# Expansion Dual To Channeled The Sample In A Series Of Shift-Invariant Space

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**Abstract:** In this article we show sampling expansion formulations on a series of shift-invariant closed sub space $\sum_{j=1}^{\infty} V\,(\varphi(t_j))$ of $L^2(\mathbb{R})$ generated by a Riesz generator series $\sum_{j=1}^{\infty} \varphi(t_j)$ or frames. Moreover we illustration a single channel sampling on a series $\sum_{j=1}^{\infty} V\,(\varphi(t_j))$ . Finally, examples are given to support our results.

**Index Terms:** Channeling, Expansion, Frames, Sampling, Shift-invariant spaces.

## 1 INTRODUCTION

The channeled sampling in a series of shift-invariant spaces is a very common problem-solving tool in many branches of mathematical analysis. In recent decades, Expansion Dual to channeled the Sample has been studied and there are many interesting articles published in this field of research(see for example Hong, et al. [1], Kim and Kwon [2], Zakria [3]). Nashed and Sun [4] studied the function spaces for sampling expansions. The generalized stable sample in a shift-invariant space was derived using some special double frames or Riesz bases in $L^2(0,1)$ by García and Pérez-Villalón [5], García, et al. [6], Han, et al. [7]. In addition, Kim and Kwon [2] studied the celebrated WSK (Whittaker–Shannon–Kotel'nikov)-sampling theorem says that any signal $f(t)$ of finite energy with band-width $\pi$, that is, $f \in PW_\pi$ can be reconstructed via its regularly spaced discrete sample values $\{f(n) : n \in \mathbb{Z}\}$ as $f(t) = \sum_{n\in\mathbb{Z}} f\,(n)\ sinc(t-n)$, which converges both in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$.This study is an attempt to study theories of double expansion to guide the sample in a series of shift-invariant spaces and we illustration that the sampling expansion methods and a single channel sampling on a series of shift-invariant closed subspace $\sum_{j=1}^{\infty} V\,(\varphi(t_j))$ of $L^2(\mathbb{R})$ generated via a Riesz generator series $\sum_{j=1}^{\infty} \varphi(t_j)$ or frames .

## 1. Series of shift-invariant spaces

The mathematical results of this section can be displayed succinctly as following lemmas

### Lemma (2.1)

For any $c = \{c(n)\}_{n\in\mathbb{Z}}$ and $d = \{d(n)\}_{n\in\mathbb{Z}}$ in $l^2$, let

$$c * d = \left\{(c * d)(n) = \sum_{k\in\mathbb{Z}} c(k)\ d(n-k)\right\}_{n\in\mathbb{Z}}$$

be the discrete complication product of c and d. Then

$$\hat{c}^*(\xi)\ \hat{d}^*(\xi)\ \sim \sum_{n\in Z}(\ c * d)(n)e^{-in\xi}$$
(1)

where $\sum_{n\in\mathbb{Z}}(\ c * d)(n)e^{-in\xi}$ is the Fourier series expansion of $\hat{c}^*(\xi)\ \hat{d}^*(\xi)\ \in\ L^1[0,2\pi]$, $c * d\ \in\ c_0$ and

$$\int_0^{2\pi}\left|\hat{c}^*(\xi)\hat{d}^*(\xi)\right|^2\ d\xi\ =\ 2\pi\sum_{n\in\mathbb{Z}}|\ (c * d)(n)|^2.$$
(2)

Proof . $\hat{\boldsymbol{c}}^*(\xi)$ and $\hat{d}^*(\xi)\ \in L^2[0,2\pi]$, $\hat{\boldsymbol{c}}^*(\xi)\ \hat{d}^*(\xi)\ \in\ L^1[0,2\pi]$ of which the Fourier series is

$$\hat{c}^*(\xi)\hat{d}^*(\xi)\ \sim\ \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\langle\hat{c}^*(\xi)\hat{d}^*(\xi), e^{-in\xi}\rangle_{L^2[0,2\pi]}\ e^{-in\xi}$$

from which (1) follows. Then $c * d\ \in\ c_0$ by Riemann Lebesgue lemma and (2) is a direct consequence of the Parseval's identity (see Kim and Kwon [2], Bhandari and Zayed [8]). In particular, (2) implies that $\hat{c}^*(\xi)\ \hat{d}^*(\xi)\ \in L^2[0,2\pi]$ iff $c * d\ \in\ l^2$.

