Mixed Estimation In Seemingly Unrelated Regression Equation Model Some Finite Sample Properties Results

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Abstract: in (1964) Nagar and Kakwani analysis the results of the first and second moment that proposed by Theil and Goldberger (1961) ,derive the bias estimators and second moment matrix of mixed model estimators consider with the order of magnitude criteria ,derived biased estimators to order $O_P(T^{-1})$ that refer to big (O) in probability where (T) begin the number of observations. In this paper we derive mixed seemingly unrelated regression equations (SURE) by combining the prior information and sample information in a single model, derive the bias estimator to order $O_P(T^{-1})$ and the moment matrix to order $O_P(T^{-2})$ by using the methodology of Nagar's expansion for the moment of estimator. the bias has been derived conceder the normality distribution assumption of the random disturbances.

Keywords: mixed seemingly unrelated regression model, moment matrix, Nagar's expansion, prior information

1. Introduction

There are two popular types of the prior information. Exact linear prior information on the coefficient according to economic theorem, and the stochastic linear prior information it is involved exact part and stochastic part.Thus we have two types of estimators. First type is pure estimator that depends upon the data of original sample, second type that the mixed estimator obtained from unifying data of exact information and stochastic information. The problem of incorporating the two types of prior information discuss by Theil and Goldberger (1961) assumes additional information of the coefficient comes from previous statistical investigation, economic theorem and previous experience about the phenomena. , combining stochastic prior information with the available sample information when estimating the parameters of linear regression model to obtain the estimator of mixed regression model. In Section (2) presents the description of the model with their assumption and properties, estimating (sure) model under prior information. Section (3) drive mixed (sur) estimator. we are going to derive the feasible mixed (SUR) model and estimator, this paper ended with section (4) that we are going to derive the bias to order (T^{-1})& variance covariance matrix of the feasible mixed (SUR) regression estimator to order (T^{-2}) . This approximation in view of Nagar's procedure.

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(2)THE MODEL & ASSUMPTION

Let us consider the following model of (GLS)

$$\underset{(T.1)}{Y_i} = \underset{(T.k_i)}{X_i}\,\underset{(K_i.1)}{B_i} + \underset{(T.1)}{U_i} \qquad i=1,2 \qquad (2\text{-}1)$$

$$K = k_1 + k_2$$

That is the sample information

Where y_i is a $(\mathrm{T.1})$ vector of observations on the $i-th$ dependent variable (the variable to be "explained" by the $i-th$ regression equation). ,X_i is a $(T.k_i)$block diagonal matrix of observation on (K_i)nonstochastic independent variable, each column of which consists of T observation on a regressors (explanatory variable) in the $i-th$ equation of the model, with rank $(K_i \le T)$. ,B_i is a $(k_i.1)$vector of regression coefficients that unknown parameters in the $i-th$ equation of the model. u_i is the corresponding $(\mathrm{T.1})$ vector of random disturbances term in the $i-th$ regression equation. We can write (2-1) as

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}\begin{pmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{pmatrix} + \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

We have the following assumption that the disturbance terms have a normally distribution with zero mean vector and variance $\sigma_{ii} I$, $U_{T.1} \sim N_T(0, \sigma_{ii} I)$

$$\left.\begin{aligned} &E(u_i)=0_{(T.1)} \\ &Var(u_i)=E(u_i^2)=\sigma_{ii}I \qquad ,i=1,2 \\ &Cov(u_i,u_j)=E(u_i,u_j')=\sigma_{ij}I \qquad ,i\neq j \end{aligned}\right\} \quad (2\text{-}2)$$

$$\Sigma = E\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\begin{pmatrix} u_1' & u_2' \end{pmatrix} = \begin{pmatrix} \sigma_{11}I & \sigma_{12}I \\ \sigma_{21}I & \sigma_{22}I \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \otimes I = (\Sigma \otimes I)_{(T.T)}$$

Σ Nonsingular matrix

Where:

σ_{ii} is scalar and Represents the variance of the random disturbance in the $i-th$ equation for each observation in the sample, I_T is an identity matrix of order T ,and $E(.)$ denotes the usual expectation operation. , σ_{ij} Represents the covariance between the disturbances of $i-th$ equations and $j-th$ equations for each observation in the sample Assume that the regressors in two equations are orthogonal

$$x_1'x_2 = x_2'x_1 = 0$$

3. ESTIMATING MIXED (SURE) MODEL

Suppose the following stochastic linear extraneous prior information " it called stochastic linear restriction" on the regression coefficients of (2-1) is available and it can be written as.

$$\underset{(qx1)}{r} = \underset{(qxk)}{R}\ \underset{(kx1)}{\beta} + \underset{(qx1)}{V} \qquad (3\text{-}1)$$

Where:

r is a $(qx1)$ column random vector of "Known" elements , R is a (qxk) matrix of prior information of "known" elements with rank being $q(\le k)$, V is a $(qx1)$ disturbance random vector with (q) component which are "Independently" distributed of the element of u random vector.

