Minimum Dominating Extended Energy

M. R. Rajesh Kanna, S. Roopa

Abstract: In this manuscript we defined a new type of matrix called minimum dominating extended matrix and hence energy by using degree of an adjacent vertices. In this manuscript we have calculated minimum dominating extended energies for some standard graphs. Latterly, the manuscript lower and upper bounds for this extended energy are also obtained.

Index Terms: Minimum dominating set, Extended matrix, Extended energy, Eigenvalues.

1. INTRODUCTION

The perception of energy of a graph was led through I. Gutman [3] during 1978. G is a graph containing \( \eta \) vertices and \( \mu \) edges. Let \( A = (a_{ij}) \) is an adjacency matrix of a graph. Eigenvalues \( \kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_\eta \) of \( A \), expected in decreasing direction, be the eigenvalues of \( G \). Energy of a graph \( G \) is denoted by \( e(G) \) and is represented by

\[
e(G) = \sum_{i=1}^{\eta} |\kappa_i|.
\]

For particulars on the mathematical features of the theory of graph energy observe the documents [4] along with references mentioned in that.

1.1 Extended Energy

Extended matrix of \( G \) is the \( \eta \times \eta \) matrix defined by

\[
A_{ex}(G) := (a_{ij}),
\]

where

\[
a_{ij} = \begin{cases} \frac{1}{2} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) & \text{if } v_iv_j \in E(G) \\ 0 & \text{otherwise} \end{cases}
\]

The eigenvalues of \( A_{ex}(G) \) are called extended eigenvalues.

The extended energy [8] of \( G \) is described as

\[
E_{ex}(G) := \sum_{i=1}^{\eta} |\kappa_i| \quad \text{where } \kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_\eta \text{ are the eigenvalues of } A_{ex}(G).
\]

1.2 Minimum Dominating Extended (MDE) Energy:

MDE matrix of \( G \) is the \( \eta \times \eta \) matrix described by

\[
A_{ex}^D(G) := (x_{ij}),
\]

wherever

\[
x_{ij} = \begin{cases} \frac{1}{2} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) & \text{if } v_iv_j \in E(G) \\ 1 & \text{if } i = j \quad \text{& } v_i \in D \\ 0 & \text{otherwise} \end{cases}
\]

Here \( D \) is Minimum dominating collection of vertices of \( G \). MDE eigenvalues of the graph \( G \) are the eigenvalues of \( A_{ex}^D(G) \). If \( \kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_\eta \) are the eigenvalues of the matrix \( A_{ex}^D(G) \) then MDE energy of \( G \) is demarcated as

\[
E_{ex}^D(G) := \sum_{i=1}^{\eta} |\kappa_i|.
\]

Example 1: Probable minimum dominating collections for the succeeding graph \( G \) [Figure 1] are

1) \( D_1 = \{v_1, v_3\} \) 2) \( D_2 = \{v_2, v_5\} \) 3) \( D_3 = \{v_2, v_6\} \)

\[
\begin{align*}
\text{FIGURE 1:} \\
&
\begin{pmatrix}
1 & 17/8 & 17/8 & 0 & 0 & 0 \\
17/8 & 0 & 5/4 & 5/4 & 1 & 0 \\
0 & 5/4 & 0 & 0 & 5/4 & 0 \\
0 & 5/4 & 0 & 0 & 5/4 & 0 \\
0 & 1 & 5/4 & 5/4 & 1 & 17/8 \\
0 & 0 & 0 & 0 & 17/8 & 0 \\
\end{pmatrix}
\end{align*}
\]

Characteristic equation is

\[
\kappa^n - 2\kappa^{n-1} - \frac{489}{32}\kappa^4 + \frac{421}{32}\kappa^2 + \frac{211921}{4096}\kappa^2 - \frac{7225}{512} = 0.
\]

Minimum dominating extended eigenvalues are

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\[
\begin{align*}
\kappa_1 &= 0.0, \quad \kappa_2 = 0.26083960 \quad 87572508, \quad \kappa_3 = -2.19739302 \quad 1509463, \\
\kappa_4 &= -2.56552006 \quad 1822621, \quad \kappa_5 = 2.26474223 \quad 6256991, \\
\kappa_6 &= 4.23733123 \quad 8371842.
\end{align*}
\]

