Extending A Consensus Measure To The Fuzzy Sets

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Abstract: William J. Tastle and et al. (Proc ISECON 2005, V22 (Columbus OH)) presented a new consensus measure for ranking sets of Likert scale data (ordinal data). In this paper we extend the consensus measure to the fuzzy sets. This measure called the strength of consensus is a modification of both the Shannon entropy, an equation common to the foundation of information theory, and the standard consensus measure.

Index Text: Likert scale; Fuzzy sets; Consensus measure; Fuzzy Consensus measure.

1. Introduction

In a classical group decision making situation there is a problem to solve, a solution set of possible alternatives, and a group of two or more experts, who express their opinions about this solution set of alternatives. These problems consist in multiple individuals interacting to reach a decision. Each expert may have unique motivations or goals and may approach the decision process from a different angle, but have a common interest in reaching eventual agreement on selecting the best option(s) citechen, fodor. To do this, experts have to express their preferences by means of a set of evaluations over a set of alternatives. However, the study of consensus in groups is fundamental to the understanding of group processes and the psychological experiences of individuals within groups and therefore, measuring consensus in groups is tricky. William J. Tastle and et al.(2005) have presented a new consensus measure for ranking sets of Likert scale data(ordinal data). In this paper we extend the consensus measure to the fuzzy sets, such that a new and interesting measure is defined for the consensus in fuzzy sets. This consensus measure can apply to comparing fuzzy sets. The paper is organized as follows: Section 2 and its subsections introduces some definitions and notion about entropy measure and consensus on the ordinal data, which will be used in the rest of this paper. In section 3, we extend consensus measure of ordinal data to the fuzzy sets. Finally, discussion and conclusion is mentioned in the last section.

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2. Basic definitions and notions

In this section we state some of the Basic definitions and notions.

2.1. Fuzzy sets

A fuzzy set is characterized by a generalized characteristic function $\mu_A : X \to [0,1]$ called the membership function of A and defined over a universe of discourse X. We restrict X to be a bounded subset of the real numbers (line). The set of all elements that have a nonzero degree of membership in A is called the support of A, i.e.

$$\sup p(A) = \{x \in X \,|\, \mu_A(x) > 0\}.$$

A notation convention for fuzzy sets when the universe of discourse, X, is discrete and finite, is as follows for a fuzzy set A:

$$A = \left\{ \frac{\mu_A(x_i)}{x_i}, i = 1,2,...,n \right\}.$$

$\mu_A(x)$ can be represented by real number ranging from zero to unity (including 0 and 1).while element x dose not completely belong to A and not completely un-belong to A, the membership function $\mu_A(x)$ with respect to this element will be a decimal fraction between 0 and 1. The bigger $\mu_A(x)$ is, the higher degree x belongs to A and vice versa. Hence membership extends the range of crisp set from {0,1} to [0,1]. In other word, membership represent the degree of certain element belong to the fuzzy set. It is subjective and variant depending on human perception; however, we still can describe the regularity of taking by objective statistical methods. Let A be a fuzzy set in the universe of discourse X with the membership function μ_A if $\exists x_i \in X$, such that $\mu_A(x_i) = 1$, then the fuzzy set A is called a normal fuzzy set.

If $\exists x_1, x_2 \in X$, such that

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)),$$

where $\lambda \in [0,1]$, then the fuzzy set A is called a convex fuzzy set. A fuzzy number is a fuzzy set in the universe of discourse X which is both normal and convex.

2.2. Membership function and probability density function

Definition 1. The membership function μ_A of a fuzzy number A can be defined as follows:

$$\mu_A(x) = \begin{cases} \mu_A^L(x), & x \in [a,b] \\ 1, & x \in [b,c] \\ \mu_A^R(x), & x \in [c,d] \\ 0 & otherwise. \end{cases} \quad \text{(I)}$$

Where $\mu_A^L : [a,b] \xrightarrow[R]{} [0,1]$, $\mu_A^R : [a,b] \to [0,1]$, μ_A^L denotes the left membership function of the fuzzy number A, and μ_A^R denotes the right membership function of the fuzzy number A.