Assume that.

$$E(V) = \underset{(qX1)}{0} ,$$

$$COV(V) = E(VV') = \underset{(qxq)}{\Psi} ,$$

Ψ is a (qxq) matrix of "Known" elements and nonsingular by unifying the sample information model (2-1) and the extraneous prior information (3-1) in a single model. We get

$$\underset{(TM+q)x1}{\begin{bmatrix} \underset{TMx1}{Y} \\ \underset{qx1}{r} \end{bmatrix}} = \underset{(TM+q)xk}{\begin{bmatrix} \underset{TMx}{X} \\ \underset{qxk}{R} \end{bmatrix}} \underset{(kx1)}{\beta} + \underset{(TM+q)x1}{\begin{bmatrix} \underset{TMx1}{u} \\ \underset{qx1}{V} \end{bmatrix}} \qquad (3\text{-}2)$$

rewrite model as

$$Y^* = ZB + W$$

Where

$$Y^* = \underset{(TM+q)x1}{\begin{bmatrix} \underset{TMx1}{Y} \\ \underset{qx1}{r} \end{bmatrix}} ,Z = \underset{(TM+q)xk}{\begin{bmatrix} \underset{TMx}{X} \\ \underset{qxk}{R} \end{bmatrix}} ,W = \underset{(TM+q)x1}{\begin{bmatrix} \underset{TMx1}{u} \\ \underset{qx1}{V} \end{bmatrix}}$$

We have assumption

$$E(w) = \underset{(qxk)}{0}$$

$$E(W) = E\begin{bmatrix} u \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underset{(TM+q)x1}{0}$$

$$E(WW') = \Phi$$

$$COV(W) = COV\begin{bmatrix} u \\ V \end{bmatrix} = E\begin{bmatrix} u \\ V \end{bmatrix}\begin{bmatrix} u' & V' \end{bmatrix} = \begin{bmatrix} \Sigma\otimes I & 0 \\ 0 & \Psi \end{bmatrix} = \Phi$$

In view of above assumption. Applying (GLS) we get

$$\hat{\beta}_M = \begin{pmatrix} b_{1M} \\ b_{2M} \end{pmatrix} = (Z'\Phi^{-1}Z)^{-1}(Z'\Phi^{-1}y^*)$$

$$\left(\begin{pmatrix} x' & R' \end{pmatrix} \begin{pmatrix} \Sigma\otimes I & 0 \\ 0 & \psi^{-1} \end{pmatrix} \begin{pmatrix} x \\ R \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} x' & R' \end{pmatrix} \begin{pmatrix} \Sigma\otimes I & 0 \\ 0 & \psi^{-1} \end{pmatrix} \begin{pmatrix} y \\ r \end{pmatrix} \right)^{-1}$$

Where

$$\Sigma^{-1} = \begin{pmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{pmatrix}$$

Applying Aitken's (GLS) procedure to obtain the Mixed (SUR) estimator $\hat{\beta}_M$ we get

$$\hat{\beta}_M = \begin{pmatrix} b_{1M} \\ b_{2M} \end{pmatrix} = \begin{bmatrix} \left(X'(\Sigma^{-1}\otimes I)X + R'\Psi^{-1}R\right)^{-1} \\ .\left(X'(\Sigma^{-1}\otimes I)y + R'\Psi^{-1}r\right) \end{bmatrix}$$

$$\widehat{\beta}_M=\begin{pmatrix}b_{1M}\\ b_{2M}\end{pmatrix}=\left[\begin{pmatrix}x_1' & 0\\ 0 & x_2'\end{pmatrix}\begin{pmatrix}\sigma^{11} & \sigma^{12}\\ \sigma^{21} & \sigma^{22}\end{pmatrix}\begin{pmatrix}x_1 & 0\\ 0 & x_2\end{pmatrix}+\begin{pmatrix}R_1' & 0\\ 0 & R_2'\end{pmatrix}\begin{pmatrix}\Psi_{11}^{-1} & 0\\ 0 & \Psi_{22}^{-1}\end{pmatrix}\begin{pmatrix}R_1 & 0\\ 0 & R_2\end{pmatrix}\right]^{-1}$$

$$.\left(\begin{pmatrix}x_1' & 0\\ 0 & x_2'\end{pmatrix}\begin{pmatrix}\sigma^{11} & \sigma^{12}\\ \sigma^{21} & \sigma^{22}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}+\begin{pmatrix}R_1' & 0\\ 0 & R_2'\end{pmatrix}\begin{pmatrix}\Psi_{11}^{-1} & 0\\ 0 & \Psi_{22}^{-1}\end{pmatrix}\begin{pmatrix}r_1\\ r_2\end{pmatrix}\right)$$