Minimum dominating extended energy,
\[
E_{D}^{\eta} (G) = 11.786657 \quad 7542142.
\]

\[ A_{\eta}^D (G) = \begin{cases}
0 & 17/8 \quad 0 \quad 0 \quad 0 \quad 0 \\
17/8 & 1 \quad 5/4 \quad 5/4 \quad 1 \quad 0 \\
0 & 5/4 \quad 0 \quad 0 \quad 5/4 \quad 0 \\
0 & 5/4 \quad 0 \quad 0 \quad 5/4 \quad 0 \\
0 & 1 \quad 5/4 \quad 5/4 \quad 1 \quad 17/8 \\
0 & 0 \quad 0 \quad 0 \quad 17/8 \quad 0
\end{cases}
\]

Characteristic equation is
\[
k^6 - 2k^5 - \frac{489}{32}k^4 + \frac{289}{32}k^3 + \frac{199121}{4096}k^2 = 0.
\]

Minimum dominating extended eigenvalues are
\[
k_{\eta} = 0.0, \quad \kappa_{\eta} = 0.0, \quad \kappa_{\eta} = 2.125, \quad \kappa_{\eta} = -2.12500000 \quad 0000001,
\]
\[
k_{\eta} = -2.43010568 \quad 350306, \quad \kappa_{\eta} = 4.43010568 \quad 350306.
\]

Minimum dominating extended energy,
\[
E_{D}^{\eta} (G) = 11.1102113 \quad 6706612.
\]

Minimum dominating extended energy is influenced by the dominating set.

2 Assets of Minimum Dominating Extended(MDE) Eigenvalues

Theorem 1. If \( \kappa_1, \kappa_2, \kappa_3, ..., \kappa_\eta \) are the eigenvalues of MDE matrix \( A_{\eta}^D (G) \) then

\[(i) \sum_{i=1}^{\eta} \kappa_{i} = |D| \quad (ii) \sum_{i=1}^{\eta} \kappa_{i}^2 = |D| + \sum_{i \neq j}^{\eta} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2
\]

where \( D \) is the MDE set.

Proof :- (a) We see that the totality of the eigenvalues of \( A_{\eta}^D (G) \) is the trace of \( A_{\eta}^D (G) \). \( \sum_{i=1}^{\eta} \kappa_{i} = \sum_{i=1}^{\eta} a_{ij} = |D| \).

b) Corresponding totality of squares of the eigenvalues of \( A_{\eta}^D (G) \) is the trace of \( \left( A_{\eta}^D (G) \right)^2 \).

\[
\sum_{i=1}^{\eta} \kappa_{i}^2 = \sum_{i=1}^{\eta} \sum_{j=1}^{\eta} a_{ij}^2 = \sum_{i=1}^{\eta} \sum_{j=1}^{\eta} \frac{d_i}{d_j} + \frac{d_j}{d_i} \frac{d_i}{d_j} + \frac{d_j}{d_i} = \sum_{i=1}^{\eta} \sum_{j=1}^{\eta} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2
\]

3. Restrictions For Minimum Domination Extended Energy

Parallel to McClelland’s [6] restriction for energy of a graph, restriction for \( E_{D}^{\eta} (G) \) are specified in the succeeding paragraph.

Theorem: (3.1) If \( D \) is Minimum dominating collection & \( R = |\det A_{\eta}^D (G) | \) then

\[
\sqrt{\left| D \right| + \frac{1}{2} \sum_{i \neq j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2} \leq \text{E}_{D}^{\eta} (G) \leq \eta \left| D \right| + \frac{1}{2} \sum_{i \neq j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2.
\]

Proof: Cauchy Schwarz variation stands

\[
\left( \sum_{i=1}^{\eta} a_{ij} \right)^2 \leq \left( \sum_{i=1}^{\eta} a_{ij}^2 \right) \left( \sum_{i=1}^{\eta} b_{ij}^2 \right)
\]

If \( a_j = 1, b_i = \kappa_i \) then \( \sum_{i=1}^{\eta} \kappa_{i} \leq \sum_{i=1}^{\eta} \kappa_{i}^2 \)

\[
\left[ E_{D}^{\eta} (G) \right]^2 \leq \eta \left| D \right| + \frac{1}{2} \sum_{i \neq j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2
\]

\[
E_{D}^{\eta} (G) \leq \sqrt{\eta \left| D \right| + \frac{1}{2} \sum_{i \neq j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2}
\]