### Lemma (2.2)

Set $\boldsymbol{c} = \{c(n)\}_{n\in\mathbb{Z}}\ \in\ l^2, \varphi(t_j)\ \in\ L^2(\mathbb{R})$ , and assume that $(\boldsymbol{c} * \varphi)(t_j)$ converges in $L^2(\mathbb{R})$. If $c \in l^2$ or $\{\varphi(t_j - n) : n \in \mathbb{Z}, j = 1,2,\dots,\infty\}$ is a Bessel sequence, then

$$\mathcal{F}[c * \varphi](\xi)\ =\ \hat{c}^*(\xi)\ \hat{d}^*.$$
(3)

Proof . Since $(\boldsymbol{c} * \varphi)(t_j)\ =\ \sum_{n\in\mathbb{Z}} c\ (n)\varphi(t_j - n)$ converges in $L^2(\mathbb{R})$, $\mathcal{F}[\boldsymbol{c} * \varphi](\xi)\ = \sum_{n\in\mathbb{Z}}(\ c(n)\ e^{-in\xi}\ \hat{\varphi}(\xi))$ Converges in $L^2(\mathbb{R})$, that is, $\hat{\boldsymbol{c}}_n\ (\xi)\ \hat{\varphi}(\xi) = \sum_{|k|\le n} c\ (k)\ e^{-ik\xi}\ \hat{\varphi}(\xi)$ converges to $\mathcal{F}[\boldsymbol{c} * \varphi](\xi)$ in $L^2(\mathbb{R})$. Hence to show (3), it is sufficient to show that $\hat{\boldsymbol{c}}_n\ (\xi)\ \hat{\varphi}(\xi)$ converges to $\hat{\boldsymbol{c}}^*(\xi)\ \hat{\varphi}(\xi)$ in $L^2(\mathbb{R})$. If $\boldsymbol{c}\ \in\ l^1$ or $\{\varphi(t_j - n) : n \in \mathbb{Z}, j = 1,2,\dots,\infty\}$ is a Bessel sequence. Now

$$\left\|\hat{c}^*{}_n\,(\xi)\hat{\varphi}(\xi) - \hat{c}^*(\xi)\hat{\varphi}(\xi)\right\|^2$$
$$= \int_{-\infty}^{\infty} |\hat{c}^*{}_n\,(\xi) - \hat{c}^*(\xi)|^2|\hat{\varphi}(\xi)|^2\ d\xi$$
$$= \int_0^{2\pi} |\ \hat{c}^*{}_n\,(\xi) - \hat{c}^*(\xi)|^2 G_\varphi(\xi)d\xi$$
$$\le \left\|\hat{c}^*{}_n\,(\xi)\ -\ \hat{c}^*(\xi)\right\|^2_{L^\infty\,[0,2\pi]} \int_0^{2\pi} G_\varphi\,(\xi)d\xi$$
$$\le\ \left\|G_\varphi(\xi)\right\|_{L^\infty\,[0,2\pi]} \int_0^{2\pi} \left|\hat{c}^*{}_n\,(\xi) - \hat{c}^*(\xi)\right|^2 d\xi.$$

Therefore $\|\hat{c}^*{}_n\,(\xi)\ \hat{\varphi}(\xi)\ -\ \hat{c}^*(\xi)\ \hat{\varphi}(\xi)\|\ =\ 0$ provided that either $c\ \in\ l^1$ so that $\hat{c}^*{}_n$ converges to $\hat{c}^*(\xi)$ uniformly on

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$[0,2\pi]$ or $\{\varphi(t_j - n) : n \in \mathbb{Z}, j = 1,2,\ldots,\infty\}$ is a Bessel sequence hence $G_\varphi(\xi) \in L^\infty [0,2\pi]$ by Kim and Kwon [2]

## 2. SAMPLING EXPANSION IN A SERIES OF SHIFT-INVARIANT SPACES

In this section we give explicit formulas for the sampling expansion in a series of shift-invariant spaces and for this we prove the following theorems.

