A Prey-Predator Fishery Model With A Relative Size Of Reserved Area

Kulbhushan Agnihotri, Sheenu Nayyer

Abstract: A prey-predator fishery model having reserved and unreserved area, with prey dispersal in a two-patch environment, has been proposed and investigated in this work. The logistic growth is considered for the fish species in each area. Holling type-II predator functional response has been considered. Relative size of the reserve and unreserved area is responsible for increase as well as decrease the density of the fishes. The harvesting is applied on both prey in an unreserved area and on predator. The dynamics of the proposed system has been explored locally. The thresholds for existence of biological equilibrium points are obtained. Optimal harvesting policy has been examined by the Pontryagin's Maximum Principle. Finally, theoretical results acquired and verified with the assistance of numerical simulations through MATLAB.

Keywords: Relative size; Holling type II; Migration; Optimal harvesting; Pontryagin's Maximum Principle; MPAs.

1. Introduction

The depletion of renewable and non-renewable resources at an alarming rate is causing a great concern in all over the world. These resources are being depleting due to over exploitation, industrialization, fast urbanization and increasing population etc. It is also well known that many species have already become extinct and many others are at the verge of extinction due to several natural or man-made reasons like over exploitation, indiscriminate harvesting, over predation, environmental pollution, loss of habitat due to mismanagement of natural resources, etc. But for the ecological balance, it is very important to keep these resources at an optimal level. So, Marine protected areas (MPAs) are a developing instrument for handling marine assets. The inspiration for setting up MPAs initially originated from nations, for example, Australia, New Zealand, and Seychelles. Many countries recognized the economic potential of their marine assets. As the most recent examinations reveal that there is an expanding require the amplification of marine protected areas in light of the fact that MPAs can improve yield as well as secure stocks and sustain fishery attainability. As the countries start to look more towards the ocean for financial development and new food sources, due to this fact it leads to huge pressure on fishing in the world's Ecosystem. Consequently over exploitation of marine assets is a serious and immediate worldwide issue. Hence, formation of marine reserves or no fishing zones epitomise a significant instrument for future fisheries administration and supportable improvement of biological system. Hartmann et al. [1] allowed a better management strategy to investigate the economic optimality of implementing an MPA. Sumalia [2] indicated that MPAs can protect the discounted economic rent from the fishery.

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Takashina et al. [3] have analyzed the potential impact of building up MPAs on marine biological systems and their analyse revealed that it can sometimes result in an extensive decline or even extinction of a species. T.K. Kar and H. Matsuda [4] inspected the effects of MPAs and harvesting from both economic and biological perspectives on resource populations. They demonstrated that protected patches are successful methods for conserving resource populations, however elimination can't be forestalled. Kunal Chakraborty et al. [5] have considered a prey–predator type fishery model with prey reserved area. Relative size of the reserve is taken as control to contemplate optimal sustainable yield policy. B. Dubey et al. [6] have proposed a scientific model and examined to study the dynamics of fishery resource, have two zones of the aquatic ecosystem, one free fishing zone and the other reserved zone. They have investigated that appropriate equilibrium level never disturbed under continuous harvesting of fish species outside the reserved zone. Yunfei Lv a et al. [7] proposed a prey-predator harvesting fishery resource, model. Holling type II predator functional response is considered. The results demonstrate that so long as the prey population in the reserved zone does not extinct, the both prey dependably exist. B. Dubey [8] examined that the reserve zone has a stabilizing effect on predator-prey species. Keeping this in view and literature considered earlier, the present paper deals with a prey predator framework where a fractional part ($0 < s < 1$) of the prey population is considered to regulate the system.

2. The Mathematical Model

A habitat is considered in an ecosystem with prey (fishes) dispersal in a two-patch environment, one is assumed to be a free fishing zone and other is a reserved zone. Both zones are supposed to be homogeneous. Also, there is a predator (fish) in the ecological system that is sustaining on prey (fish) in the unreserved zone. Holling type II predator functional response is considered. Here, it is assumed that the prey (fishes) species migrate between the two zones randomly. Harvesting of both species, prey in unreserved area and predator is taken. Unit population distribution area is considered in this research work. Here s ($0<s<1$) and 1-s is assumed for reserved and unreserved area respectively.