Meanwhile arithmetic mean is greater than geometric mean, we take
\[
\frac{1}{\eta (\eta - 1)} \sum_{i,j} |\kappa_i| |\kappa_j| \geq \left[ \prod_{i,j} |\kappa_i| |\kappa_j| \right]^{1/(\eta - 1)} \\
= \left[ \prod_{i=1}^{\eta} |\kappa_i| \right]^{2/(\eta - 1)} \\
= \left[ \sum_{i=1}^{\eta} |\kappa_i| \right]^{2/(\eta - 1)} \\
= \text{det} \ A^{n\times n}(G) = R^2/2 \\
\therefore \sum_{i,j} |\kappa_i| |\kappa_j| \geq \eta (\eta - 1) R^2/2 . \tag{3.1}
\]

Now reflect,\[ E^e_{\nu}(G) ^2 = \left( \sum_{i,j} |\kappa_i| \right)^2 \\
= \sum_{i,j} |\kappa_i| ^2 + \sum_{i,j} |\kappa_i| |\kappa_j| \]
\[ \therefore \ E^e_{\nu}(G) ^2 \geq \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2 + \eta (\eta - 1) R^2/2 \quad \text{[From 3.1]}
\]
i.e., \[ E^e_{\nu}(G) \geq \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2 + \eta (\eta - 1) R^2/2 
\]

Theorem 3.2 If \( \kappa_1(G) \) is the biggest minimum dominating extended eigenvalue of \( \Lambda^D_{ex}(G) \), then
\[ \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \]
\[ \kappa_1(G) \geq \frac{\left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{\eta} . \]

Proof: Let \( Y \) be some nonzero vector. Then by [1], we have
\[ \kappa_1(A) = \max_{Y \neq 0} \frac{Y^T AY}{Y^T Y} \]
\[ \therefore \kappa_1(A) = \frac{\left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{\eta} . \]
Where \( J \)
remains a unit matrix \([1, 1, 1, ..., 1]\).

Alike to Koolen and Moulton's [5] upper bound for energy of a graph, higher bound for \( E^e_{\nu}(G) \) is specified in next theorem.

Theorem 3.3 If
\[ \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \geq \eta \]
then \[ E^e_{\nu}(G) \leq \frac{\left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{\eta} + \left( \eta - 1 \right) \left[ \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \right] \]

Proof. Cauchy-Schwartz variation stands
\[ \left( \sum_{i=1}^{\eta} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{\eta} a_i^2 \right) \left( \sum_{i=1}^{\eta} b_i^2 \right) \]
Put \( a_i = 1, b_i = \kappa_i \) then
\[ \left[ \sum_{i=1}^{\eta} |\kappa_i| \right]^2 = \left( \sum_{i=1}^{\eta} 1 \right) \left( \sum_{i=1}^{\eta} \kappa_i^2 \right) \]
\[ \geq \left( \sum_{i=1}^{\eta} |\kappa_i| \right)^2 \leq \left( \eta - 1 \right) \left( \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2 - \kappa_i^2 \right) \]
\[ \therefore \ E^e_{\nu}(G) \leq \kappa_1 + \sqrt{\left( \eta - 1 \right) \left( \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2 - \kappa_1^2 \right)} \]

Let \( f(x) = x + \sqrt{\left( \eta - 1 \right) \left( \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2 - x^2 \right)} \)
\[ \therefore \ x \geq \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \]

Since \[ \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) = \eta \]
we have
\[ \sqrt{\left( \eta - 1 \right) \left( \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \right) \leq \eta} \]
[From theorem 3.2]
\[ \therefore \ f(\kappa_1) = \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \]
\[ \therefore \ E^e_{\nu}(G) \leq f \left( \kappa_1 \right) = \left[ \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \right] \]

i.e., \[ E^e_{\nu}(G) \leq \left[ \left| D \right| + 2 \sum_{i,j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \right] \]

\[ \text{Milovanovic [7] bounds for MDE energy is proved below.} \]

Theorem 3.4 Let \( \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n \) be a decreasing directive of minimum dominating extended eigenvalues of \( \Lambda^{\nu}_{ex}(G) \) and \( D \) is minimum dominating collection then
Wherever \( \alpha(\eta) = \eta \left( \frac{\eta}{2} \right) \left( 1 - \left[ \frac{\eta}{2} \right] \right) \) in addition \([x]\) denotes integral portion of a real number.