**Theorem(3.1)**

Let $\varphi(t_j)$ is a frame generator of $V(\varphi(t_j))$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$.

(a) If $S(t)$ in $V(\varphi(t_j))$ such that $\{S(t_j - n) : n \in \mathbb{Z}, j = 1,2,\ldots,\infty\}$ is a Bessel sequence of $V(\varphi(t_j))$ and the sampling expansion formula

$$\sum_{j=1}^{\infty} f(t_j) = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} f(n) S(t_j - n), f \in V(\varphi(t_j)) \quad (4)$$

holds in the $L^2$ sense, then

$$supp\hat{\varphi} = supp\, \hat{S} \subset suppG_\varphi = suppG_S \subset supp\hat{\varphi}^* \quad (5)$$

and a constant $\beta \geq \alpha > 0$ such that

$$\alpha \leq |\hat{\varphi}^*(\xi)| (\alpha \leq |\hat{\varphi}^*(\xi)| \leq \beta) a.e. on\, suppG_\varphi. \quad (6)$$

Furthermore

$$\hat{S}(\xi) = \frac{(\hat{\varphi}(\xi)}{(\hat{\varphi}^*(\xi)} \chi suppG_\varphi(\xi) a.e. on\, \mathbb{R}\,. \quad (7)$$

(b) If $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$ and $S(t_j) \in V\left(\varphi(t_j)\right) : j = 1,2,\ldots,\infty$ such that (4) holds, then (5), (7) hold and

$$\frac{1}{(\hat{\varphi}^*(\xi)} \chi suppG_\varphi(\xi) \in L^2 [0,2\pi]\,. \quad (8)$$

**Proof.** (a) Let $\{S(t_j - n) : n \in \mathbb{Z}: j = 1,2,\ldots,\infty\}$ is a Bessel sequence of $V(\varphi(t_j))$ with a Bessel bound $B_S$ and the sampling expansion formula (4) holds .Then

$\sum_{j=1}^{\infty} S(t_j) = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} a(n) \varphi(t_j - n)$ and

$\sum_{j=1}^{\infty} \varphi(t_j) = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \varphi(n) S(t_j - n)$ ,

$a = \{a(n)\}_{n\in\mathbb{Z}}$ in $l^1$. Using Lemma (2.2) we get

$$\hat{S}(\xi) = \hat{a}^*(\xi)\hat{\varphi}(\xi)\, and\, \hat{\varphi}(\xi) = \hat{\varphi}^*(\xi)\, \hat{S}(\xi) \quad (9)$$

and thus

$$G_S(\xi) = |\hat{a}^*(\xi)|^2 G_\varphi(\xi) \text{ and } G_\varphi(\xi) = |\hat{\varphi}^*(\xi)|^2 G_S(\xi), \quad (10)$$

since (5) follows immediately. Also we have from (9)

$\hat{S}(\xi) = 0$ a.e. on (supp$\hat{\varphi}$)c and

$\hat{S}(\xi) = \frac{(\hat{\varphi}(\xi)}{(\hat{\varphi}^*(\xi)}$ a.e. on supp $(\hat{\varphi}^*(\xi)$ therefore (7) holds by (5).

Now (10) implies

$$|\hat{\varphi}^*(\xi)|^2 = \frac{G_\varphi(\xi)}{G_S(\xi)}\, a.e. on\, suppG_\varphi \quad (11)$$

therefore $\frac{A_S}{B_S} \leq \left|(\hat{\varphi}^*(\xi)\right|^2$ a.e. on $suppG_\varphi$,

where $(A_S, B_S)$ are frame bounds of

$\{\varphi(t - n) : n \in \mathbb{Z}\}$. If $\{S(t - n) : n \in \mathbb{Z}\}$ is also a frame of $V(\varphi(t_j))$ with frame bounds $(A_S, B_S)$, then (11) implies

$\frac{A_S}{B_S} \leq |\hat{\varphi}^*(\xi)|^2 \leq \frac{B_S}{A_S}$ a.e. on supp$G_\varphi$. Hence (6) holds.