Keeping every one of the presumptions in view, a model is governed by the following system of ordinary differential equations.

$$\frac{dx}{dt} = r_1 x\left(1 - \frac{x}{K(1-s)}\right) - m_1 x + m_2 y - \frac{\beta_1 xz}{a+x} - q_1 E_1 x$$

$$\frac{dy}{dt} = r_2 y\left(1 - \frac{y}{Ks}\right) + m_1 x - m_2 y$$

$$\frac{dz}{dt} = -dz + \frac{\beta_2 xz}{a+x} - q_2 E_2 z \qquad (2.1)$$

Table 1
Variables and parameters used in proposed model

Variables/ Parameters	Description
$x(t)$	Biomass density of the prey species inside the unreserved area
$y(t)$	Biomass density of the prey species inside the reserved area
$z(t)$	Biomass density of the predator
r_1	The intrinsic growth rate of the prey species inside the unreserved area
r_2	The intrinsic growth rate of the prey species inside the reserved area
K	Environmental carrying capacity of prey inside the unreserved and reserved area respectively
m_1, m_2	Migration rates of the fishes from the unreserved area to reserved and *vice versa*
E_1, E_2	Harvesting efforts applied to the prey (fishes) in unreserved area and the predator respectively
d	Death rate of a predator
a	Half saturation level coefficient /Michaelis-Menten constant
β_1, β_2	Capturing and Conversion rate of prey and predator in unreserved area
q_1, q_2	Catchability coefficient of prey in unreserved area and predator respectively

In absence of migration of fish population from the reserved zone to the unreserved zone

i.e. $(m_2 = 0)$ and $r_1 - m_1 - q_1 E_1 < 0$, then $\frac{dx}{dt} < 0$,

Correspondingly, if there is no relocation of the fish population from unreserved area to reserved area *i.e.* $(m_1 = 0)$ and $r_2 - m_2 < 0$, then $\frac{dy}{dt} < 0$

This implies the fish species will be extinct from both the unreserved area and reserved area. To protect the prey (fishes) from annihilation, movement of fish species from both the patches are essential. Subsequently, all through our investigation, it is assumed that

$$r_1 - m_1 - q_1 E_1 > 0 \text{ and } r_2 - m_2 > 0 \qquad (2.2)$$

3. Boundedness and Stability Analysis

Theorem 3.1 *All the solutions of system (2.1) are always bounded.*

Proof: Let $\eta = x + y + \frac{\beta_1}{\beta_2} z$ and $\eta > 0$

Then

$$\frac{d\eta}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{\beta_1}{\beta_2}\frac{dz}{dt}$$

$$\frac{d\eta}{dt} = r_1 x\left(1 - \frac{x}{K(1-s)}\right) - q_1 E_1 x + r_2 y - \frac{r_2 y^2}{Ks} - \frac{\beta_1}{\beta_2}(d + q_2 E_2) z$$

$$\frac{d\eta}{dt} + \eta(d + q_2 E_2) \le G \qquad (3.1)$$

Where

$$G = \frac{K(1-s)}{4r_1}(r_1 - q_1 E_1 + d + q_2 E_2)^2 + \frac{Ks}{4r_2}(r_2 + d + q_2 E_2)^2$$

Applying the theory of differential inequality (Birkoff and Rota, 1982), the following inequality will be obtained

$$0 < \eta(t) < e^{-(d+q_2E_2)t}\left(\eta(0) - \frac{G}{d + q_2 E_2}\right) + \frac{G}{d + q_2 E_2}$$

As $t \to \infty$, from above expression we may conclude that the solution space (x, y, z) is bounded in the region R_+^3. Hence the theorem.

3.1 Analysis of the model at equilibrium points:

Possible equilibria of the system (2.1) are:

I. $P_0(0, 0, 0)$, (extinction of all species)

II. $P_1(x_1, y_1, 0)$, (Predator-free equilibrium point)

III. $P_2(x_2, y_2, z_2)$, (The nonzero -equilibrium point)

The predator-free equilibrium point $P_1(x_1, y_1, 0)$

From Ist equation of system (2.1), we obtain

$$y = \frac{1}{m_2}\left[(m_1 + q_1 E_1 - r_1)x + \frac{r_1 x^2}{K(1-s)}\right] \qquad (3.2)$$

Substitute the value of *y* in 2nd eq. of (2.1)

After simplification

$$Ax^3 + Bx^2 + Cx + D = 0 \qquad (3.3)$$

Where

$$A = r_2 r_1^2,\ B = 2r_2 r_1 K(1-s)(m_1 - r_1 + q_1 E_1)$$

$$C = K^2(1-s)\left(r_2(m_1 - r_1 + q_1 E_1)^2(1-s) - r_1 \alpha m_2(r_2 - m_2)\right)$$

$$D = K^3 \alpha m_2 (1-s)^2 (r_2 - m_2)(r_1 - m_1 - q_1 E_1)$$

There is one change of sign in above cubic equation, so one positive value say (x_1) will be obtained. Thus value of y_1 can be computed from (3.2), further y_1 exist if following inequality hold

$$x_1 > \frac{K(1-s)(r_1 - m_1 - q_1E_1)}{r_1} \qquad (3.4)$$

Hence, $P_1(x_1, y_1, 0)$ exist, provided condition (3.4) is satisfied.

