Generalizations Of Dunkl-Williams Inequality In Hilbert C*-Modules

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Abstract: In this paper we deduce some inequalities and get a generalization of the Dunkl-Williams inequality in the framework of Hilbert C*-modules.

Key words and phrases: absolute value operator, C*-algebra, Dunkl-Williams inequality, Hilbert C*-module, operator inequality.

1 INTRODUCTION

In [2], Dunkl and Williams showed that for any nonzero elements u, v in a normed linear space X

$$||u/||u|| - v/||v|| \leq 4||u-v||/(||u||+||v||) \quad (1)$$

Pecaric and Rajic [6] gave the following refinement of "(1)": For any nonzero elements u, v in a normed linear space X

$$u/||u|| - v/||v|| \leq \sqrt{2||u-v||^2 + 2(||u|| - ||v||)^2} / \max\{||u||, ||v||\} \quad (2)$$

Also in [6] they generalized the inequality "(2)" to the operators A, B belong to the algebra $B(H)$ such that $|A|, |B|$ are invertible as follows:

Theorem 1. Let $x, y \in B(H)$ of all bounded linear operators acting on a complex Hilbert space H such that $|x|$ and $|y|$ are invertible, and let $r, s > 1$ with $1/r + 1/s = 1$. Then

$$|x|x|^{-1} - y|y|^{-1}|^2 \leq |x|^{-1}(r|x-y|^2 + s(|x| - |y|^2))|x|^{-1}$$

The equality holds if and only if $(r-1)(x-y)|x|^{-1} = y(|x|^{-1} - |y|^{-1})$. In 2011, Dadipour and Moslehian [1] introduced operator version of the Dunkl-Williams inequality with respect to the p-angular distance as a generalization of the Theorem 1 as follows:

Theorem 2. Let a, b in C*-algebra A such that $|a|$ and $|b|$ are invertible, $1/r + 1/s = 1$, $(r > 1)$ and p be an element in real numbers. Then

$$|a|a|^{p-1} - b|b|^{p-1}|^2 \leq |a|^{p-1}[r|a-b|^2 + s(|a|^{1-p}|b|^p - |b|)(|b|^{1-p}|a|^p - |b|)]|a|^{p-1}.$$

The equality holds if and only if $(r-1)(a-b)|a|^{p-1} = b(|a|^{p-1} - |b|^{p-1})$. In this paper they extend Theorem 2 to the Hilbert C*-modules case.

Theorem 3. Let u, v be elements of a Hilbert C*-module H such that $|u|$ and $|v|$ are invertible, $1/r + 1/s = 1$, $(r > 1)$ and p be an element in real numbers. Then

$$|u|u|^{p-1} - v|v|^{p-1}|^2 \leq |u|^{p-1}[r|u-v|^2 + s(|u|^{1-p}|v|^p - |v|)(|v|^{1-p}|u|^p - |v|)]|u|^{p-1}.$$

The equality holds if and only if $(r-1)(u-v)|u|^{p-1} = v(|u|^{p-1} - |v|^{p-1})$.

We improved the Theorem 3 without the assumption of the invertibility of the absolute value of operator $|v|$.

3 PRELIMINARIES

Throughout this paper, N, R, C are natural, real and complex numbers, respectively, and J is a subset of N. Let us recall some definitions and basic properties of C*-algebras and Hilbert C*-modules that we need in the rest of the paper. A Banach *-algebra A is called a C*-algebra if it satisfies $\|a^*a\|^2 = \|a\|$ for any $a \in A$ An element a of a C*-algebra A is positive if there exists $b \in A$ such that $a = b^*b$. We write $a \geq 0$ to mean that a is positive. The relation " $\leq$ " given by $a \leq b$ if and only if $b - a$ is positive defines a partial ordering on A. Let A be a C*-algebra then the absolute value of a is defined by $|a| = (a^*a)^{1/2}$. For undefined notions and more details on C*-algebra theory, we refer to [5]. Let A be a C*-algebra and let H be a right A-module. H is a pre-Hilbert A-module if H is equipped with an A-valued inner product $\langle .,. \rangle : H \times H \to A$ that possesses the following properties,

(i) $\langle u, u \rangle \geq 0$, for all $u \in H$ and $\langle u, u \rangle = 0$ if and only if $u = 0$;