With simply calculation and Using orthogonal condition we get

$$b_{1M}=\left[\begin{matrix}\left(\sigma^{11}x_1'x_1+R_1'\Psi_{11}^{-1}R_1\right)^{-1}\\ .\left(\sigma^{11}x_1'y_1+\sigma^{12}x_1'y_2+R_1'\Psi_{11}^{-1}r_1\right)\end{matrix}\right] \quad (3\text{-}3)$$

b_{1M} is the Mixed (SUR) estimator.

Thus consider the alternative estimator s_{ij} derived by replacing σ_{ij} by s_{ij} then we get

$$\because s^{11}=s_{22}(s_{11}s_{22}-s_{12}^2)^{-1} \qquad s^{22}=s_{11}(s_{11}s_{22}-s_{12}^2)^{-1}$$

$$s^{21}=s^{12}=-s_{12}(s_{11}s_{22}-s_{12}^2)^{-1}$$

$$let\ \phi=(s_{11}s_{22}-s_{12}^2)^{-1}$$

$$b_{1\,FM}=\left(x_1'x_1+\frac{\phi}{s_{22}}R_1'\Psi_{11}^{-1}R_1\right)^{-1}\left(x_1'y_1-\frac{\phi s_{12}}{\phi s_{22}}x_1'y_2+\frac{\phi}{s_{22}}R_1'\Psi_{11}^{-1}r_1\right) \quad (3\text{-}4)$$

This is called the Feasible Mixed (SUR) Model (FMSUR), $b_{1\,FM}$ is the Feasible Mixed (SUR) estimator.

Now we are going to find $\frac{\phi}{s_{22}}$ where $s=(\sigma+\varepsilon)$

$$\frac{\phi}{s_{22}}=(\sigma_{11}+\varepsilon_{11})-(\sigma_{12}+\varepsilon_{12})^2(\sigma_{22}+\varepsilon_{22})^{-1} \quad (3\text{-}5)$$

$$(\sigma_{22}+\varepsilon_{22})^{-1}=\frac{1}{\sigma_{22}}\left(1+\frac{\varepsilon_{22}}{\sigma_{22}}\right)^{-1}$$

$$=\frac{1}{\sigma_{22}}\left(1-\frac{\varepsilon_{22}}{\sigma_{22}}+\frac{\varepsilon_{22}^2}{\sigma_{22}^2}-\frac{\varepsilon_{22}^3}{\sigma_{22}^3}+\ldots\ldots\right) \quad (3\text{-}6)$$

Substitute (3-6) in (3-5) and rearrange we get

$$=(\sigma_{11}+\varepsilon_{11})-\left(\frac{\sigma_{12}^2}{\sigma_{22}}-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}+\frac{\sigma_{12}^2\varepsilon_{22}^2}{\sigma_{22}^3}\right)+\left(\frac{\varepsilon_{12}^2}{\sigma_{22}}-\frac{\varepsilon_{12}^3}{\sigma_{22}^2}+\frac{\varepsilon_{12}^4}{\sigma_{22}^3}\right)+\left(\frac{2\sigma_{12}\varepsilon_{12}}{\sigma_{22}}-\frac{2\sigma_{12}\varepsilon_{12}\varepsilon_{22}}{\sigma_{22}^2}+\frac{2\sigma_{12}\varepsilon_{12}\varepsilon_{22}^2}{\sigma_{22}^3}\right)$$

$$=(\sigma_{11}+\varepsilon_{11})-\left(\begin{matrix}\frac{\sigma_{12}^2}{\sigma_{22}}+\left(-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}+\frac{2\sigma_{12}\varepsilon_{12}}{\sigma_{22}}\right)\\ +\left(\frac{\sigma_{12}^2\varepsilon_{22}^2}{\sigma_{22}^3}+\frac{\varepsilon_{12}^2}{\sigma_{22}}-\frac{2\sigma_{12}\varepsilon_{12}\varepsilon_{22}}{\sigma_{22}^2}\right)\\ +\left(-\frac{\varepsilon_{12}^3}{\sigma_{22}^2}+\frac{2\sigma_{12}\varepsilon_{12}\varepsilon_{22}^2}{\sigma_{22}^3}+\frac{\varepsilon_{12}^4}{\sigma_{22}^3}\right)\end{matrix}\right)$$