Definition 2. A fuzzy number A= (a, b, c, d) is called a trapezoidal fuzzy number if its membership function A(x) has the following form:

$$A(x) = \begin{cases} \dfrac{x-a}{b-a} & x \in [a,b] \\ 1 & x \in [b,c] \\ \dfrac{d-x}{d-c} & x \in [c,d] \\ 0 & otherwise. \end{cases} \quad \text{(II)}$$

The membership functions of a fuzzy set can also be based on statistical data. In this case, it can be determined by

$$\mu(x) = kp(x), \qquad k = \frac{1}{\max\left[p(x)\right]}, \quad (2.1)$$

where p(x) is the probability density function or its estimate obtained from the histogram of the feature (X) considered for defining the fuzzy set. The above equation satisfies the possibility-probability consistency principle: the degree of possibility of an element is greater than or equal to its degree of probability. If the membership function is used as a grade for possibility, the consistency principle implies that:

$$\max_{x \in B}\left[\frac{\mu(x)}{\max(\mu(x))}\right] \geq \int_B p(x)dx, \quad for\ any\ set\ B \quad (2.2)$$

This way the probability density function can used to obtain the membership function approximating its overall shape. Also, if $\mu(x)$ be a membership function, then the probability density function is

$$p(x) = \frac{\mu(x)}{\int \mu(x)dx},$$

$$\left(p(x)\, is\ defined\ on\ the\ interval\ [a,b]\right),$$

$$p(x) = \frac{\mu(x)}{\sum \mu(x)dx},$$

$$\left(p(x)\, is\ defined\ on\ the\ set\ X = \{x_1, x_2, ..., x_n\}\right).$$

We shortly recall that probability distributions are defined through a probability vector $\pi = (p_1, p_2, ..., p_k)$:

$$p_i = P(x_i);\ 0 \leq 1;\quad \sum_{i=1}^{n} p_i = 1$$

2.3. Entropy

According to the Information theory, the more expanded the width of a fuzzy point gets, the bigger the entropy of the fuzzy point. The concept of entropy was used in thermodynamics first, and then it has been redefined in statistical Physics and mathematics (Shanon 1948). Then it has gotten a more abstract definition in the Information theory. For a given probability distribution

$$\pi = \{P(X_i) = p_i, i = 1, ..., n\} \text{ with } \sum p_i = 1,$$

the entropy of this distribution is defined as

$$H(\pi) = -\left(\frac{1}{ln2}\right)\sum p_i \ln p_i,$$

$$\left(p_i \ln p_i = 0\ when\ ever\ \ p_i = 0\right).$$

Here $H(\pi)$ is the weighted mean

$$I\left(P(X_i)\right) = \frac{-\ln p_i}{\ln 2},$$

which is called the information capacity of the event Xi. People are often confused on the meaning of entropy and think that the larger the entropy H, the larger the information capacity I, and the more the information has gotten. This is a mistake. Indeed, it is known that $I\left(P(X_i)\right)$ attains its maximum when the probability distribution, distributes uniformly: $\pi = (1/n, ..., 1/n)$; while I attains its minimum whenever the probability 1 is concentrated on one event:

$\pi = (0,...,1,...,0)$. It means that when our knowledge has grasped the total information about something to ensure that $P(X_i) = 1$ and $P(X_j) = 0$ ($when\ j \neq i$) the entropy is minimum; when our knowledge has no information on something, we only represent our knowledge by means of a uniform distribution; in this case, the entropy is maximum. Entropy reflects not the information we have grasped, but the information we want to grasp or that has been lost. The quantity I is also called the remained information. It is well known that the concept of entropy has been introduced into fuzzy sets theory: for a given fuzzy subset A on the real line $(-\infty, +\infty)$, the entropy of A is defined as

$$H(A) = -k\int \mu_A(x)\ln \mu_A(x)\,dx,$$

and for the discreet set X = {x_1, x_2, ..., x_n},

$$H(A) = -k\sum \mu_A(x)\ln \mu_A(x),$$

where k > 0 is a parameter. $H(A)$ is called the fuzzy entropy of a fuzzy subset in real line, which is the one of the measures of the fuzziness on fuzzy sets. The larger the fuzzy entropy, the more fuzzy its subset gets. For triangle fuzzy numbers, the wider the width, the more fuzzy the number and the bigger the information-loss gets. However, this measure is fundamental to the study of information theory and has played a strong role in fuzzy mathematics (Klir and Wierman, 1997, 1998).

2.4. Ordinal data and consensus measure

Ordinal scales are merely ordered categories, but the ordering makes comparisons implicitly pos-sible. A Likert scale is a type of psychometric response scale often used in questionnaires, and is the most widely used scale in survey research. When responding to a Likert questionnaire item, respon-dents specify their level of agreement to a statement. The scale is named after Rensis Likert, who published a report describing its use (Likert, 1932). Sample Question presented using a five-point Likert Scale A typical test item in a Likert scale is a statement, the respondent is asked to indicate their degree of agreement with the statement. Tra-ditionally a five-point scale is used, however many psychometricians advocate using a seven or nine point scale. Ice cream is good for breakfast.