For the Interior equilibrium point $P_2(x_2, y_2, z_2)$

From the 3rd equation of (2.1), we get

$$x_2 = \frac{a(d + q_2E_2)}{\beta_2 - (d + q_2E_2)} \qquad (3.5)$$

x_2 will be positive provided

$$\beta_2 > (d + q_2E_2) \qquad (3.6)$$

Substitute the value of x_2 in 2nd eq. of (2.1), Following quadratic equation will be obtained

$$A_1y_2^2 + A_2y_2 + A_3 = 0$$

Where

$$A_1 = \frac{r_2}{Ks}, \quad A_2 = (m_2 - r_2), \quad A_3 = \frac{m_1 a(d + q_2E_2)}{(d + q_2E_2) - \beta_2}$$

$$y_2 = \frac{Ks}{2r_2}\left[(r_2 - m_2) + \left\{ (m_2 - r_2)^2 + \frac{4r_2m_1x_2}{Ks} \right\}^{\frac{1}{2}}\right] \qquad (3.7)$$

y_2 will be positive, provided

$$\beta_2 > (d + q_2E_2), \quad r_2 - m_2 > 0 \qquad (3.8)$$

Using (3.5) and (3.7), z_2 will be calculated

$$z_2 = \frac{a + x_2}{\beta_1}\left[(r_1 - m_1 - q_1E_1)x_2 - \frac{r_1x_2^2}{K(1-s)} + m_2y_2\right] \qquad (3.9)$$

z_2 will exist, provided

$$\Rightarrow (r_1 - m_1 - q_1E_1)x_2 + m_2y_2 > \frac{r_1x_2^2}{K(1-s)} \qquad (3.10)$$

Hence, $P_2(x_2, y_2, z_2)$ exists provided conditions (3.8) and (3.10) are satisfied.

3.2 Stability Analysis

The variational matrix of the system of equations (2.1) is given below.

$$J(x,y,z) = \begin{bmatrix} r_1 - \frac{2r_1x}{K(1-s)} - m_1 - q_1E_1 - \frac{\beta_1 az}{(a+x)^2} & m_2 & -\frac{\beta_1 x}{a+x} \\ m_1 & r_2 - m_2 - \frac{2r_2y}{Ks} & 0 \\ \frac{\beta_2 az}{(a+x)^2} & 0 & -d - q_2E_2 + \frac{\beta_2 x}{a+x} \end{bmatrix}$$

Theorem 3.2 *If the equilibrium point* $P_0(0, 0, 0)$ *exist, then it will be always unstable.*

Proof: One of the eigenvalues is

$$\lambda_1 = -(d + q_2E_2) < 0 \text{ at } P_0(0, 0, 0)$$

Other two eigenvalues are given by

$$\left[\lambda^2 - \lambda(r_1 - m_1 - q_1E_1 + r_2 - m_2) + (r - q_1E_1)(r_2 - m_2) - m_1r_2\right] = 0$$

$$(r_1 - m_1 - q_1E_1) + (r_2 - m_2) > 0 \text{ (By assumption)}$$

So, all the eigenvalues of the above characteristics equation have not negative real parts as there is at least one change of sign. Therefore, the equilibrium point $P_0(0, 0, 0)$ is always unstable.

Biological Meaning: It is concluded that regardless of whether the system is misused ceaselessly in the unreserved zone, the prey or the predator populace persist and don't wiped out for sufficiently long time.

Theorem 3.3 *If the Equilibrium point* $P_1(x_1, y_1, 0)$ *exists then it will be locally asymptotically stable provided*

$$x_1 < \frac{a(d + q_2E_2)}{\beta_2 - d - q_2E_2}$$

Proof: One of the eigenvalue is

$$\lambda_1 = -d - q_2E_2 + \frac{\beta_2x_1}{a + x_1},$$

It will be negative

$$\text{if } x_1 < \frac{a(d + q_2E_2)}{\beta_2 - d - q_2E_2} \qquad (3.11)$$

Other two eigenvalues are given by

$$\lambda^2 + \left(\frac{2r_1x_1}{K(1-s)} + \frac{2r_2y_1}{Ks} - r_1 + m_1 + q_1E_1 - r_2 + m_2\right)\lambda$$
$$+ \left[\left(r_1 - m_1 - q_1E_1 - \frac{2r_1x_1}{K(1-s)}\right)\left(r_2 - m_2 - \frac{2r_2y_1}{Ks}\right) - m_1m_2\right] = 0$$

Both the eigens values of above quadratic equation will be negative

$$\frac{2r_1x_1}{K(1-s)} + \frac{2r_2y_1}{Ks} > r_1 - m_1 - q_1E_1 + r_2 - m_2,$$

$$\left(r_1 - m_1 - q_1E_1 - \frac{2r_1x_1}{K(1-s)}\right)\left(r_2 - m_2 - \frac{2r_2y_1}{Ks}\right) > m_1m_2 \qquad (3.12)$$

Thus equilibrium point $P_1(x_1, y_1, 0)$ of the system (2.1) is locally asymptotically stable provided, (3.11) and (3.12) hold.