Then we have

$$\frac{\phi}{s_{22}}=\left[\begin{matrix}\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)_{O(T^0)}+\left(\sigma_{11}-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}\right)_{O(T^{\frac{-1}{2}})}\\ +\left(\sigma_{11}-\frac{\sigma_{12}^2\varepsilon_{22}^2}{\sigma_{22}^3}\right)_{O(T^{-1})}+\ldots\ldots\end{matrix}\right] \quad (3\text{-}7)$$

From (3-4) we have

$$b_{1\,FM}=B+\left(x_1'x_1+\frac{\phi}{s_{22}}R_1'\Psi_{11}^{-1}R_1\right)^{-1}.\left(x_1'u_1+\frac{\phi}{s_{22}}R_1'\Psi_{11}^{-1}V-\frac{s_{12}}{s_{22}}x_1'x_2B_2-\frac{s_{12}}{s_{22}}x_1'u_2\right)$$

Using orthogonal condition $x_1'x_2=x_2'x_1=0$

$$\left(b_{1\,FM}-B\right)=\left[\begin{matrix}\left(x_1'x_1+\frac{\phi}{s_{22}}R_1'\Psi_{11}^{-1}R_1\right)^{-1}\\ \left(x_1'u_1+\frac{\phi}{s_{22}}R_1'\Psi_{11}^{-1}V-\frac{s_{12}}{s_{22}}x_1'u_2\right)\end{matrix}\right]$$

Using (3-7) we get

$$\left(\underset{FM}{b_1}-B\right)=\begin{pmatrix} x_1'x_1+\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1 \\ +\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}R_1 \end{pmatrix}^{-1}$$

$$\cdot\begin{pmatrix} x_1'u_1+\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}V \\ +\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}V-\dfrac{s_{12}}{s_{22}}x_1'u_2 \end{pmatrix}$$

$$=\left(x_1'x_1+2\sigma_{11}R_1'\Psi_{11}^{-1}R_1-\frac{\sigma_{12}^2}{\sigma_{22}}R_1'\Psi_{11}^{-1}R_1-\frac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}R_1'\Psi_{11}^{-1}R_1\right)^{-1}$$

$$\cdot\begin{pmatrix} x_1'u_1+\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}V+\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}V \\ -\dfrac{\sigma_{12}}{\sigma_{22}}x_1'u_2 \end{pmatrix}$$

$$=\begin{pmatrix} \left(x_1'x_1+\left(2\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right) \\ -\dfrac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}R_1'\Psi_{11}^{-1}R_1 \end{pmatrix}^{-1}$$

$$\cdot\begin{pmatrix} x_1'u_1+\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}V+ \\ \left(\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}V-\dfrac{\sigma_{12}}{\sigma_{22}}x_1'u_2\right) \end{pmatrix}$$

Take a common factor $\left(x_1'x_1+\left(2\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)$

$$=\left[\left(x_1'x_1+\left(2\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)\left(I-\frac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}R_1'\Psi_{11}^{-1}R_1\left(x_1'x_1+\left(2\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)^{-1}\right)\right]^{-1}$$

$$\cdot\left(x_1'u_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}V+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}V-\frac{\sigma_{12}}{\sigma_{22}}x_1'u_2\right)$$

By using $(AB)^{-1}=B^{-1}A^{-1}$
Then

$$\left(\underset{FM}{b_1}-B\right)=\begin{pmatrix} I-\dfrac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}R_1'\Psi_{11}^{-1}R_1 \\ \left(x_1'x_1+\left(2\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)^{-1} \end{pmatrix}^{-1}$$

$$\cdot\left(x_1'x_1+\left(2\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)^{-1}$$

$$\cdot\begin{pmatrix} x_1'u_1+\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}V \\ +\left(\sigma_{11}-\dfrac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}V-\dfrac{\sigma_{12}}{\sigma_{22}}x_1'u_2 \end{pmatrix} \quad (3\text{-}8)$$

4. BIAS AND MOMENT MATRIX OF (FMSUR) ESTIMATOR

we are going to derive the bias to order (T^{-1}) & variance covariance matrix of the feasible mixed (SUR) regression estimator to order (T^{-2})