(b) Assume $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^1$ and (4) holds on $V(\varphi(t_j))$ for some

$S(t_j) \in V(\varphi(t_j))$. Then (5) and (7) hold by the same arguments as in the proof of (a). Now we have from (7) and

$\chi suppG_\varphi(\xi) = \chi suppG_\varphi(\xi + 2\pi)$,

$$\infty > \int_{-\infty}^{\infty} \left|\hat{S}(\xi)\right|^2 d\xi = \int_{-\infty}^{\infty} \left|\frac{\hat{\varphi}(\xi)}{\hat{\varphi}^*(\xi)}\right|^2 \chi suppG_\varphi(\xi) d\xi$$

$$= \int_0^{2\pi} \frac{\hat{\varphi}(\xi)}{|\hat{\varphi}^*(\xi)|^2} \chi suppG_\varphi(\xi) d\xi$$

$$\geq \frac{A_\varphi}{2\pi} \int_0^{2\pi} \frac{1}{|\hat{\varphi}^*(\xi)|^2} \chi suppG_\varphi(\xi) d\xi$$

hence (8) holds. Theorem (3.1) gives some needed conditions for the sampling expansion formula (4) to hold. Conversely, we have:

**Theorem (3.2)**

Suppose $\varphi(t_j)$ is a frame generator of $V(\varphi(t_j))$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$. If β ≥ α > 0 therefore

$$\alpha \leq |\hat{\varphi}^*(\xi)| \leq \beta\, a.e.\, on\, suppG_\varphi(\xi), \quad (12)$$

and there exist a frame generator $S(t)$ of $V\left(\varphi(t_j)\right)$ for which

$$\sum_{j=1}^{\infty} f(t_j) = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} f(n) S(t_j - n) \quad (13)$$

holds for any $f(t) = (\boldsymbol{c} * \varphi)(t) \in V(\varphi(t_j))$ satisfying

$$\hat{\boldsymbol{c}}^*(\xi)\hat{\varphi}^*(\xi)(\xi) \in L^2[0,2\pi]\,. \quad (14)$$

If in addition $|\hat{\varphi}^*(\xi)| \leq \beta$ a.e. on $\mathbb{R}$, thus (4)–(7) hold and $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ for any $\in V(\varphi(t_j))$ .

**Proof .** Inequality (12) implies that

$\frac{1}{\hat{\varphi}^*(\xi)} \chi suppG_\varphi(\xi) \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$ thus

$\frac{1}{\hat{\varphi}^*(\xi)} \chi suppG_\varphi(\xi) = \sum_{n\in\mathbb{Z}} a(n) e^{-in\xi} = \hat{a}^*(\xi)$,

$a = \{a(n)\}_{n\in\mathbb{Z}} \in l^2$. Describe $\hat{S}(\xi)$ by (7) , that is,

$\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{\hat{\varphi}^*(\xi)} \chi suppG_\varphi = \hat{a}^*(\xi)\hat{\varphi}(\xi)$. Then

$\int_{-\infty}^{\infty} \left|\hat{S}(\xi)\right|^2 d\xi = \int_0^{2\pi} |\hat{a}^*(\xi)|^2\, G_\varphi(\xi) d\xi$

$\leq \left\|G_\varphi(\xi)\right\|_\infty |\hat{a}^*(\xi)|^2 d\xi < \infty$ hence $\hat{S}(\xi) \in L^2(\mathbb{R})$. Since

$$\hat{S}(\xi) = \hat{a}^*(\xi)\hat{\varphi}(\xi) = \sum_{n\in\mathbb{Z}} a(n) e^{-in\xi}\, \hat{\varphi}(\xi) \quad (15)$$

by Lemma (2.2) , we get by Fourier inversion

$\sum_{j=1}^{\infty} S(t_j) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{\infty} a(n)\varphi(t_j - n) \in V(\varphi(t_j))$ Now (15) implies $supp\hat{S} \subset supp\hat{\varphi} \subset suppG_\varphi$ th

$$\hat{\varphi}(\xi) = \hat{\varphi}^*(\xi)\hat{S}(\xi) a.e. \text{ on } \mathbb{R} \quad (16)$$

then (16) holds on $suppG_\varphi$ by (7) and $\hat{\varphi}(\xi) = \hat{S}(\xi) = 0$ a.e. on $(suppG_\varphi)^c$. Then (10) holds from Theorem (3.1), therefore

$$G_S(\xi) = \frac{G_\varphi(\xi)}{|\hat{\varphi}^*(\xi)|^2} \text{ on } supp\, \hat{\varphi}^* \supset suppG_\varphi = suppG_S. \quad (17)$$