Theorem 3.4 *For the system (2.1), if the Interior equilibrium point* $P_2(x_2, y_2, z_2)$ *exists, is always locally asymptotically stable provided*

$C_1 > 0, C_3 > 0$ and $C_1C_2 - C_3 > 0$ where C_1, C_2 and C_3 are given in the proof.

***Proof*:** The characteristic equation of variational matrix of the system (2.1) at P_2 is

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0 \qquad (3.13)$$

Where

$$C_1 = \frac{m_1 x_2}{y_2} + \frac{r_2 y_2}{Ks} + \frac{r_1 x_2}{K(1-s)} + \frac{m_2 y_2}{x_2} - \frac{\beta_1 z_2}{a + x_2} + \frac{\beta_1 a z_2}{(a+x_2)^2}$$

$$C_2 = \left(\frac{m_1 x_2}{y_2} + \frac{r_2 y_2}{Ks}\right)\left(\frac{r_1 x_2}{K(1-s)} + \frac{m_2 y_2}{x_2} - \frac{\beta_1 z_2}{a+x_2} + \frac{\beta_1 a z_2}{(a+x_2)^2}\right) + \frac{\beta_1 \beta_2 z_2 x_2 a}{(a+x_2)^3} - m_1 m_2$$

$$C_3 = \left(\frac{m_1 x_2}{y_2} + \frac{r_2 y_2}{Ks}\right)\frac{\beta_1 \beta_2 z_2 x_2 a}{(a+x_2)^3}$$

Therefore by the Routh-Hurwitz criteria (3.13) will have negative real parts iff $C_1 > 0, C_3 > 0$ and $C_1 C_2 - C_3 > 0$

Provided $\frac{2 r_1 x_2}{K(1-s)} + \frac{2 r_1 x_2}{K(1-s)} > (r_1 - m_1 - q_1 E_1) + (r_2 - m_2)$ holds

Hence the interior equilibrium point $P_2(x_2, y_2, z_2)$ will be locally asymptotically stable.

Biological Meaning: It is examined through the equations (3.5) and (3.7) that if the conversion coefficient of prey to predator and migration rate are sufficiently high, then all the three species will coexit and system will stable.

4. Bionomic Equilibrium

The bionomic equilibrium is said to be achieved if the revenue received by selling the harvested biomass is equal to the total cost involved in harvesting. The economic rent (revenue at any time) is given by

$$\Pi = TR(E) - TC(E)$$

Let c_1 and c_2 be the harvesting cost per unit effort, p_1 and p_2 be the price per unit biomass of the prey in the unreserved area and predator respectively. Therefore, the net economic revenue at any time *t* is given by

$$\Pi = (p_1 q_1 x - c_1) E_1 + (p_2 q_2 z - c_2) E_2$$

The Bionomic equilibrium point will be obtained by solving the following simultaneous equations

$$r_1 x\left(1 - \frac{x}{K(1-s)}\right) - m_1 x + m_2 y - \frac{\beta_1 x z}{a + x} - q_1 E_1 x = 0 \qquad (4.1)$$

$$r_2 y\left(1 - \frac{y}{Ks}\right) + m_1 x - m_2 y = 0 \qquad (4.2)$$

$$-dz + \frac{\beta_2 x z}{a + x} - q_2 E_2 z = 0 \qquad (4.3)$$

$$\Pi = (p_1 q_1 x - c_1) E_1 + (p_2 q_2 z - c_2) E_2 = 0 \qquad (4.4)$$

Case 1: *If* $c_2 > p_2 q_2 z$ and $c_1 < p_1 q_1 x$

Here the cost of harvesting of predator is greater than the revenue received and cost of harvesting of prey (fish) is less than the revenue. Hence the harvesting of a predator will be stopped and the only prey (fish) harvesting (in an unreserved area) remains operational. Thus we have

$$E_{2\infty} = 0, \; x_\infty = \frac{c_1}{p_1 q_1}$$

$$y_\infty = \frac{Ks}{2 r_2}\left[(r_2 - m_2) + \left((m_2 - r_2)^2 + \frac{4 r_2 m_1 c_1}{Ks p_1 q_1}\right)^{\frac{1}{2}}\right] \qquad (4.5)$$

$$y_\infty > 0 \text{ provided } r_2 > m_2 \qquad (4.6)$$

Since $c_1 < p_1 q_1 x < p_1 q_1 K(1-s)$

Hence $1 - \frac{c_1}{p_1 q_1 K(1-s)} > 0$

$(z_\infty, E_{1\infty})$ will be any point on the following line

$$m_1 + \beta_1 z + q_1 E_1 = r_1\left(1 - \frac{c_1}{p_1 q_1 K(1-s)}\right) + \frac{m_2 K s p_1 q_1}{2 r_2 c_1}\left[(r_2 - m_2) + \left((m_2 - r_2)^2 + \frac{4 r_2 m_1 c_1}{K s p_1 q_1}\right)^{\frac{1}{2}}\right]$$