THEOREM (1)

The bias to order $O_P(T^{-1})$ is zero under the assumptions of the disturbance term that follow normal distribution

PROOF

From (3-8) we have

$$let\ \left(A_{\frac{-1}{2}}\right)^{-1}=\left(\frac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}R_1'\Psi_{11}^{-1}R_1\left(x_1'x_1+\left(2\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)\right)^{-1}$$

$$let\quad B_{-1}=\left(x_1'x_1+\left(2\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)^{-1}$$

$$let\quad C_{\frac{1}{2}}=\left(x_1'u_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}V\right)$$

$$let\quad D_0=\left(\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}V-\frac{\sigma_{12}}{\sigma_{22}}x_1'u_2\right)$$

$$\left(\underset{FM}{b_1}-B\right)=\left(I+A_{\frac{-1}{2}}\right)^{-1}B_{-1}\left(C_{\frac{1}{2}}+D_0\right)$$

Expanding the right hand side of this equation by using binomial expansion we get

$$=\left(I-A_{\frac{-1}{2}}+A_{\frac{-1}{2}}A_{\frac{-1}{2}}-A_{\frac{-1}{2}}A_{\frac{-1}{2}}A_{\frac{-1}{2}}\right)\left(B_{-1}C_{\frac{1}{2}}+B_{-1}D_0\right)$$

$$=\begin{pmatrix} \left(B_{-1}C_{\frac{1}{2}}\right)_{O(T^{\frac{-1}{2}})}+\left(B_{-1}D_0-A_{\frac{-1}{2}}B_{-1}C_{\frac{1}{2}}\right)_{O(T^{-1})} \\ +\left(A_{\frac{-1}{2}}A_{\frac{-1}{2}}B_{-1}C_{\frac{1}{2}}-A_{\frac{-1}{2}}B_{-1}D_0\right)_{O(T^{\frac{-3}{2}})} \\ +\left(A_{\frac{-1}{2}}A_{\frac{-1}{2}}B_{-1}D_0-A_{\frac{-1}{2}}A_{\frac{-1}{2}}A_{\frac{-1}{2}}B_{-1}C_{\frac{1}{2}}\right)_{O(T^{-2})} \\ +\left(A_{\frac{-1}{2}}A_{\frac{-1}{2}}A_{\frac{-1}{2}}B_{-1}D_0\right)_{O(T^{\frac{-5}{2}})} \end{pmatrix}$$

To derive the bias of estimator $\underset{FM}{b_1}$ to order $O(T^{-1})$ we are going to calculate the expectation of the first and second terms

$$E\left(B_{-1}C_{\frac{1}{2}}\right)=\left(x_1'x_1+\left(2\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)^{-1}$$

$$E\left(x_1'u_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}V\right)$$

$$E(u)=\underset{(TX1)}{0}\quad,\quad E(V)=\underset{(qX1)}{0}$$

$$E\left(B_{-1}C_{\frac{1}{2}}\right)=0$$

$$E\left(B_{-1}D_0\right)=\left(x_1'x_1+\left(2\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\right)^{-1}$$

$$\left(\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}^2}\varepsilon_{22}\right)R_1'\Psi_{11}^{-1}V-\frac{\sigma_{12}}{\sigma_{22}}x_1'u_2\right)$$

$$E(u)=\underset{(TX1)}{0}\quad,\quad E(V)=\underset{(qX1)}{0}$$

$$E\left(B_{-1}D_0\right)=0$$

From above expectation we get

$$E\left(\underset{FM}{b_1}-B\right)=E\left(\left(I+A_{-\frac{1}{2}}\right)^{-1}B_{-1}\left(C_{\frac{1}{2}}+D_0\right)\right)=0$$

Then $\left(\underset{FM}{b_1}\right)$ is unbiased estimator for (B)

THEOREM (2)

