A Method To Calculate Determinants, With Computer Algorithm Interpretation

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Abstract: In this paper we present the new algorithm to calculate determinants of nth order using Rezaifar's method of reducing the order of determinants to second order. We have implemented the Dodgson's algorithm within Rezaifar's method to calculate sub matrices and developed a new method. Within the paper we have also developed the computer algorithm to calculate the determinant using this new method. While comparing the computer execution time with the Rezaifar's method, we have seen that this new algorithm presented is executed faster.

Index Terms: Determinants, computer agorithm, determinant calculation, time comparison.

1. INTRODUCTION

LET A be a $n \times n$ matrix:

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Its determinant is the sum:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_1 j_2 \dots j_n} \varepsilon_{j_1 j_2 \dots j_n} \cdot a_{j_1} \cdot a_{j_2} \cdot \dots \cdot a_{j_n},$$

ranging over the symmetric permutation group $j_1 j_2 \dots j_n$, where:

$$\varepsilon_{j_1 j_2 \dots j_n} = \begin{cases} +1, \text{if } j_1 j_2 \dots j_n, \text{is an even permutation} \\ -1, \text{if } j_1 j_2 \dots j_n, \text{is an odd permutation.} \end{cases}$$

Lemma 1: [2] Let $B_n = [b_{ij}]_{n\times n}$ be a square real matrix of order *n*. then the Dodgson's condensation of matrix B_n is a $(n-1) \times (n-1)$ matrix defined as:

$$DC(B_n) = \begin{bmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} & \cdots & \begin{vmatrix} b_{1(n-1)} & b_{1n} \\ b_{2(n-1)} & b_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} b_{(n-1)1} & b_{(n-1)2} \\ b_{n1} & b_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} b_{(n-1)(n-1)} & b_{(n-1)n} \\ b_{n(n-1)} & b_{nn} \end{vmatrix} \end{bmatrix}_{(n-1)\times(n-1)}.$$

Lemma 2: [6] The determinant of the square matrix $A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ is equal to:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \frac{\begin{vmatrix} \begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} \end{vmatrix} & \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} \\ \begin{vmatrix} a_{21} & \cdots & a_{2(n-1)} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} \end{vmatrix} & \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} \end{vmatrix}}.$$

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2. A METHOD TO CALCULATE DETERMINANTS

Theorem 1: Given a square matrix of *n*th order:

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}, \text{ where, }$$

$$\begin{bmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix} \neq 0, \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \neq 0 \text{ and } \begin{vmatrix} a_{22} & \cdots & a_{2,n-2} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n-2} \end{vmatrix} \neq 0.$$

Then:

$$|A_{n\times n}| = \frac{\begin{vmatrix} |[FDC(A_n)_1]^*|_{2\times 2} & |[FDC(A_n)_2]^*|_{2\times 2} \\ |[FDC(A_n)_3]^*|_{2\times 2} & |[FDC(A_n)_4]^*|_{2\times 2} \end{vmatrix}}{\begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & \cdots & a_{2,n-2} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n-2} \end{vmatrix}^2}.$$

Proof: By using Lemma 2, Rezaifar's method see equation 3.1 in [6], we know that:

$$|A_{n\times n}|$$

$$= \frac{\begin{vmatrix} \begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} \end{vmatrix} & \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} \\ \begin{vmatrix} a_{21} & \cdots & a_{2(n-1)} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} \end{vmatrix} & \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} \end{vmatrix}}.$$

By using determinant properties to changing locations of rows/columns, we have:

$$|A_{n\times n}| = \frac{\begin{vmatrix} \begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} \end{vmatrix} & \begin{vmatrix} a_{1n} & a_{12} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,n} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \\ \begin{vmatrix} a_{n1} & \cdots & a_{n,n-1} \\ a_{21} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix} & \begin{vmatrix} a_{nn} & a_{n2} & \cdots & a_{n,n-1} \\ a_{1n} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,n} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & \cdots & a_{2(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} \end{vmatrix}}.$$

eq. 1.

Implementing Lemma 1, Dodgson algorithm on sub matrices in formula (1) until we reach the second order, or directly implementing Lemma 2, Rezaifar's method, we have:

$$|[FDC(A_n)]^*|_{2\times 2} = \frac{\begin{vmatrix} |X_{11}| & |X_{12}| \\ |X_{21}| & |X_{22}| \end{vmatrix}}{\begin{vmatrix} a_{22} & \cdots & a_{2,n-2} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n-2} \end{vmatrix}}$$

eq. 2.

Since internal matrix of all sub matrices in formula (1) is the same: $\begin{vmatrix} a_{22} & \cdots & a_{2,n-2} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n-2} \end{vmatrix}$, implementing the formula (2) in formula (1) and using determinant properties, we get the proof of Theorem 1.