Hence, we have by (12) and (17)

$$\frac{A_\varphi}{2\pi\beta_2} \leq GS(\xi) \leq \frac{B_\varphi}{2\pi\alpha_2} \text{ a.e. on } suppG_S \quad (18)$$

thus $\{S(t_j - n) : j = 1,2,\ldots\infty, n \in \mathbb{Z}\}$ is at least a Bessel sequence of $V(\varphi(t_j))$. Now for any $f(t) = (\boldsymbol{c} * \varphi)(t)$ in $V(\varphi(t_j))$ with $\boldsymbol{c} = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$,

$$\hat{f}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\, \hat{\varphi}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\, \hat{\varphi}^*(\xi)\, \hat{S}(\xi) \quad (19)$$

by (16). If $\hat{\boldsymbol{c}}^*(\xi)\, \hat{\varphi}^*(\xi) \in L^2[0,2\pi]$ , then $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ and $\hat{\boldsymbol{c}}^*(\xi)\, \hat{\varphi}^*(\xi) = \hat{f}^*(\xi) = \sum_{n\in\mathbb{Z}} f(n) e^{-in\xi}$ in $L^2[0,2\pi]$ by Lemma (2.1). Hence, we get

$$\hat{f}(\xi) = \hat{f}^*(\xi)\, \hat{S}(\xi) = \sum_{n\in\mathbb{Z}} (f(n) e^{-in\xi}\, \hat{S}(\xi)) \quad (20)$$

by Lemma (2.2) since $\{S(t - n) : n \in \mathbb{Z}\}$ is a Bessel sequence. Then we have (13) by taking Fourier inversion on (20).

Also we have from (19)

$\hat{f}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\hat{\varphi}^*(\xi)\hat{S}(\xi) = \hat{\boldsymbol{c}}^*(\xi)\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi)\hat{S}(\xi)$ since $supp\hat{S} \subset suppG_\varphi$ . Let $\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi) = \hat{d}^*(\xi) = \sum_{n\in\mathbb{Z}}(f(n)e^{-in\xi}$ be the Fourier series expansion of $\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi) \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$. Then

$\hat{c}^*(\xi)\hat{\varphi}^*(\xi)\chi suppG_\varphi(\xi) = \hat{c}^*(\xi)\hat{d}^*(\xi) = \sum_{n\in\mathbb{Z}}(c * d)(n)e^{-in\xi}$

thus $\hat{f}(\xi) = \sum_{n\in\mathbb{Z}}(c * d)(n)e^{-in\xi}\,\hat{S}(\xi)$ and therefore $\sum_{j=1}^{\infty} f(t_j) = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{\infty}(\boldsymbol{c} * d)(n)S(t_j - n), f \in V(\varphi(t_j))$.

Hence $V(S) = V(\varphi(t_j))$ thus (18) implies $\{S(t_j - n) : j = 1,2,\dots,\infty, n \in \mathbb{Z}\}$ is a frame of $V(\varphi(t_j))$. Finally, assume

$\alpha\chi suppG_\varphi(\xi) \le |\hat{\varphi}^*(\xi)| \le \beta\ a.e.\ on\ \mathbb{R}$. (25)

Then (14) holds for any $c = \{c(n)\}_{n\in\mathbb{Z}}$ in $l^2$ since $\hat{\varphi}^*(\xi) \in L^\infty[0,2\pi]$. Hence $\{f(n)\}_{n\in\mathbb{Z}} \in l^2$ for any $f \in V(\varphi(t_j))$ and (13) holds on $V(\varphi(t_j))$ that is, (4) holds. (5),(6),(7) then follows from (4) by Theorem (3.1).