**Proof:** Let \( r, r_1, r_2, \ldots, r_n, R \) and \( s, s_1, s_2, \ldots, s_n, S \) be real quantities such that \( r \leq r_1 \leq R \) and \( s \leq s_i \leq S \) \( \forall \ i = 1, 2, \ldots, n \). Then the subsequent variation is effective.

\[
\left| \sum_{i=1}^{n} r_i s_i - \sum_{i=1}^{n} r_i \sum_{i=1}^{n} s_i \right| \leq \alpha(\eta) (R - r) (S - s)
\]

where

\[
\alpha(\eta) = \eta \left( \frac{\eta}{2} \right) \left( 1 - \left[ \frac{\eta}{2} \right] \right)
\]

and equivalence clutches if and only if \( r_1 = r_2 = \ldots = r_n \) and \( s_1 = s_2 = \ldots = s_n \). If \( r_i = |\kappa_i|, s_i = |\kappa_i| \), \( r = s = |\kappa| \) and \( R = S = |\kappa| \), then

\[
\eta \left| \sum_{i=1}^{n} |\kappa_i|^2 - \left( \sum_{i=1}^{n} |\kappa_i| \right)^2 \right| \leq \alpha(\eta) \left( |\kappa| - |\kappa| \right)^2
\]

However,

\[
\sum_{i=1}^{n} |\kappa_i|^2 = |D| + \frac{1}{2} \sum_{i,j} \left( \frac{d_{ij}}{d_i} + \frac{d_{ij}}{d_j} \right)^2
\]

\[
E_{ns}^{\theta}(G) \leq \sqrt{\eta \left( \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_{ij}}{d_i} + \frac{d_{ij}}{d_j} \right)^2 \right)}
\]

then the overhead variation turn out to be

\[
\eta \left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_{ij}}{d_i} + \frac{d_{ij}}{d_j} \right)^2 \leq \alpha(\eta) \left( |\kappa| - |\kappa| \right)^2
\]

**Theorem 3.5.** Let \( |\kappa_1| \geq |\kappa_2| \geq \ldots \geq |\kappa_\eta| > 0 \) remain a decreasing direction of eigenvalues of \( A_{ns}^{\theta}(G) \) then

\[
\left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_{ij}}{d_i} + \frac{d_{ij}}{d_j} \right)^2 + \eta |\kappa_1| |\kappa_\eta| \geq \frac{E_{ns}^{\theta}(G)}{\left( |\kappa_1| + |\kappa_\eta| \right)}
\]

**Proof:** Let \( f_1 \neq 0, g_1, h \) and \( H \) stand actual numbers nourishing \( hf_1 \leq g_1 \leq Hf_1 \), then the succeeding variation grips. [Theorem 2, [7]]

\[
\sum_{i=1}^{n} g_i^2 + hH \sum_{i=1}^{n} f_i \leq (h + H) \sum_{i=1}^{n} f_i g_i
\]

Put \( g_i = |\kappa_i|, f_i = 1, h = |\kappa_\eta| \) and \( H = |\kappa_1| \) then

\[
\sum_{i=1}^{n} |\kappa_i|^2 + |\kappa_\eta|^2 \sum_{i=1}^{n} \leq \left( \sum_{i=1}^{n} |\kappa_i| + |\kappa_\eta| \right) \sum_{i=1}^{n} |\kappa_i|
\]

i.e., \( |D| + \frac{1}{2} \sum_{i,j} \left( \frac{d_{ij}}{d_i} + \frac{d_{ij}}{d_j} \right)^2 + |\kappa_1| |\kappa_\eta| \eta \leq \left( |\kappa_1| + |\kappa_\eta| \right) E_{ns}^{\theta}(G)
\]

\[
E_{ns}^{\theta}(G) \geq \frac{\left| D \right| + \frac{1}{2} \sum_{i,j} \left( \frac{d_{ij}}{d_i} + \frac{d_{ij}}{d_j} \right)^2 + |\kappa_1| |\kappa_\eta| \eta}{\left( |\kappa_1| + |\kappa_\eta| \right)}
\]

Bapat and S. Pati [2] demonstrated that if the graph energy is a rational number then it is an even integer. Comparable outcome for MDE energy is specified in the ensuing deduction.

