A Method To Calculate Determinants, With Computer Algorithm Interpretation

Armend Salihu, Fatlinda Salihu

Abstract: In this paper we present the new algorithm to calculate determinants of nth order using Rezaifar’s method of reducing the order of determinants to second order. We have implemented the Dodgson’s algorithm within Rezaifar’s method to calculate sub matrices and developed a new method. Within the paper we have also developed the computer algorithm to calculate the determinant using this new method. While comparing the computer execution time with the Rezaifar’s method, we have seen that this new algorithm presented is executed faster.

Index Terms: Determinants, computer algorithm, determinant calculation, time comparison.

1. INTRODUCTION

Let $A$ be a $n \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Its determinant is the sum:

$$\det(A) = |A| = \sum_{j_1 \neq j_2 \neq \cdots \neq j_n} e_{j_1j_2\ldots j_n} \cdot a_{j_1} \cdot a_{j_2} \cdots \cdot a_{j_n}$$

ranging over the symmetric permutation group $j_1j_2\ldots j_n$, where:

$$e_{j_1j_2\ldots j_n} = \begin{cases} +1, & \text{if $j_1j_2\ldots j_n$ is an even permutation} \\ -1, & \text{if $j_1j_2\ldots j_n$ is an odd permutation}. \end{cases}$$

Lemma 1: [2] Let $B_n = [b_{ij}]_{n \times n}$ be a square real matrix of order $n$, then the Dodgson’s condensation of matrix $B_n$ is a $(n-1) \times (n-1)$ matrix defined as:

$$DC(B_n) = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1(n-1)} & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2(n-1)} & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{(n-1)1} & b_{(n-1)2} & \cdots & b_{(n-1)(n-1)} & b_{(n-1)n} \\ b_{1n} & b_{2n} & \cdots & b_{(n-1)n} & b_{nn} \end{bmatrix}$$

Lemma 2: [6] The determinant of the square matrix

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

is equal to:

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

2. A METHOD TO CALCULATE DETERMINANTS

Theorem 1: Given a square matrix of nth order:

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where,

$$A_{n-1,2} \neq 0, \quad A_{n-1,n-1} \neq 0 \quad \text{and} \quad A_{n-2,2} \neq 0.$$

Then:

$$|A_{n \times n}| = \begin{bmatrix} |FDC(A_n)_{11}| & \cdots & |FDC(A_n)_{1n}| \\ \vdots & \ddots & \vdots \\ |FDC(A_n)_{n1}| & \cdots & |FDC(A_n)_{nn}| \end{bmatrix}. \quad \ast$$

Proof: By using Lemma 2, Rezaifar’s method see equation 3.1 in [6], we know that:

$$|A_{n \times n}|$$

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Algorithm 1: Rezaifar’s Method see Table 5 in [6]:

Step 1: Insert order of determinant
Step 2: Insert the determinant
Step 3: Calculating submatrices using same algorithm

\[
d_0 = \text{det}_\text{Rezaifar}(A(2 : n - 1, 2 : n - 1))
\]
\[
d_1 = \text{det}_\text{Rezaifar}(A(1 : n - 1, 1 : n - 1))
\]
\[
d_2 = \text{det}_\text{Rezaifar}(A(1 : n - 1, 2 : n))
\]
\[
d_3 = \text{det}_\text{Rezaifar}(A(2 : n, 1 : n - 1))
\]
\[
d_4 = \text{det}_\text{Rezaifar}(A(2 : n, 2 : n))
\]

Step 4: Calculate the determinant
\[
d = (d_1 * d_4 - d_2 * d_3) / d_0
\]

Algorithm 2: Dodgson’s Method

Step 1: Insert order of determinant
Step 2: Insert the determinant
Step 3: Dodgson’s Method

Create loop: for \( i = 1 \) : \( m - 2 \)
Checking size of reduction step: \([n,n] = \text{size}(A)\);
Create loop: for \( j = 1 \) : \( n - 1 \)
Create loop: for \( j = 1 \) : \( n - 1 \)
Calculation Dodgson’s elements:
\[
B(i, j) = A(i, j) * A(i + 1, j + 1) - A(i, j + 1) * A(i + 1, j)
\]

Conditions for the divisor in Dodgson’s algorithm:
if \( m = n \)
\[
X = A; A = B; B = 0
\]
else
\[
C = B; X(2 : n, 2 : n); X = A; A = C; B = 0
\]
end
end
end

Step 4: Display reduced matrix \( A \) in second order

\[
d = A
\]

Algorithm 3: Salihu’s Method

Step 1: Insert order of determinant
Step 2: Insert the determinant
Step 3: Calculating submatrices using Dodgson’s algorithm

\[
ed_0 = \text{det}_\text{Dodgson‘}(A([2 : n - 1, [2 : n - 1]))
ed_1 = \text{det}_\text{Dodgson‘}(\text{det}_\text{Dodgson‘}(A([1 : n - 1, [1 : n - 1])))
ed_2 = \text{det}_\text{Dodgson‘}(\text{det}_\text{Dodgson‘}(A([1 : n - 1, [n - 2 : n - 1])))
ed_3 = \text{det}_\text{Dodgson‘}(\text{det}_\text{Dodgson‘}(A([n - 2 : n - 1, [1 : n - 1])))
ed_4 = \text{det}_\text{Dodgson‘}(\text{det}_\text{Dodgson‘}(A([n - 2 : n - 1, [n - 1 : n - 1])))
ed_5 = \text{det}_\text{Dodgson‘}(A([2 : n - 2, [2 : n - 2]))
\]