Case 2: *If* $c_1 > p_1 q_1 x$ and $c_2 < p_2 q_2 z$

Here the harvesting cost of prey (in unreserved area) is greater than the revenue received and harvesting cost of predator is less than revenue. Hence, the harvesting of prey will be ceased and only predator harvesting remains operational. Thus, we have

$$E_{1\infty} = 0, \; z_\infty = \frac{c_2}{p_2 q_2} \qquad (4.7)$$

After simplification

$$x = \frac{1}{m_1}\left[\frac{r_2 y^2}{Ks} - (r_2 - m_2)\right] \qquad (4.8)$$

Substitute the value of (4.8) in (4.1), following 6[th] degree equation in *y* will be:

$$B_1 y^6 + B_2 y^4 + B_3 y^3 + B_4 y^2 + B_5 y + B_6 = 0 \qquad (4.9)$$

Where $E_3 = a m_1 - r_2 + m_2$

$$B_1 = \left[\frac{r_1 r_3^3}{K^3 m_1^2 (1-s) s^2}\right]$$

$$B_2 = -\frac{r_2^2}{m_1 K s}\left(\frac{2 r_1 (r_2 - m_2)}{K m_1 (1-s)} + (r_1 - m_1)\right)$$

$$B_3 = -m_2 r_2$$

$$B_4 = \begin{pmatrix} \dfrac{r_1 r_2 (r_2 - m_2)}{K m_1^2 (1-s)} \left(-2E_3 + (r_2 - m_2)\right) \\ + \dfrac{(r_1 - m_1) r_2}{m_1} (r_2 - m_2 - E_3) + \dfrac{\beta_1 c_2 r_2}{p_2 q_2} \end{pmatrix}$$

$$B_5 = -m_2 E_3 K s y$$

$$B_6 = \left(\frac{r_1 E_3 K s}{K m_1^2 (1-s)} \left((r_2 - m_2)^2 + \frac{r_2^2}{K^2 s^2} \right) \right) + K s (r_2 - m_2) \left(\frac{(r_1 - m_1) E_3}{m_1} - \frac{\beta_1 c_2}{p_2 q_2} \right)$$

Thus, at least one positive value of y exists and substitutes this value in (4.8) , (4.4), x and E_2 will be obtained.

Case 3: *If* $c_1 > p_1 q_1 x$ and $c_2 > p_2 q_2 z$

In this case, the fishing cost exceeds the revenue for both the prey (fish) in unreserved area and predator (bird), then we will obtain negative economic rent. Thus no harvesting will be done.

Case 4: *If* $c_1 < p_1 q_1 x$ and $c_2 < p_2 q_2 z$

Here, the revenue received is greater than the cost of harvesting of both the species *i.e*. prey (fish) in the unreserved area and predator. Hence, the harvesting of prey in unreserved area and predator is possible. Thus the nontrivial Bionomic equilibrium point

$B_{1\infty}(x_\infty, y_\infty, z_\infty, E_{1\infty}, E_{2\infty})$ will exist.

Hence

$$x_\infty = \frac{c_1}{p_1 q_1}$$

$$z_\infty = \frac{c_2}{p_2 q_2}$$

$$y_\infty = \frac{K s}{2 r_2} \left[(r_2 - m_2) + \left((m_2 - r_2)^2 + \frac{4 r_2 m_1 c_1}{K s p_1 q_1} \right)^{\frac{1}{2}} \right]$$

$$E_{1\infty} = \frac{p_1}{c_1} \begin{pmatrix} \dfrac{r_1 c_1}{p_1 q_1} \left(1 - \dfrac{c_1}{K p_1 q_1 (1-s)} \right) - \dfrac{m_1 c_1}{p_1 q_1} \\ + m_2 y_\infty - \dfrac{\beta_1 c_1 c_2}{p_2 q_2 (a p_1 q_1 + c_1)} \end{pmatrix}$$

$$E_{2\infty} = \left[-\frac{d}{q_2} + \frac{\beta_2 c_1}{q_2 (a p_1 q_1 + c_1)} \right]$$

5. Optimal Harvesting Policy

In this section, the objective is to maximize the present value of "J" of a continuous time stream of revenue given by

$$J = \int_0^\infty e^{-\delta t} \left((p_1 q_1 x - c_1) E_1(t) + (p_2 q_2 z - c_2) E_2(t) \right) dt \quad (5.1)$$

Here δ denotes the instantaneous annual rate of discount. By using Pontryagin's maximal principle (Clark [1]), we propose to maximize (5.1) subject to the state equations (2.1). The Hamiltonian function for the problem is given by

$$H = e^{-\delta t} \left((p_1 q_1 x - c_1) E_1(t) + (p_2 q_2 z - c_2) E_2(t) \right) + \lambda_1 \left[r_1 x \left(1 - \frac{x}{K(1-s)} \right) - m_1 x + m_2 y - \frac{\beta_1 x z}{a + x} - q_1 E_1 x \right] + \lambda_2 \left[r_2 y \left(1 - \frac{y}{K s} \right) + m_1 x - m_2 y \right] + \lambda_3 \left[-d z + \frac{\beta_2 x z}{a + x} - q_2 E_2 z \right]$$