**THEOREM(3.3)**

Assume that $\varphi(t_j)$ is a frame generator of $V(\varphi(t_j))$ and $\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$.Then there is a Riesz generator $S(t_j)$ of $V(\varphi(t_j))$ for which (4) holds iff $\varphi(t)$ is also a Riesz generator of $V(\varphi)$ and

$0 < \|\hat{\varphi}^*(\xi)\|_0 \le \|\hat{\varphi}^*(\xi)\|_\infty < \infty$. (23)

Furthermore in this case, we have, in addition to (21) and (22);

$S(t)$ is cardinal, i.e. $S(n) = \delta_{0,n}$ for $n \in \mathbb{Z}$ . (24)

Proof . Suppose that (4) holds on $V(\varphi(t_j))$ for some Riesz generator $S(t)$ of $V(\varphi(t_j))$. Then we have (9), (10) and thus (5). Since $suppG_\varphi = suppG_S = \mathbb{R}$, $\{\varphi(t_j - n): j = 1,2,\dots,\infty, n \in \mathbb{Z}\}$ must be a Riesz basis of $V(\varphi(t_j))$ therefore (21) and (22) hold by Lemma (2.3) in Kim and Kwon [2]. And (23) derives from (11):

$|\hat{\varphi}^*(\xi)|^2 = \frac{G_\varphi(\xi)}{G_S(\xi)}$ a.e. on $\mathbb{R}$ and (24) comes immediately from $S(t_j) = \sum_{n\in\mathbb{Z}} S(n)S(t_j - n) : j = 1,2,\dots,\infty$. Conversely, assume that $\varphi(t_j)$ is a Riesz generator of $V(\varphi(t_j))$and (23) hold. Explain $\hat{S}(\xi)$ by (22). Then $\hat{S}(\xi) = \hat{a}^*(\xi)\,\hat{\varphi}(\xi) \in L^2(\mathbb{R})$, where $\hat{a}^*(\xi) = \hat{\varphi}^*(\xi)^{-1} \in L^\infty[0,2\pi]$ hence $S(t_j) = (a * \varphi)(t_j) \in V\left(\varphi(t_j)\right) : j = 1,2,\dots,\infty$. The remainder of the proof is the same as in Theorem(3.2) .

**Theorem (3.4)**

Let $\varphi(t)$ is a frame generator of $V\left(\varphi(t_j)\right)$ and$\{\varphi(n)\}_{n\in\mathbb{Z}} \in l^2$ for some $\sigma$ in $[0,1)$.

(a) If there exist a frame generator $S_\sigma(t_j)$ of $V(\varphi)$ for which the regular shifted sampling expansion formula

$\sum_{j=1}^{\infty} f(t_j) = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{\infty} f(\sigma + n)S_\sigma(t_j - n)$, (25)

holds, then there are constants $\beta \ge \alpha > 0$ such that $\alpha \le |Z_\varphi(\sigma,\xi)| \le \beta$ a.e. on $suppG_\varphi$, $supp\,\hat{\varphi} = supp\ \hat{S}_\sigma \subset suppG_\varphi = suppG_{S_\sigma} \subset suppZ_\varphi(\sigma,\xi)$, and

$\hat{S}_\sigma(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma,\xi)}\chi suppG_\varphi(\xi)$.

(26)

(b) Conversely, if β ≥ α > 0 such that $\alpha\chi suppG_\varphi(\xi) \le |Z_\varphi(\sigma,\xi)| \le \beta$ a.e. on $\mathbb{R}$ then there exist a frame generator $S_\sigma(t)$ of $V(\varphi)$ for which (25) and (26) hold.

(c) There exist a Riesz generator $S_\sigma(t)$ of $V(\varphi)$ for which (25) holds iff $\varphi(t)$ is a Riesz generator and $0 < \|Z_\varphi(\sigma,\xi)\|_0 \le \|Z_\varphi(\sigma,\xi)\|_\infty < \infty$. Also, in this case, we have $S_\sigma(\sigma + n) = \delta_{0,n}$ for $n \in \mathbb{Z}$ and

$\hat{S}_\sigma(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\varphi(\sigma,\xi)}$ a.e. on $\mathbb{R}$.

