called the Cantor middle-2/5 set and K_3 consists of 3^3 intervals of length $\frac{1}{5^3}$. In general, K_n consists of 3^n intervals, each of length $\frac{1}{5^n}$. Thus the dimension of the Cantor middle-2/5 set is

$$\dim(C_5) = \lim_{n\to\infty} \frac{\ln 3^n}{\ln 5^n} = \frac{\ln 3}{\ln 5}.$$

(b) When $\omega = 3/7$, construction of the Cantor middle-3/7 set:
In this case, we remove the middle second, fourth and sixth of 7 portions of the unit interval $I = [0,1]$. Then we have $K_1 = \left[0, \frac{1}{7}\right] \cup \left[\frac{2}{7}, \frac{3}{7}\right] \cup \left[\frac{4}{7}, \frac{5}{7}\right] \cup \left[\frac{6}{7}, 1\right]$ and $K_2 = \left[0, \frac{1}{49}\right] \cup \left[\frac{2}{49}, \frac{3}{49}\right] \cup \dots \cup [\frac{48}{49}, 1]$.

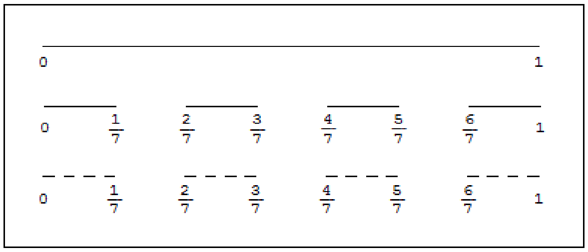

Figure 6. Construction of the middle-3/7 Cantor set.

Repeating this process, the limiting set $C_7 = K_\infty$ can be called the Cantor middle-3/7 set and K_3 consists of 4^3 intervals of length $\frac{1}{7^3}$. In general, K_n consists of 4^n intervals, each of length $\frac{1}{7^n}$. Thus the dimension of the Cantor middle-3/7 set is

$$\dim(C_7) = \lim_{n\to\infty} \frac{\ln 4^n}{\ln 7^n} = \frac{\ln 4}{\ln 7}.$$

(c) Generalization: When $\omega = (m-1)/(2m-1)$ and $m \geq 2$, construction of the Cantor middle- (m-1)/(2m-1) set:

As above K_n consists of m^n intervals, each of length $1/(2m-1)^n$. Thus the dimension of the Cantor middle-$(m-1)/(2m-1)$ set is

$$\dim(C_{2m-1}) = \lim_{n\to\infty} \frac{\ln m^n}{\ln(2m-1)^n} = \frac{\ln m}{\ln(2m-1)}$$

where $m \geq 2$.

6.2. Removing the middle segments (except two end segments).

(a) When $\omega = 2/4$, construction of the Cantor middle-2/4 set: In this case, we delete the middle two of four portions of the unit interval $I = [0,1]$. That is, we remove the open interval $(\frac{1}{4}, \frac{3}{4})$ from the unit interval $I = [0,1]$. Then we have $K_1 = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$ and $K_2 = \left[0, \frac{1}{16}\right] \cup \left[\frac{3}{16}, \frac{1}{4}\right] \cup \left[\frac{3}{4}, \frac{13}{16}\right] \cup \left[\frac{15}{16}, 1\right]$.

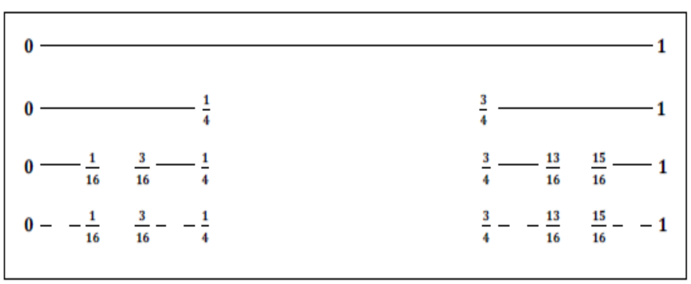

Figure 7. Construction of the middle-$2/4$ Cantor set.