Where $\lambda_1, \lambda_2, \lambda_3$ denoted as the adjoint variables and the control variables E_1 and E_2 appear linearly in the Hamiltonian function *H.*

According to Pontryagin's maximum principle

$$\frac{\partial H}{\partial E_1} = 0, \quad \frac{\partial H}{\partial E_2} = 0, \quad \frac{\partial \lambda_1}{\partial t} = -\frac{\partial H}{\partial x}, \quad \frac{\partial \lambda_2}{\partial t} = -\frac{\partial H}{\partial y}, \quad \frac{\partial \lambda_3}{\partial t} = -\frac{\partial H}{\partial z} \quad (5.2)$$

On taking the interior equilibrium $F(x_2, y_2, z_2)$

From the equations of (5.2)

$$\frac{\partial H}{\partial E_1} = 0 \Rightarrow \lambda_1 = e^{-\delta t} \left(p_1 - \frac{c_1}{q_1 x} \right) \quad (5.3)$$

$$\frac{\partial H}{\partial E_2} = 0 \Rightarrow \lambda_3 = e^{-\delta t} \left(p_2 - \frac{c_2}{q_2 z} \right) \quad (5.4)$$

$$\frac{\partial \lambda_2}{\partial t} - N_1 \lambda_2 = -N_2 e^{-\delta t}$$

$$\Rightarrow \lambda_2 = \frac{N_2}{N_1 + \delta} e^{-\delta t} \quad (5.5)$$

$$\frac{\partial \lambda_1}{\partial t} - \lambda_1 B_1 = -e^{-\delta t} B_2$$

$$\Rightarrow \lambda_1 = \frac{B_2}{B_1 + \delta} e^{-\delta t} \quad (5.6)$$

Where

$$N_1 = m_2 - r_2 + \frac{2r_2 y}{K\alpha}, \quad N_2 = m_2\left(p_1 - \frac{c_1}{q_1 x}\right)$$

$$B_1 = -r_1 + \frac{2r_1 x}{K(1-s)} + m_1 + q_1 E_1 + \frac{\beta_1 az}{(a+x)^2}$$

$$B_2 = \frac{m_1 N_2}{N_1 + \delta} + p_1 q_1 E_1 + \frac{\beta_2 az}{(a+x)^2}\left(p_2 - \frac{c_2}{q_2 z}\right)$$

From (5.3) and (5.6) equation of the singular path is obtained.

$$\left(p_1 - \frac{c_1}{q_1 x_2}\right) = \frac{B_2}{B_1 + \delta} \Rightarrow p_1 q_1 x_2 - c_1 = \frac{B_2 q_1 x_2}{B_1 + \delta} \qquad (5.7)$$

Where equation (5.7) can be written as

$$p_1 q_1 x_2 - c_1 = h(x_2)$$

$$\text{Where } h(x_2) = \frac{\left(\frac{m_1 N_2}{N_1 + \delta} + p_1 q_1 E_1 + \frac{\beta_2 az}{(a+x_2)^2}\left(p_2 - \frac{c_2}{q_2 z_2}\right)\right) q_1 x_2}{\delta - r_1 + \frac{2r_1 x_2}{K(1-s)} + m_1 + q_1 E_1 + \frac{\beta_1 az_2}{(a+x_2)^2}}$$

$$\text{Let } F(x_2) = h(x_2) - (p_1 q_1 x_2 - c_1) \qquad (5.8)$$

Then positive root of $F(x_2) = 0$ gives the optimal level of fish population in unreserved area at $x_2 = x_\delta$ in the interval $0 < x_\delta < K(1-s)$

If the following inequalities hold

$$F(0) < 0, \quad F(K) > 0, \quad F'(x_2) > 0, \quad \text{for } x_2 > 0 \qquad (5.9)$$

Knowing the value of $x_2 = x_\delta$, then y_δ and z_δ will be obtained from (3.7) and (3.9). Further optimal level of harvesting efforts will be computed from following equations.

$$E_{2\delta} = \frac{1}{q_2}\left[-d + \frac{\beta_2 x_\delta}{a + x_\delta}\right]$$

$$E_{1\delta} = \frac{1}{q_1 x_\delta}\left[r_1 x_\delta\left(1 - \frac{x_\delta}{K(1-s)}\right) - m_1 x_\delta + m_2 y_\delta - \frac{\beta_2 x_\delta z_\delta}{a + x_\delta}\right]$$

Here $E_{1\delta} > 0$ and $E_{2\delta} > 0$

If $x_\delta > \frac{da}{(\beta_2 - d)}$, $r_1 x_\delta + m_2 y_\delta > \frac{r x_\delta^2}{K(1-s)} + \frac{\beta_2 x_\delta z_\delta}{a + x_\delta} + m_1 x_\delta$ holds.