Under the assumptions of normal distribution of the disturbance term .the moment of feasible mixed (SUR) estimator $\left(\underset{FM}{b_1}\right)$ to order $\left(T^{-2}\right)$ is given by

$$\left(I-2A_{-\frac{1}{2}}'\right)(\varpi+2\Omega)+\mathrm{H}$$

Proof

Note the order of magnitude of

$$\left(\frac{\sigma_{12}^2}{\sigma_{22}}\right)\ is\ O(T^0)\ ,\quad \left(-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}+\frac{2\sigma_{12}\varepsilon_{12}}{\sigma_{22}}\right)\ is\ O(T^{\frac{-1}{2}})\ ,\quad \left(\frac{\varepsilon_{12}^4}{\sigma_{22}^3}\right)\ is\ O(T^{-2})\ ,$$

$$\left(\frac{\sigma_{12}^2\varepsilon_{22}^2}{\sigma_{22}^3}+\frac{\varepsilon_{12}^2}{\sigma_{22}}-\frac{2\sigma_{12}\varepsilon_{12}\varepsilon_{22}}{\sigma_{22}^2}\right)\ is\ O(T^{-1})\ ,$$

$$\left(-\frac{\varepsilon_{12}^3}{\sigma_{22}^2}+\frac{2\sigma_{12}\varepsilon_{12}\varepsilon_{22}^2}{\sigma_{22}^3}\right)\ is\ O(T^{\frac{-3}{2}})$$

The moment matrix of $\left(\underset{FM}{b_1}\right)$ estimator can be derive as

$$E\left(\underset{FM}{b_1}-B\right)\left(\underset{FM}{b_1}-B\right)'=E\left[\begin{array}{l}\left(\left(I+A_{-\frac{1}{2}}\right)^{-1}B_{-1}\left(C_{\frac{1}{2}}+D_0\right)\right)\\ \left(\left(I+A_{-\frac{1}{2}}\right)^{-1}B_{-1}\left(C_{\frac{1}{2}}+D_0\right)\right)'\end{array}\right]$$

Expanding the right hand side of this equation by using binomial expansion we get

$$=\left(\left(I-A_{-\frac{1}{2}}+A_{-\frac{1}{2}}A_{-\frac{1}{2}}\right)\left(B_{-1}C_{\frac{1}{2}}+B_{-1}D_0\right)\right)\left(\left(I-A_{-\frac{1}{2}}+A_{-\frac{1}{2}}A_{-\frac{1}{2}}\right)\left(B_{-1}C_{\frac{1}{2}}+B_{-1}D_0\right)\right)'$$

$$=\begin{pmatrix}B_{-1}C_{\frac{1}{2}}+B_{-1}D_0-A_{-\frac{1}{2}}B_{-1}C_{\frac{1}{2}}-A_{-\frac{1}{2}}B_{-1}D_0\\ +A_{-\frac{1}{2}}A_{-\frac{1}{2}}B_{-1}C_{\frac{1}{2}}+A_{-\frac{1}{2}}A_{-\frac{1}{2}}B_{-1}D_0\end{pmatrix}.\begin{pmatrix}B_{-1}C_{\frac{1}{2}}+B_{-1}D_0-A_{-\frac{1}{2}}B_{-1}C_{\frac{1}{2}}-A_{-\frac{1}{2}}B_{-1}D_0\\ +A_{-\frac{1}{2}}A_{-\frac{1}{2}}B_{-1}C_{\frac{1}{2}}+A_{-\frac{1}{2}}A_{-\frac{1}{2}}B_{-1}D_0\end{pmatrix}'$$

Expanding the right hand side and Rearrange these values according to the order of magnitude we get 12 items required to expected values of it,

$$=E\left(\begin{array}{l}\left(B_{-1}C_{\frac{1}{2}}C_{\frac{1}{2}}'B_{-1}\right)_{O(T^{-1})}+\begin{pmatrix}B_{-1}C_{\frac{1}{2}}D_0'B_{-1}-B_{-1}C_{\frac{1}{2}}C_{\frac{1}{2}}'B_{-1}A_{-\frac{1}{2}}'\\ +B_{-1}D_0C_{\frac{1}{2}}'B_{-1}-A_{-\frac{1}{2}}B_{-1}C_{\frac{1}{2}}C_{\frac{1}{2}}'B_{-1}\end{pmatrix}_{O(T^{\frac{-3}{2}})}\\ \begin{pmatrix}-B_{-1}C_{\frac{1}{2}}D_0'B_{-1}A_{-\frac{1}{2}}'+B_{-1}C_{\frac{1}{2}}C_{\frac{1}{2}}'B_{-1}A_{-\frac{1}{2}}'A_{-\frac{1}{2}}'\\ +B_{-1}D_0D_0'B_{-1}-B_{-1}D_0C_{\frac{1}{2}}'B_{-1}A_{-\frac{1}{2}}'\\ -A_{-\frac{1}{2}}B_{-1}C_{\frac{1}{2}}D_0'B_{-1}+A_{-\frac{1}{2}}B_{-1}C_{\frac{1}{2}}C_{\frac{1}{2}}'B_{-1}A_{-\frac{1}{2}}'-A_{-\frac{1}{2}}B_{-1}D_0C_{\frac{1}{2}}'B_{-1}\end{pmatrix}_{O(T^{-2})}+O(T^{\frac{-5}{2}})\end{array}\right)$$

Note that.