Repeating this process, the limiting set $C_4 = K_\infty$ can be called the Cantor middle-2/4 set and K_3 consists of 2^3 intervals of length $\frac{1}{4^3}$. In general, K_n consists of 2^n intervals, each of length $\frac{1}{4^n}$. Thus the dimension of the Cantor middle-2/4 set is

$$\dim(C_4) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln 4^n} = \frac{\ln 2}{\ln 4}.$$

(b) When $\omega = 3/5$, construction of the Cantor middle-3/5 set: In this case, we delete the middle three of five portions of the unit interval $I = [0,1]$. Then we have $K_1 = \left[0, \frac{1}{5}\right] \cup \left[\frac{4}{5}, 1\right]$ and $K_2 = \left[0, \frac{1}{25}\right] \cup \left[\frac{4}{25}, \frac{1}{5}\right] \cup \left[\frac{4}{5}, \frac{21}{25}\right] \cup \left[\frac{24}{25}, 1\right]$.

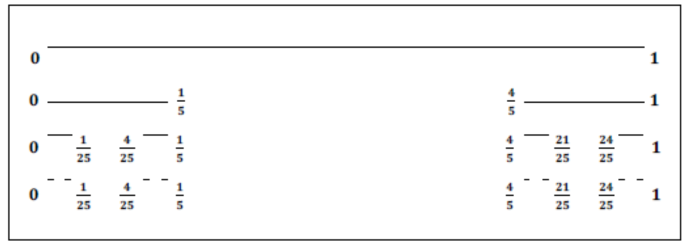

Figure 8. Construction of the middle-$3/5$ Cantor set.

Repeating this process as above, K_n consists of 2^n intervals, each of length $\frac{1}{5^n}$. Thus the dimension of the Cantor middle-$3/5$ set C_5 as

$$\dim(C_5) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln 5^n} = \frac{\ln 2}{\ln 5}$$

(c) When $\omega = 4/6$, construction of the Cantor middle-4/6 set: In this case, we delete the middle four of the six portions of the unit interval $I = [0,1]$. Then we have

$K_1 = \left[0, \frac{1}{6}\right] \cup \left[\frac{5}{6}, 1\right]$ and $K_2 = \left[0, \frac{1}{36}\right] \cup \left[\frac{5}{36}, \frac{1}{6}\right] \cup \left[\frac{5}{6}, \frac{31}{36}\right] \cup \left[\frac{35}{36}, 1\right]$.

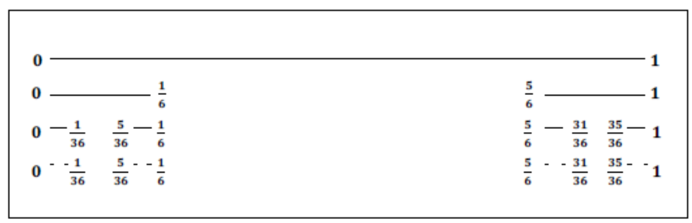

Figure 9. Construction of the middle-$4/6$ Cantor set.

In general, K_n consists of 2^n intervals, each of length $\left(\frac{1}{6}\right)^n$. Thus the dimension of the Cantor middle-4/6 set C_6 as

$$dim(C_6) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln 6^n} = \frac{\ln 2}{\ln 6}$$

(d) Generalization: When $\omega = m/(m+2)$ and $m \in \mathbb{Z}^+$, construction of the Cantor middle- $m/(m+2)$ set: As above, K_n consists of 2^n intervals, each of length $\frac{1}{(m+2)^n}$. Thus the dimension of the Cantor middle-$m/(m+2)$ set C_{m+2} as

$$\dim(C_{m+2}) = \lim_{n\to\infty} \frac{\ln 2^n}{\ln(m+2)^n} = \frac{\ln 2}{\ln(m+2)} ,$$

where $m \in \mathbb{Z}^+$.

7 CONCLUSION

We construct the generalized Cantor sets in three phases with self-similarity and find their fractal dimensions in each case. Although our construction of the Cantor set used the typical “middle-thirds” or ternary rule, we can easily generalize this one-dimensional idea to any length other than $\frac{1}{3}$, excluding of course the degenerate cases of 0 and 1. After decomposing the typical Cantor set into two distinct subsets, the portion of the set in $[0,1/3]$ and the portion in $[2/3,1]$, we see that each of these pieces resembles the original Cantor set. The only difference is the original interval is smaller by a factor of $1/3$. In the same manner, the magnifications of our generalized Cantor sets resemble the original set.

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