From (5.3), (5.4) and (5.5), it is observed that for optimal equilibrium, $\lambda_i(t)(i = 1, 2, 3)$ is independent of the time. Hence, they remain bounded as $t \to \infty$

6. Numerical Simulations

We choose different set of parameters.

$$\text{Let } r_1 = 6;\ r_2 = 3.7;\ m_1 = 0.5;\ m_2 = 0.7;$$
$$\beta_1 = 0.5;\ q_1 = 4;\ q_2 = 1;\ E_1 = 1.5;\ E_2 = 0.08;$$
$$a = 0.1;\ d = 0.05;\ K = 50;\ \beta_2 = 0.3;\ s = 0.4; \qquad (6.1)$$

For this set of parameters, equilibrium point P_2(0.0765, 16.229, 52.2501) exist and it is locally asymptotically stable by stability Theorem 3.4. Further it will always be globally stable in the absence of limit cycles. Here P_1 exist but unstable and P_0 is always unstable (**Fig. 1).**

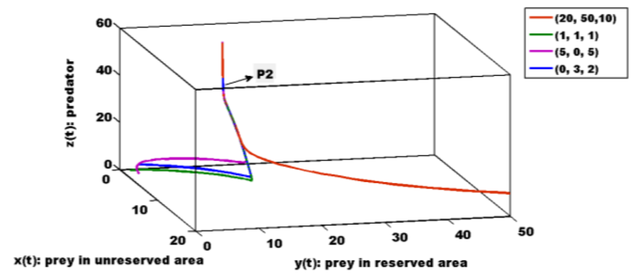

Fig. 1: The Phase diagram showing the global stability of P_2 for data set (6.1)

Now on considering the variation in the relative size of area (s), keeping all other parameters same as (6.1) (**Fig. 2, 3, 4).**

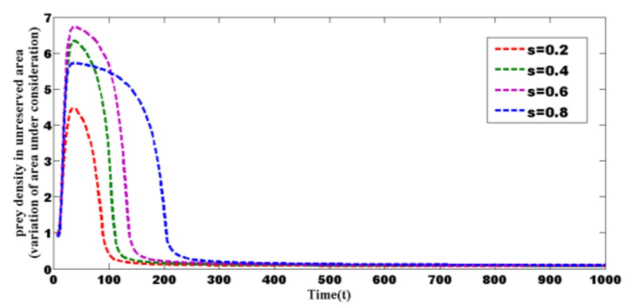

Fig. 2: Time series plot of $x(t)$, for same data set as (6.1) changing relative size of area (s)

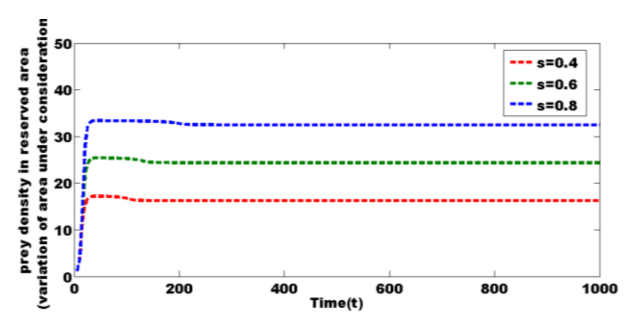

Fig. 3: Time series plot of $y(t)$, for same data set as (6.1) changing relative size of area (s)

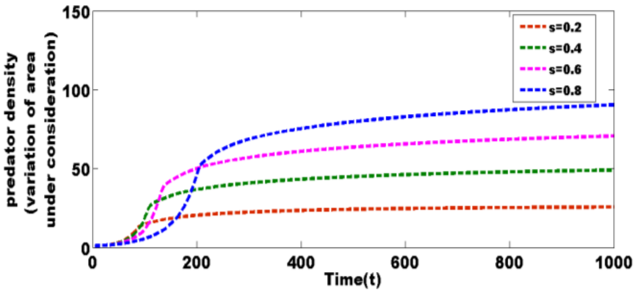

Fig. 4: Time series plot of $z(t)$, having same data set as (6.1) changing relative size of area (s)

Now on changing migration coefficients, keeping all other parameters same as (6.1) (**Fig. 5, 6,7**).