Step 4: Calculate the determinant
\[
d = (d_1 * d_4 - d_2 * d_3) / (d_0 * (d_5)\^2)
\]

\text{det}_\text{Dodgson‘} in Algorithm 3, is the same as Algorithm 2, with the change in step 3 that the loop is created for \( i = m - 1 \).

3. COMPUTER INTERPRETATION

In the following we present computer algorithm of Rezaifar’s method and Salihu’s method:

3.1. Computer comparison of determinants calculation methods

To compare the execution time of determinant calculation we have used the software presented in table 1.

<table>
<thead>
<tr>
<th>OS</th>
<th>Windows 10 Pro 64-bit, Version 1703 (OS Build 15063.483</th>
</tr>
</thead>
<tbody>
<tr>
<td>Software</td>
<td>MATLAB, Version 9.0.0321247 (R2016a), 64-bit (win64)</td>
</tr>
</tbody>
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While the computer hardware used for this simulation is presented in table 2.

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<td>MATLAB, Version 9.0.0321247 (R2016a), 64-bit (win64)</td>
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While the computer hardware used for this simulation is presented in table 2.
Table 2

Computer Hardware Used to Simulate the Calculation of Determinants:

<table>
<thead>
<tr>
<th>Name:</th>
<th>Lenovo</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model:</td>
<td>Ideapad 700-15ISK</td>
</tr>
<tr>
<td>CPU:</td>
<td>Intel Core i7 6700HQ 2.6Ghz</td>
</tr>
<tr>
<td>RAM:</td>
<td>16 GB DDR4</td>
</tr>
<tr>
<td>GPU:</td>
<td>FULL HD Display 15.6&quot; 1920x1080, nVidia GTX 950 4096 mb dedicated graphics</td>
</tr>
<tr>
<td>HDD:</td>
<td>256 GB SSD</td>
</tr>
</tbody>
</table>

The results of execution time of Rezaifar's Method Algorithm and Salihu's Method Algorithm, to simulate the execution time of determinant calculation are presented in table 3.

Table 3

Execution Time of Determinant Calculation Using Matlab Functions for Rezaifar's Method and Salihu's Method, for the Order 3 x 3 to 15 x 15 Present in Seconds:

<table>
<thead>
<tr>
<th>Order of det.</th>
<th>Salihu's Method</th>
<th>Rezaifar's Method</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 x 3</td>
<td>0.0021</td>
<td>0.0011</td>
<td>-0.0010</td>
</tr>
<tr>
<td>4 x 4</td>
<td>0.0021</td>
<td>0.0012</td>
<td>-0.0009</td>
</tr>
<tr>
<td>5 x 5</td>
<td>0.0059</td>
<td>0.0017</td>
<td>-0.0043</td>
</tr>
<tr>
<td>6 x 6</td>
<td>0.0036</td>
<td>0.0036</td>
<td>0.0000</td>
</tr>
<tr>
<td>7 x 7</td>
<td>0.0024</td>
<td>0.0058</td>
<td>0.0034</td>
</tr>
<tr>
<td>8 x 8</td>
<td>0.0297</td>
<td>0.0369</td>
<td>0.0072</td>
</tr>
<tr>
<td>9 x 9</td>
<td>0.0029</td>
<td>0.0750</td>
<td>0.0721</td>
</tr>
<tr>
<td>10 x 10</td>
<td>0.0046</td>
<td>0.3091</td>
<td>0.3045</td>
</tr>
<tr>
<td>11 x 11</td>
<td>0.0302</td>
<td>1.2591</td>
<td>1.2289</td>
</tr>
<tr>
<td>12 x 12</td>
<td>0.0052</td>
<td>5.1166</td>
<td>5.1114</td>
</tr>
<tr>
<td>13 x 13</td>
<td>0.0022</td>
<td>21.7667</td>
<td>21.7645</td>
</tr>
<tr>
<td>14 x 14</td>
<td>0.0020</td>
<td>91.9499</td>
<td>91.9480</td>
</tr>
<tr>
<td>15 x 15</td>
<td>0.0055</td>
<td>410.9499</td>
<td>410.9444</td>
</tr>
</tbody>
</table>

From the comparison of the results presented in Table 3, it shows that Rezaifar's Method is slightly more effective for computer computations of determinants of the order lower than or equal to 6 x 6, whereas for calculating computer determinants starting from the order 7 x 7 it is seen that the Salihu's Method is much more effective. Since Rezaifar's Algorithm is more effective than the Laplace method (see table 7 in [6]), we can conclude that also this new algorithm is more effective than Laplace method.

References