Proof . Proofs of (a) , (b) and (c) are essentially the same as the ones in Theorems (3.1) and (3.2) respectively .

**Example (3.5)**

The Shannon function $\varphi(t) = sin\pi t/\pi t$ is a continuous real-valued Riesz generator and $\{\varphi(n)\}_{n\in\mathbb{Z}} = \{\delta_{0,n}\}_{n\in\mathbb{Z}}$. Since $\hat{\varphi}^*((\xi) = 1$ on $[0,2\pi]$ (see [2 ,4])

**Theorem (3.6)**

Let $\varphi(t) \in L^2(\mathbb{R})$ be a frame generator and $H(\xi) \in L^\infty(supp(\hat{\varphi}))$ a transfer function such that either $H(\xi) \in L^2(supp(\hat{\varphi}))$ or $\hat{\varphi}(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Set $\{C(\varphi)(n)\}_{n\in\mathbb{Z}} \in l^2$. Then there is a Riesz generator $S(t)$ of $V\left(\varphi(t_j)\right)$ for which the channeled sampling expansion formula

$\sum_{j=1}^{\infty} f(t_j) = \sum_{n\in\mathbb{Z}}\sum_{j=1}^{\infty} C(f)(n)S(t_j - n)\ , f \in V\left(\varphi(t_j)\right)$

holds iff $\varphi(t_j)$ is a Riesz generator of $V\left(\varphi(t_j)\right)$ and $0 < ||\widehat{C(\varphi)}^*(\xi)||_0 \le ||\widehat{C(\varphi)}^*||_\infty < \infty$. Furthermore in this case, $C(S)(t)$ is interpolatory, i.e $C(S)(n) = \delta_{0,n}$ for $n \in \mathbb{Z}$.

And $\hat{S}(\xi) = \frac{\hat{\varphi}(\xi)}{\widehat{C(\varphi)}^*(\xi)}(\xi)$ a.e. on $\mathbb{R}$.

**Example (3.7)**

Let $\varphi(t_j) = t\chi_{[0,1)}(t) + (2 - t)\chi_{[1,2)}(t_j) : j = 1,2,\dots,\infty$ be the cardinal B-spline of degree 1. Then $\varphi(t)$ is a continuous Riesz generator and

$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1 - e^{-i\xi}}{i\xi}\right)^2 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Take a transfer function $H(\xi) = e^{i\sigma\xi}$ with $0 \le \sigma < 1$. Then $C(\varphi)(t) = \varphi(t + \sigma)$ so that $C(\varphi)(\sigma) = \sigma$, $C(\varphi)(\sigma + 1) = 1 - \sigma$, and $C(\varphi)(\sigma + n) = 0$ for $n = 0,1$ .

Therefore $\widehat{C(\varphi)}^*\xi) = Z_\varphi(\sigma,\xi) = \sigma + (1 - \sigma)e^{-i\xi}$

Thus $\left|\widehat{C(\varphi)}^*\xi\right|_0 = |2\sigma - 1|$and $|\widehat{C(\varphi)}^*\xi(\xi)|_\infty = 1$.

Hence, by Theorem (3.4), for any $\sigma \in [0,1)\backslash\left\{\frac{1}{2}\right\}$, there is a Riesz generator $S(t)$ of $V\left(\varphi(t_j)\right)$ that we have the sampling expansion

$\sum_{j=1}^{\infty} f(t_j) = \sum_n\sum_{j=1}^{\infty} f(n + \sigma)S(t_j - n)$ on $V\left(\varphi(t_j)\right)$,

which converges not only in $L^2(\mathbb{R})$ but also uniformly on $\mathbb{R}$ since $sup_{\mathbb{R}}\ \sum_{n\in\mathbb{Z}}|\varphi(t_j - n)|2 < \infty$.