(ii) $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$, for all $\alpha, \beta \in C$ and $u, v, w \in H$;

(iii) $\langle u, v \rangle = \langle v, u \rangle^*$, for all $u, v \in H$;

(iv) $\langle u, va \rangle = \langle u, v \rangle a$, for all $a \in A$ and $u, v \in H$;

The action of A on H is C- and A-linear, i.e., $\mu(ua) = u(\mu a) = (\mu u)a$ for every $\mu \in C$, $a \in A$ and $u \in H$. For $u \in H$, we define $\|u\| = \|\langle u, u \rangle\|^{1/2}$. If H is complete with $\|.\|$, it is called a Hilbert A-module or a Hilbert C*-module over A. The C*-algebra A itself can be recognized as a Hilbert A-module with the inner product $\langle a, b \rangle = a^*b$ for any $a, b \in A$. For a C*-algebra A the standard Hilbert A-module $l^2(A)$ is defined by

$$\ell^2(A) = \left\{ \{a_j\}_{j \in N} : \sum_{j \in N} a_j^* b_j \ \ converges\ in\ A \right\}$$

With A-inner product $\langle \{a_j\}_{j \in N}, \{b_j\}_{j \in N} \rangle = \sum_{j \in N} a_j^* b_j$.

For every u in Hilbert C*-module H we define the absolute

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value of u as the unique positive square root of $\langle u,u\rangle$, that is, $|u| = \langle u,u\rangle^{1/2}$. We refer the reader to [3, 7] for more information on Hilbert C*-modules. In the sequel we denote H and K as Hilbert modules over a unital C*-algebra with unit e.

3 Main Results

We have the following generalization of the Dunkl-Williams type inequality [2] in the framework of Hilbert C*-modules. As a result, Theorem 3 is extended without demanding the invertibility of $|v|$. Theorem 4. Let u, v be two elements of H. If A is a unital and a, b are elements in A such That a is invertible, $\lambda \geq 0$, then

$$|ua - vb|^2 \leq a^* \left((1+\lambda)|u-v|^2 + \left(1+\frac{1}{\lambda}\right)|vba^{-1} - v|^2\right)a.$$

The reverse inequality is valid for $\lambda < 0$. The equality holds if and only if $\lambda(u-v)a = v(a-b)$.

Proof. **.** First observe that

$a^*|u-v|^2a = a^*\langle u-v, u-v\rangle a \ = \langle (u-v)a, (u-v)a\rangle = |(u-v)a|^2$ (3)

Also,

$|vba^{-1} - v|^2 = \langle vba^{-1} - v, vba^{-1} - v\rangle = \langle vba^{-1}, vba^{-1}\rangle - \langle vba^{-1}, v\rangle - \langle v, vba^{-1}\rangle + \langle v, v\rangle = (a^*)^{-1}b^*|v|^2ba^{-1} - (a^*)^{-1}b^*|v|^2 - |v|^2ba^{-1} + |v|^2,$ (4)

By multiplying a^* and a from the left and right “(4)”, respectively, we have

$a^*|vba^{-1} - v|^2a = b^*|v|^2b - b^*|v|^2a - a^*\ |v|^2b + a^*|v|^2a = \langle vb, vb\rangle - \langle vb, va\rangle - \langle va, vb\rangle + \langle va, va\rangle = -\langle vb, va - vb\rangle + \langle va, va - vb\rangle = \langle va - vb, va - vb\rangle = |v(a-b)|$^2. (5)

Using “(3)”, “(5)” and the part (i) of Theorem 3.7 of [4], we obtain

$|ua - vb|^2 = |(u-v)a + v(a-b)|^2 \leq (1+\lambda)|(u-v)a|^2 + (1+1/\lambda)|v(a-b)|^2 = (1+\lambda)a^*|u-v|^2a + (1+1/\lambda)a^*|vba^{-1} - v|^2a = a^*\ ((1+\lambda)|u-v|^2 + (1+1/\lambda)|vba^{-1} - v|^2)a).$

The reverse inequality followed from “(3)”, “(5)” and the part (ii) of Theorem 3.7 of [4]. The equality case is obtain by Theorem 3.7 of [4] •