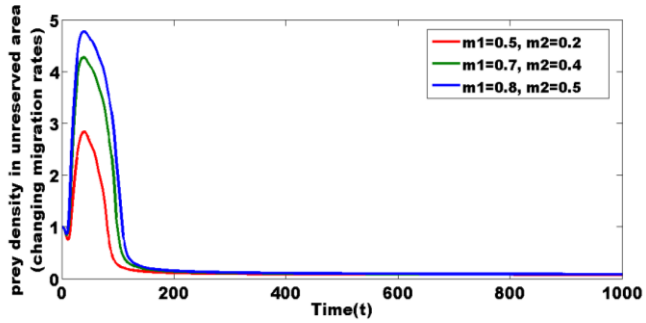

Fig. 5: Time series plot of x(t), having same data set as (6.1) changing migration coefficients

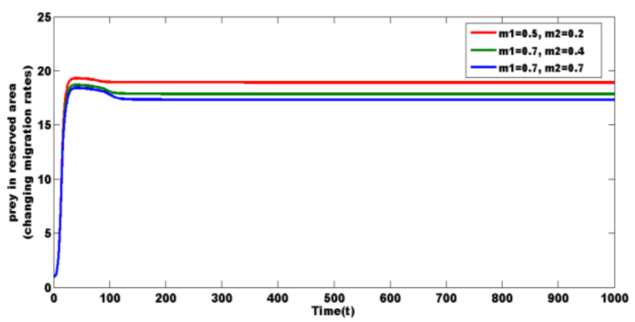

Fig. 6: Time series plot of y(t), having same data set as (6.1) changing migration coefficients

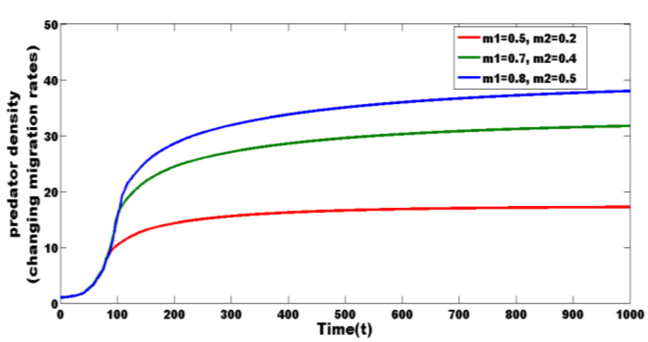

Fig. 7: Time series plot of z(t), having same data set as (6.1) changing migration coefficients

Further changing harvesting efforts rates (E_1, E_2), keeping all other parameters same as (6.1) (**Fig. 8, 9**).

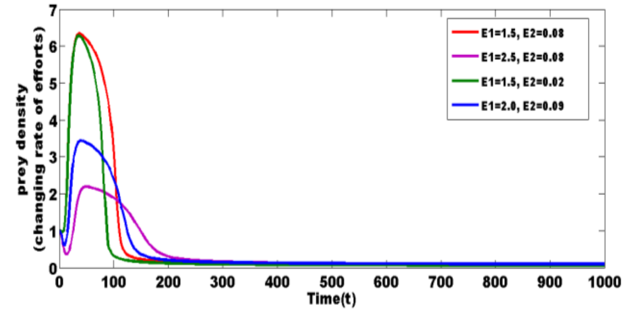

Fig. 8: Time series plot of x (t), having same data set as (6.1) changing harvesting efforts rates

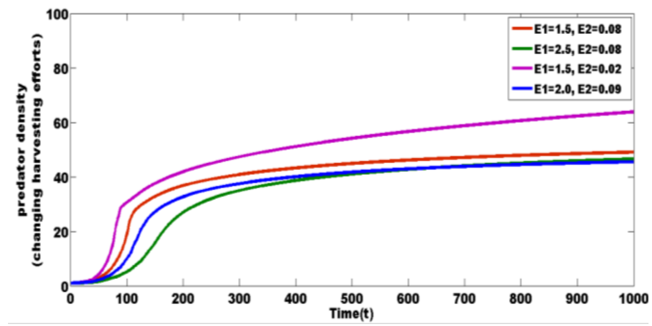

Fig. 9: Time series plot of z(t) having same data set as (6.1) changing harvesting efforts rates.

7. Conclusion

A prey-predator fishery model, with prey dispersal in a two-patch environment, have been proposed and analysed in the present paper. We observe that as the area under reserve region increases, the density of prey in reserved area as well as predator increases whereas density of prey in unreserved area remains same (shown in fig 2, 3, 4). Further, we observe that as the migration coefficients from unreserved area to reserved area and *vice versa* decrease, the density of prey in unreserved area remain stable, density of prey in reserved area increases and density of predator decreases (shown in fig 5, 6, 7). It is concluded that both increase in area size and increasing migration rate are responsible for the survival of predators otherwise predator will eliminate from the system. It is also examined that as the harvesting efforts rate for predator increases then predator population will extinct (shown in fig 8, 9). It is also examined that if the conversion coefficient of prey to predator and migration rate is sufficiently high, then all the three species will coexist and system will stable.
Optimal harvesting policy has been discussed by Pontryagin's maximum principle. Further It has been concluded that with the increase in discount rate, the economic rent decreases and as discount rate tends to infinity, then the economic rent even may tend to zero.

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