1. Strongly disagree
2. Disagree
3. Neither agree nor disagree
4. Agree
5. Strongly agree

Likert scaling is a bipolar scaling method, measuring either positive and negative response to a statement. Sometimes Likert scales are used in a forced choice method where the middle option of "Neither agree nor disagree" is not available. Likert scales may be subject to distortion from several causes. Respondents may avoid using extreme response categories (central tendency bias); agree with statements as presented (acquiescence response bias); or try to portray themselves or their group in a more favorable light (social desirability bias). Also in example: Testing the temperature of a cup of tea would permit someone to use the words cold(1), tepid(2), lukewarm(3), warm(4), moderately hot(5), hot(6), and very hot(7) as their scale of comparative measure. There is no sense of interval scale in this measure and hence, equations such as cold + lukewarm = moderately hot, or the average of hot and very hot is hot and a half, is both impractical and illogical. Likert scales fall into this category of measures. The number of choices that may be selected as categories of a Likert scale are virtually without limit, although the five or seven category scales are the most prominent. With an increase in the number of categories, it may be argued that the accuracy of the category selected becomes crisper. However, regardless of the granularity of the categories, ordinal scales are merely ordered. The distance between each category, sometimes referred to as the interval, is incorrectly assumed to be equal. Interval scales possess a definite and fixed. In the following a measure for comprising ordinal data is introduced (Jennifer M. Tastle and William J. Tastle, 2006 EDSIG).

Definition 2.1. [5] The equation of the consensus measure is defined as:

$$C_{ns}(X) = 1 + \sum_{i=1}^{n} p_i \log_2\left(1 - \frac{|X_i - \bar{X}|}{d_X}\right) \quad (2.3)$$

where X is represented as the Likert scale (X = {1, 2, 3, 4, 5}), pi is the probability of the frequency associated with each X, d_X is the width of $\bar{X}$, X_i is the particular Likert attribute, and $\bar{X}$ is the mean of X. This measure developed for use in group decision-making activities, has emerged as a useful tool in understanding dispersion of ordinal data by transforming the unit interval values to percentages. Hence, the consensus measure can be interpreted to possess the same information as a weighted standard deviation measure, except that it is much easier to understand and utilize dispersion when represented as a percentage. Low consensus values can also be interpreted as having a high dispersion about the weighted mean value, and high consensus values have a low dispersion about the weighted mean. Utilizing the Likert scale has been a continuing problem when seeking ways in which to tabulate and compare the values of ordinal scales.

3. Extending consensus measure to the fuzzy sets

In this section the consensus measure to fuzzy sets and discuses on it.

Definition 3.1. Let

$$A = \left\{\frac{\mu_A(x_i)}{x_i}, i = 1, 2, ..., n\right\},$$

be a fuzzy set. We define the fuzzy consensus measure of A as

$$FC_{ns}(A)=1+\sum_{i=1}^{n}\frac{\mu_A(x_i)}{\sum\mu_A(x_i)}\ln\left(1-\frac{\left|x_i-M(A)\right|}{d_X^A}\right) \quad (3.1)$$

Where,

$$M(A)=\frac{\sum x_i\mu_A(x_i)}{\sum\mu_A(x_i)},\ d_X^A=\max(\sup p(A))-\min(\sup p(A)).$$

Example 3.1. In neighborhoods there may be several storm-water ponds draining to a single down-stream trunk sewer. In this neighborhood the city monitors all ponds for height of water caused by storm events. For two storms (labeled A and B) identified as being significant based on rainfall data collected at the airport, determine the corresponding performance of the neighborhood storm-water ponds. Suppose the neighborhood has five ponds, i.e., X = {1, 2, 3, 4, 5}, and suppose that significant pond storage membership is 1.0 for any pond that is 0.7 or more to full depth. For storm A the pond performance set is

$$A=\left\{\frac{0.6}{1},\frac{0.3}{2},\frac{0.9}{3},\frac{1}{4},\frac{1}{5}\right\}$$

For storm B the pond performance set is

$$B=\left\{\frac{0.8}{1},\frac{0.4}{2},\frac{0.9}{3},\frac{7}{4},\frac{1}{5}\right\}$$