The Second term & the fourth term symmetric,

$$E\left(B_{-1}D_0C'_{\frac{1}{2}}B_{-1}\right)_{O(T^{\frac{-3}{2}})}=E\left(B_{-1}C_{\frac{1}{2}}D'_0B_{-1}\right)_{O(T^{\frac{-3}{2}})}$$

Third term & fifth term are symmetric, then

$$E\left(B_{-1}C_{\frac{1}{2}}C'_{\frac{1}{2}}B_{-1}A'_{\frac{-1}{2}}\right)_{O(T^{\frac{-3}{2}})}=E\left(A_{\frac{-1}{2}}B_{-1}C_{\frac{1}{2}}C'_{\frac{1}{2}}B_{-1}\right)_{O(T^{\frac{-3}{2}})}$$

Six term & 12-th are symmetric, then

$$E\left(A_{\frac{-1}{2}}B_{-1}C_{\frac{1}{2}}D'_0B_{-1}\right)_{O(T^{-2})}=E\left(B_{-1}D_0C'_{\frac{1}{2}}B_{-1}A'_{\frac{-1}{2}}\right)_{O(T^{-2})}$$

Ninth term & tenth are symmetric, then

$$E\left(B_{-1}C_{\frac{1}{2}}D'_0B_{-1}A'_{\frac{-1}{2}}\right)=E\left(A_{\frac{-1}{2}}B_{-1}D_0C'_{\frac{1}{2}}B_{-1}\right)$$

Then by using the expectation we get

$$E\left(\underset{FM}{b}-B\right)\left(\underset{FM}{b}-B\right)=E\left(B_{-1}C_{\frac{1}{2}}C'_{\frac{1}{2}}B_{-1}\right)_{O(T^{-1})}$$

$$+\left(2E\left(B_{-1}C_{\frac{1}{2}}D'_0B_{-1}\right)-2E\left(B_{-1}C_{\frac{1}{2}}C'_{\frac{1}{2}}B_{-1}A'_{\frac{-1}{2}}\right)\right)_{O(T^{\frac{-3}{2}})}$$

$$+E\begin{pmatrix}B_{-1}C_{\frac{1}{2}}C'_{\frac{1}{2}}B_{-1}A'_{\frac{-1}{2}}A'_{\frac{-1}{2}}+B_{-1}D_0D'_0B_{-1}\\-A_{\frac{-1}{2}}B_{-1}C_{\frac{1}{2}}C'_{\frac{1}{2}}B_{-1}A'_{\frac{-1}{2}}\end{pmatrix}_{O(T^{-2})}$$

$$-2E\left(B_{-1}D_0C'_{\frac{1}{2}}B_{-1}A'_{\frac{-1}{2}}\right)_{O(T^{-2})}-2E\left(B_{-1}C_{\frac{1}{2}}D'_0B_{-1}A'_{\frac{-1}{2}}\right)_{O(T^{-2})}$$

Then we get

$$E\left(\underset{FM}{b_1}-B\right)\left(\underset{FM}{b_1}-B\right)'=B_{-1}\left(\sigma_{11}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)^2R_1'\Psi_{11}^{-1}R_1\right)B_{-1}$$

$$+2\left(B_{-1}\begin{pmatrix}-\frac{\sigma_{12}^2}{\sigma_{22}}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)\\\left(\sigma_{11}-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}\right)R_1'\Psi_{11}^{-1}R_1\end{pmatrix}B_{-1}\right)$$

$$-2\left(B_{-1}\left(\sigma_{11}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)^2R_1'\Psi_{11}^{-1}R_1\right)B_{-1}A'_{\frac{-1}{2}}\right)$$

$$+B_{-1}\left(\sigma_{11}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)^2R_1'\Psi_{11}^{-1}R_1\right)B_{-1}A'_{\frac{-1}{2}}A'_{\frac{-1}{2}}$$

$$+B_{-1}\left(\left(\sigma_{11}-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}\right)^2R_1'\Psi_{11}^{-1}R_1+\left(\frac{\sigma_{12}^2}{\sigma_{22}}\right)x_1'x\right)B$$

$$-A_{\frac{-1}{2}}B_{-1}\left(\sigma_{11}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)^2R_1'\Psi_{11}^{-1}R_1\right)B_{-1}A'_{\frac{-1}{2}}$$

$$-2\left(B_{-1}\begin{pmatrix}-\frac{\sigma_{12}^2}{\sigma_{22}}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}\right)\\\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\end{pmatrix}B_{-1}A'_{\frac{-1}{2}}\right)$$

$$-2\left(A_{\frac{-1}{2}}B_{-1}\begin{pmatrix}-\frac{\sigma_{12}^2}{\sigma_{22}}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}\right)\\\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)R_1'\Psi_{11}^{-1}R_1\end{pmatrix}B_{-1}\right)$$