Lemma 5. Let $u \in\ H$ and a, b be two elements of C*-algebra A such that a is invertible then

$$\big||u|ba^{-1} - |u|\big|^2 = |uba^{-1} - u|^2.$$

Proof. By definition of $|u|^2 = \langle u,u\rangle$, we have

$|uba^{-1} - u|^2 = \langle uba^{-1} - u, uba^{-1} - u\rangle = \langle uba^{-1}, uba^{-1}\rangle - \langle uba^{-1}, u\rangle - \langle u, uba^{-1}\rangle + \langle u, u\rangle = (a^*)^{-1}b^*|u|^2ba^{-1} - (a^*)^{-1}b^*|u|^2 - |u|^2ba^{-1} + |u|^2 = ((a^*)^{-1}b^*|u| - |u|)(|u|ba^{-1} - |u|) = (|u|ba^{-1} - |u|)^*(|u|ba^{-1} - |u|) = ||u|ba^{-1} - |u||^2.$

Which complete the proof •

The following Theorem follows by applying Theorem 4 and Lemma 5.

Theorem 6. Let u, v be two elements of H. If A is a unital and a, b are elements in A such That a is invertible, $\lambda \geq 0$, then

$$|ua - vb|^2 \leq a^*\left((1+\lambda)|u-v|^2 + (1+1/\lambda)||v|ba^{-1} - |v||^2\right)a.$$

The reverse inequality is valid for $\lambda < 0$. The equality holds if and only if $\lambda(u-v)a = v(a-b)$.

We have the following results.

Corollary 7. Theorem 6 gives Theorem 3.

Proof. Let us put $a = |u|^{p-1}, b = |v|^{p-1}, \lambda = r-1$ in Theorem 9. Then $a^* = |u|^{p-1}$ and $1/\lambda = 1/(r-1) = s-1$, where $1/r + 1/s = 1$, so

$|u|u|^{p-1} - v|v|^{p-1}|^2 \leq\ |u|^{p-1}\big(r|u-v|^2 + s\big||v||v|^{p-1}|u|^{1-p} - |v|\big|^2\big)|u|^{p-1} = |u|^{p-1}(r|u-v|^2 + s||v|^p|u|^{1-p} - |v||^2)|u|^{p-1} = |u|^{p-1}(r|u-v|^2 + s\big[(|v|^p|u|^{1-p} - |v|)^*(|v|^p|u|^{1-p} - |v|)\big)|u|^{p-1} = |u|^{p-1}(r|u-v|^2 + s\big[(|u|^{1-p}|v|^p - |v|)^*(|v|^p|u|^{1-p} - |v|)\big)|u|^{p-1}.$

The equality holds if and only if
$\lambda(u-v)a = v(a-b) \Leftrightarrow (r-1)(u-v)|u|^{p-1} = v(|u|^{p-1} - |v|^{p-1}).$ •

Remark. Theorem 2 followed from Theorem 6 if we set $u = a, v = b, a = |a|^{p-1}, b = |b|^{p-1}$.

A special case of Theorem 6, where Hilbert module is the $B(H)$, of all bounded linear operators on a complex Hilbert space H over itself gives rise to the main result of Pecaric and Rajic [6].

Corollary 8. Theorem 6 gives Theorem 1 if we put $u = A, v = B, a = |A|^{-1},\ b = |B|^{-1}$.

Following we give an Example.

Example. Let A be the C*-algebra of the set of all diagonal matrices in $M_{2\times 2}(C)$ and suppose A is the Hilbert A-module over itself. (Here, diagonal matrix means a 2×2 matrix (a_{ij}) such that $a_{11} = a, a_{22} = b$ and $a_{12} = a_{21} = 0$, for $a, b \in C$.) Consider,

$$u = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}, \qquad v = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Then we have
$w = |u|^{-1}\big((1+\lambda)|u-v|^2 + (1+1/\lambda)(|u| - |v|)^2\big)|u|^{-1} - ||u|^{-1}u - |v|^{-1}v|^2 = \begin{pmatrix} 9(\lambda+1)^2/\lambda & 0 \\ 0 & (\lambda+1)^2/16\lambda \end{pmatrix}$
The matrix $w \geq 0$ if $\lambda > 0$ and $w \leq 0$ if $\lambda < 0$.

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