We obtain

$$M(A)=\frac{1\times0.6+2\times0.3+3\times0.9+4\times0.7+5\times1}{0.8+0.4+0.9+0.7+1}=3.3947,\ d_X^A=5-1=4$$

$$FC_{ns}(A)=1+\frac{0.6}{3.8}\ln\left(1-\frac{|1-3.3947|}{4}\right)+\frac{0.3}{3.8}\ln\left(1-\frac{|2-3.3947|}{4}\right)+\frac{0.9}{3.8}\ln\left(1-\frac{|3-3.3947|}{4}\right)$$
$$+\frac{1}{3.8}\ln\left(1-\frac{|4-3.3947|}{4}\right)+\frac{1}{3.8}\ln\left(1-\frac{|5-3.3947|}{4}\right)=0.6083,$$
$$M(B)=3.8,\ FC_{ns}(B)=0.5794$$

$$H(A)-\sum\mu_A(x_i)\ln\mu_A(x_i)=0.6\ln(0.6)+0.3\ln(0.3)$$
$$+0.9\ln(0.9)+1\ln(1)+1\ln(1)=0.7625,\ H(B)=0.8895$$

Obviously, the consensus measure of fuzzy set A is larger than of B and the entropy set A is lower than the entropy set B.

Example 3.2. Let A = (1, 2, 5), B = (1, 4, 5) and C = (1, 3, 5) be triangular fuzzy numbers. We obtain

$$M(A)=\frac{\int x\mu_A(x)dx}{\int\mu_A(x)dx}=\frac{\int_1^2 x(x-1)dx+\int_2^5 x\left(\frac{5-x}{3}\right)dx}{\int_1^2(x-1)dx+\int_2^5\frac{5-x}{3}dx}=\frac{8}{3},\ d_X=5-1=4,$$

$$FC_{ns}(A)=1+\frac{2}{5}\left\{\int_1^2(x-1)\ln\left(1-\frac{\left|X-\frac{8}{3}\right|}{4}\right)dx+\int_2^5\frac{5-x}{3}\ln\left(1-\frac{\left|x-\frac{8}{3}\right|}{4}\right)dx\right\}=0.7945.$$

Similarly for fuzzy numbers B and C we get

$$M(B)=3,\ FC_{ns}(B)=0.8068;\quad M(C)=\frac{10}{3},\ FC_{ns}(C)=0.7945$$

Also we get

$$H(A)=-\int\mu_A(x)\ln\mu_A(x)dx=-\left(\int_1^2 x\ln(x)dx+\int_2^5\frac{5-x}{3}\ln\left(\frac{5-x}{3}\right)dx\right)=0.1137,$$
$$H(B)=1,\ H(C)=1$$

One can see that the consensus measure of the fuzzy numbers A and C are equal, but the consensus measure of fuzzy number B is larger than A and C. But the entropy of sets B and C are equal, and larger than the entropy of set A (see Fig.1). However, these examples show that for comparison of the fuzzy sets, the consensus measure is more suitable from the entropy measure.

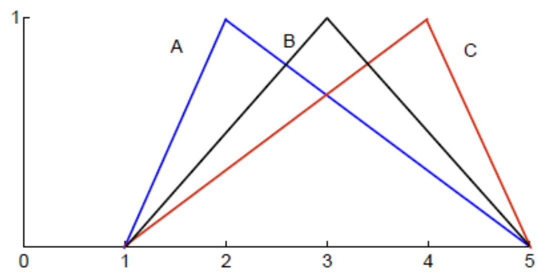

Fig.1. Fuzzy numbers A,B and C from Example 3.1.

Definition 3.2. Let $A_1, A_2, \ldots A_n$ be fuzzy numbers with finite support functions. We define the consensus measure of fuzzy numbers about point K as follows:

$$FC_{ns}^K(A_i)=1+\int\frac{\mu_{Ai}(x)}{\int\mu_{Ai}(x)dx}\ln\left(1-\frac{|x-K|}{2d_X}\right)dx \quad (3.2)$$

where,

d_X = max{max{supp(A_i)}, i = 1, 2, ...,n}– min{{min{supp(A_i)}, i = 1, 2, ..., n}, and K is a real number.