(i) Let

$$\omega=\left(\sigma_{11}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)^2R_1'\Psi_{11}^{-1}R_1\right)$$

Combine the first, third, fourth & sixth terms we get

$$\left(B_{-1}\omega\,B_{-1}\right)-2\left(B_{-1}\omega\,_{\frac{1}{2}}B_{-1}A'_{\frac{-1}{2}}\right)$$

$$+B_{-1}\omega\,B_{-1}A'_{\frac{-1}{2}}A'_{\frac{-1}{2}}-A_{\frac{-1}{2}}B_{-1}\omega\,B_{-1}A'_{\frac{-1}{2}}$$

$$=\left(B_{-1}\omega\,B_{-1}\right)\left(I-2A'_{\frac{-1}{2}}+A'_{\frac{-1}{2}}A'_{\frac{-1}{2}}-A_{\frac{-1}{2}}A'_{\frac{-1}{2}}\right)$$

$$=\left(B_{-1}\omega\,B_{-1}\right)\left(I-2A'_{\frac{-1}{2}}\right)$$

(ii) Let

$$\phi=\left(-\frac{\sigma_{12}^2}{\sigma_{22}}x_1'x_1+\left(\sigma_{11}-\frac{\sigma_{12}^2}{\sigma_{22}}\right)\left(\sigma_{11}-\frac{\sigma_{12}^2\varepsilon_{22}}{\sigma_{22}^2}\right)R_1'\Psi_{11}^{-1}R_1\right)$$

Combine the second, seventh & eighth terms we get

$$2\left(B_{-1}\phi\,B_{-1}\right)-2\left(B_{-1}\phi\,B_{-1}A'_{\frac{-1}{2}}\right)-2\left(B_{-1}\phi\,B_{-1}A'_{\frac{-1}{2}}\right)$$

$$=2\left(B_{-1}\phi\,B_{-1}\right)\left(\mathrm{I}-A'_{\frac{-1}{2}}-A'_{\frac{-1}{2}}\right)$$

$$=2\left(B_{-1}\phi\,B_{-1}\right)\left(\mathrm{I}-2A'_{\frac{-1}{2}}\right)$$

(iii) Let

$$\Phi = \left(\left(\sigma_{11} - \frac{\sigma_{12}^2 \varepsilon_{22}}{\sigma_{22}^2} \right)^2 R_1' \Psi_{11}^{-1} R_1 + \left(\frac{\sigma_{12}^2}{\sigma_{22}} \right) x_1' x \right)$$

Then the fifth term $= B_{-1} \Phi B_{-1}$

From combine these terms we get

$$= \left(B_{-1} \omega \ B_{-1} \right) \left(I - 2A'_{\frac{-1}{2}} \right) + 2\left(B_{-1} \phi \ B_{-1} \right) \left(\mathrm{I} - 2A'_{\frac{-1}{2}} \right) + B_{-1} \Phi B_{-1}$$

$$E\left(\underset{FM}{b_1} - B \right)\left(\underset{FM}{b_1} - B \right)' = \left(I - 2A'_{\frac{-1}{2}} \right) \begin{pmatrix} \left(B_{-1} \omega \ B_{-1} \right) \\ + 2\left(B_{-1} \phi \ B_{-1} \right) \end{pmatrix} + B_{-1} \Phi B_{-1}$$

$$let \ \varpi = \left(B_{-1} \omega \ B_{-1} \right), \quad \Omega = \left(B_{-1} \phi \ B_{-1} \right),$$

$$\mathrm{H} = \left(B_{-1} \Phi \ B_{-1} \right)$$

From above we get the variance covariance of feasible mixed (SUR) estimator $\left(\underset{FM}{b_1} \right)$ to order (T^{-2})

$$E\left(\underset{FM}{b_1} - B \right)\left(\underset{FM}{b_1} - B \right)' = \left(I - 2A'_{\frac{-1}{2}} \right)(\varpi + 2\Omega) + \mathrm{H}$$

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