3. Computer Interpretation

In the following we present computer algorithm of Rezaifar's method and Salihu's method:

Algorithm 1: Rezaifar's Method see Table 5 in [6]:

Step 1: Insert order of determinant
Step 2: Insert the determinant
Step 3: Calculating submatrices using same algorithm
d0 = det_Rezaifar(A(2 : n - 1, 2 : n - 1))
d1 = det_ Rezaifar (A(1 : n - 1, 1 : n - 1))
d2 = det_ Rezaifar (A(1 : n - 1, 2 : n))
d3 = det_ Rezaifar (A(2 : n, 1 : n - 1))
d4 = det_ Rezaifar (A(2 : n, 2 : n))
Step 4: Calculate the determinant
*d = (d1 * d4 - d2 * d3) / d0*

Algorithm 2: Dodgson's Method

Step 1: Insert order of determinant
Step 2: Insert the determinant
Step 3: Dodgson's Method
Create loop: for I = 1 : m - 2
Checking size of reduction step: [n,n] = size(A);
Create loop: for i = 1 : n - 1
Create loop: for j = 1 : n - 1
Calculation Dodgson's elements:
*B(i, j) = A(i, j) * A(i + 1, j + 1) - A(i, j + 1) * A(i + 1, j)*
Conditions for the divisor in Dodgson's algorithm:
if m = n
X = A; A = B; B = 0
else
C = B./ X(2 : n, 2 : n); X = A; A = C; B = 0
end
end
Step 4: Display reduced matrix A in second order
d=A

Algorithm 3: Salihu's Method

Step 1: Insert order of determinant
Step 2: Insert the determinant
Step 3: Calculating submatrices using Dodgson's algorithm
d0 = det_Dodgson`(A([2 : n - 1], [2 : n - 1]))
d1 = det_Dodgson`(det_Dodgson(A([1 : n - 1], [1 : n -1])))
d2 = det_Dodgson`(det_Dodgson(A([1 : n - 1], [n 2 : n - 1])))
d3 = det_Dodgson`(det_Dodgson(A([n 2 : n - 1], [1 : n - 1])))
d4 = det_Dodgson`(det_Dodgson(A([n 2 : n - 1], [n 2 : n - 1])))
d5 = det_Dodgson`(A([2 : n - 2], [2 : n - 2]))
Step 4: Calculate the determinant
*d = (d1 * d4 - d2 * d3) / (d0 * (d5)^2)*

det_Dodgson' in Algorithm 3, is the same as Algorithm 2, with the change in step 3 that the loop is created for *i = m - 1*.

3.1. Computer comparison of determinants calculation methods

To compare the execution time of determinant calculation we have used the software presented in table 1.

Table 1
Computer software used for the simulation:

OS	Windows 10 Pro 64-bit, Version 1703 (OS Build 15063.483
Software	MATLAB, Version 9.0.0321247 (R2016a), 64-bit (win64)

While the computer hardware used for this simulation is presented in table 2.

TABLE 2
COMPUTER HARDWARE USED TO SIMULATE THE CALCULATION OF DETERMINANTS:

Name:	Lenovo
Model:	Ideapad 700-15ISK
CPU:	Intel Core i7 6700HQ 2.6Ghz
RAM:	16 GB DDR4
GPU:	FULL HD Display 15.6" 1920x1080, nVidia GTX 950 4096 mb dedicated graphics
HDD:	256 GB SSD

The results of execution time of Rezaifar's Method Algorithm and Salihu's Method Algorithm, to simulate the execution time of determinant calculation are presented in table 3.

TABLE 3
EXECUTION TIME OF DETERMINANT CALCULATION USING MATLAB FUNCTIONS FOR REZAIFAR'S METHOD AND SALIHU'S METHOD, FOR THE ORDER 3×3 TO 15×15 PRESENT IN SECONDS:

Execution time of determinant calculation			
Order of det.	Salihu's Method	Rezaifar's Method	Difference
3×3	0.0021	0.0011	-0.0010
4×4	0.0021	0.0012	-0.0009
5×5	0.0059	0.0017	-0.0043
6×6	0.0036	0.0036	0.0000
7×7	0.0024	0.0058	0.0034
8×8	0.0297	0.0369	0.0072
9×9	0.0029	0.0750	0.0721
10×10	0.0046	0.3091	0.3045
11×11	0.0302	1.2591	1.2289
12×12	0.0052	5.1166	5.1114
13×13	0.0022	21.7667	21.7645
14×14	0.0020	91.9499	91.9480
15×15	0.0055	410.9499	410.9444

From the comparison of the results presented in Table 3, it shows that Rezaifar's Method is slightly more effective for computer computations of determinants of the order lower than or equal to 6×6, whereas for calculating computer determinants starting from the order 7×7 it is seen that the Salihu's Method is much more effective. Since Rezaifar's Algorithm is more efective than the Laplace method (see table 7 in [6]), we can conclude that also this new algorithm is more effective than Laplace method.

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