Remark 3.1. The value K can be selected as

$$K=M(A_i)=\frac{\int x\mu_A(x)dx}{\int\mu_{Ai}(x)dx},\ or\ K=\frac{\sum M(A_i)}{n}.$$

Remark 3.2. For ranking fuzzy numbers we can consider the value K as K = max{supp(Ai), i = 1, 2, ..., n}

Example 3.3. Consider the fuzzy numbers A,B and C from Example 3.2, and let K = 5. Then we get

$$FC_{ns}^K(A)=0.6441,\ FC_{ns}^K(B)=0.7028,\ FC_{ns}^K(C)=0.7568$$

One can see that the consensus measures of fuzzy numbers about the point K = 5 ordered as $FCns^K(A) < FCns^K(B) < FCns^K(C)$. Therefore the ranking order is $A \prec B \prec C$.

Note that in this example the value K is max{supp(A), supp(B), supp(C)}.

Example 3.4. Consider the symmetrical triangular fuzzy numbers A = (0, 2, 4) and B = (1, 2, 3) (see Fig.2). We get the consensus measure fuzzy numbers A and B about their means (M(A) = M(B) = 2) as FCns(A) = 0.8068 and FCns(B) = 0.9109. Also about the point K = 4, $FCns^K(A)$ = 0.7028 and $FCns^K(B)$ = 0.7099. Therefore the ranking order is $A \prec B$. Also we obtain H(A) = 1, H(B) = 0.5. Clearly, the entropy of fuzzy set A is larger than B.

Remark 3.3. We apply the Eq.3.2 for ranking fuzzy numbers, such that it is a new method and interesting to compare fussy sets.

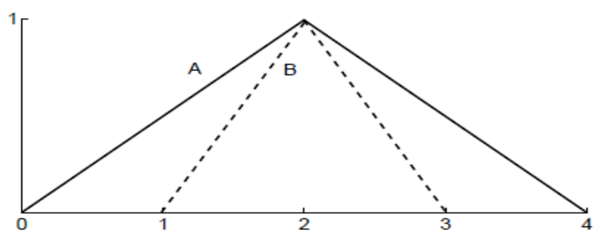

Fig.2. Fuzzy numbers A and B from Example 3.4.

Example 3.5. Consider the symmetrical triangular fuzzy numbers A = (2, 5, 8) and B = (3, 4, 9)(see Fig.3). We get the consensus measure fuzzy numbers A and B about their means $M(A) = \frac{10}{3}$, M(B)=3 as

$$FC_{ns}(A)=0.8217 \text{ and } FC_{ns}(B)=0.8385.$$

Also about the point K = 7,

$$FCns^K(A) = 0.6885 \text{ and } FCns^K(B) = 0.6559.$$

Therefore the ranking order is $B \prec A$.

Also we obtain $H(A) = 1.5, H(B) = 1.5$. Clearly, the entropy of fuzzy set A and B are equal.

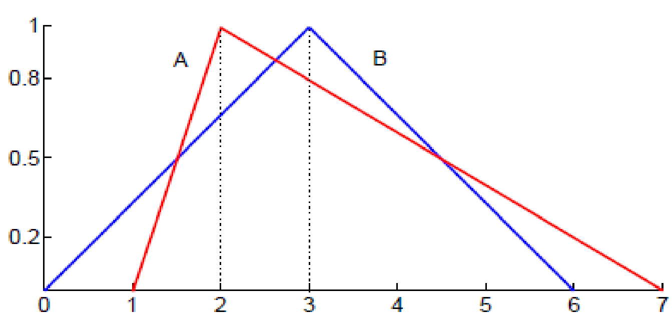

Fig.3. Fuzzy numbers A and B from Example 3.5.

4. Discussion and Conclusion

In this paper we have extended the consensus measure of ordinal data to the fuzzy sets. A new and interesting fuzzy consensus measure (FCns) is defined upon the fuzzy sets and have the following properties.

1. FCns is well defined the same as Cns.
2. FCns can be applied to compare fuzzy sets even the fuzzy numbers. Specially, ranking of fuzzy sets and numbers.
3. FCns is invariant with respect to location and scale.
4. FCns is suitable with respect to the entropy measure to compare fuzzy sets, because it use both the membership degree and the support function, but entropy use only the membership degree.
5. By assigning the K value to a focal point such as strongly degree, the FCns value is focused with respect to that point. Therefore, with change the K value, this measure can be used for comparing fuzzy sets. The examples illustrate the application of a variation in the fuzzy consensus measure.
6. FCns can be applied to compare fuzzy sets in the fuzzy analytic hieratically process (FAHP).

One can extend the FCns to compute soft consensus measures in fuzzy group decision making problems upon fuzzy data and analyzed their advantages and drawbacks in future researches.

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