**Example (3.8)**

Let $\varphi(t) = sinct$ therefore $\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}(\xi)$. Then $\varphi(t_j)$ is an orthonormal generator of $V\left(\varphi(t_j)\right) = PW_\pi$ . Take a measurable function $H(\xi)$ on $\mathbb{R}$ such that $H(\xi)$ and $H(\xi)^{-1}$ in $L^\infty[-\pi,\pi]$. Then $H(\xi) \in L^2[-\pi,\pi]$ and $C(\varphi)(t_j) = \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}H(\xi)\chi_{[-\pi,\pi]}(\xi))$ thus

$\sum_{n\in\mathbb{Z}} |C(\varphi)(n)|^2 = \left\|C(\varphi)(t_j)\right\|^2 = \frac{1}{2\pi}||H(\xi)||^2_{L^\infty[-\pi,\pi]} < \infty$,

that is, $\{C(\varphi)(n)\}_{n\in\mathbb{Z}} \in l^2$. further, by the Poisson summation formula, $\widehat{C(\varphi)}^*(\xi) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \widehat{C(\varphi)}(\xi + 2n\pi) = H(\xi)$ on $[-\pi,\pi]$. Hence by Theorem (3.4),there is a Riesz generator $S(t) = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}H(\xi)}\chi_{[-\pi,\pi]}(\xi)\right)$ of $PW_\pi$ that we have the sampling expansion $\sum_{j=1}^{\infty} f(t_j) = \sum_n \sum_{j=1}^{\infty} C(f)(n)S(t_j - n)$ on $PW_\pi$, which converges not only in $L^2(\mathbb{R})$ but also uniformly on $\mathbb{R}$. It is exactly the single channel sampling presented in [9, 10].

**Corollary (3.9)**

Suppose that $\varphi_j(t)$ is a frame generators of $V\left(\varphi(t_j)\right)$ and $\{\varphi_j(n)\}_{n\in\mathbb{Z}} \in l^2$.Now there is a Riesz generators $S_j(t)$ of $V(\varphi_j)$ for which equation (4) holds iff $\varphi_j(t)$ is also a Riesz generators of $V(\varphi_j)$ and

$$0 < \sum_{j\in\mathbb{Z}}\left\|\widehat{\varphi_j}^*(\xi)\right\|_0 \le \sum_{j\in\mathbb{Z}}\left\|\widehat{\varphi_j}^*(\xi)\right\|_\infty < \infty.$$

Proof. Assume that (4) holds on $V(\varphi_j)$ for some Riesz generator $S_j(t)$ of $V(\varphi_j)$. Now we have (9), (10) and hence (5) . Since $suppG_{\varphi_j} = suppG_{S_j} = \mathbb{R}$, $\{\varphi_j(t-n) : j,n \in \mathbb{Z}\}$ must be a Riesz basis of $V(\varphi_j)$ hence (21) and (22) hold by (2.22) and (2.23) in [2].

Now (23) comes from (11): $\sum_{j\in\mathbb{Z}}|\widehat{\varphi_j}^*(\xi)|^2 = \sum_{j\in\mathbb{Z}}\frac{G_{\varphi_j}(\xi)}{G_{S_j}(\xi)}$ a.e. on $\mathbb{R}$ and (24) comes directly from $\sum_{j\in\mathbb{Z}} S_j(t) = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} S_j(n)S_j(t-n)$.Conversely, set $\varphi_j(t)$ is a Riesz generator of $V(\varphi_j)$ and (23) hold. Define $\widehat{S_j}(\xi)$ by (22). Then $\sum_{j\in\mathbb{Z}}\widehat{S_j}(\xi) = \sum_{j\in\mathbb{Z}}\hat{a}^*(\xi)\,\widehat{\varphi_j}(\xi) \in L^2(\mathbb{R})$, where $\hat{a}^*(\xi) = \sum_{j\in\mathbb{Z}}\widehat{\varphi_j}^*(\xi)^{-1} \in L^\infty[0,2\pi]$ therefore $\sum_{j\in\mathbb{Z}} S_j(t) = \sum_{j\in\mathbb{Z}}(a*\varphi_j)(t) \in \sum_{j\in\mathbb{Z}} V(\varphi_j)$. And this finishes the proof of corollary (3.9)

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