**Theorem 3.6.** If the minimum dominating extended energy \( E_{ns}^{\theta}(G) \) is rational, then \( E_{ns}^{\theta}(G) = \left| D \right| \pmod{2} \).

**Proof:** The evidence is like the deduction 5.4 of [9].

## 4 Minimum Dominating Extended Energy of a few typical graphs

**Theorem (4.1)** For \( \psi \geq 2 \) MDE energy of \( K_\psi \) is \( (\psi - 1) + \sqrt{(\psi - 2)^2 - 2(\psi - 5)} \).

**Proof:** \( K_\psi \) is a complete graph with vertex collection \( V = \{v_1, v_2, \ldots, v_\psi\} \). Minimum dominating collection is \( D = \{v_1\} \).

MDE adjacency matrix is

\[
A_{ns}^{\theta}(K_\psi) = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2} & \ldots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1 & \ldots & \frac{1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
\end{bmatrix}
\]

Characteristic equation is

\[
(-1)\psi (\kappa + 1)^{\psi - 1} = 0.
\]

MDE spectrum is

\[
\text{Spec}_{ns}^{\theta}(K_\psi) = \begin{bmatrix}
-\psi + 1 & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \frac{\psi + 1}{2} \\
\frac{\psi + 1}{2} & \psi + 1 & \frac{\psi + 1}{2} & \ldots & \frac{\psi + 1}{2} \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \psi + 1 & \ldots & \frac{\psi + 1}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \psi + 1 \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \psi + 1 \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \psi + 1 \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \psi + 1 \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \psi + 1 \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \psi + 1 \\
\frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \frac{\psi + 1}{2} & \ldots & \psi + 1 \\
\end{bmatrix}
\]

MDE energy is

\[
\text{Spec}_{ns}^{\theta}(K_\psi) = \begin{bmatrix}
\frac{(-1)^{\psi - 1} + \sqrt{(\psi - 2)^2 - 2(\psi - 5)}}{2} \\
\frac{(-1)^{\psi - 1} + \sqrt{(\psi - 2)^2 - 2(\psi - 5)}}{2} \\
\frac{(-1)^{\psi - 1} + \sqrt{(\psi - 2)^2 - 2(\psi - 5)}}{2} \\
\vdots \\
\frac{(-1)^{\psi - 1} + \sqrt{(\psi - 2)^2 - 2(\psi - 5)}}{2} \\
\end{bmatrix}
\]
\[ \kappa_{\text{out}}(K_{\psi}) = \left| \psi - 2 \right| + \frac{(\psi - 1) + \sqrt{\psi^2 - 2\psi + 5}}{2} + \frac{(\psi - 1) - \sqrt{\psi^2 - 2\psi + 5}}{2} \]

\[ = (\psi - 2) + \sqrt{\psi^2 - 2\psi + 5}. \]

**Theorem 4.2.** MDE energy of \( K_{\psi+2} \), for \( \psi \geq 2 \) is
\[ (2\psi - 3) + \sqrt{4\psi^2 - 4\psi - 9}. \]

**Proof:** Let the vertices of \( K_{\psi+2} \) be \( V = \bigcup_{i=1}^{\psi} \{ u_i, v_i \} \).

Minimum dominating collection \( D = \{ u_1, v_1 \} \). Then
\[ A_{\text{ex}}(K_{\psi+2}) = \]

\[ A_{\text{ex}}^D(K_{\psi+2}) = \]

Characteristic equation is
\[ \kappa^{(\psi-1)(\kappa-1)(\kappa+2)}(\psi-1)^{(\psi-2)}(\kappa^2 - (2\psi - 3)\kappa - 2\psi) = 0. \]

The MDE Spectrum is
\[ \text{Spec}^D(K_{\psi+2}) = \{ 1, 0, -2 \} \]

MDE energy is
\[ c_{\text{ex}}(K_{\psi+2}) = \left| \psi - 1 \right| + \left| \psi - 2 \right| + \left( (\psi - 3) + \sqrt{4\psi^2 - 4\psi + 9} \right) \left. \right| 2 \]

\[ = 1 + 2(\psi - 2) + \sqrt{4\psi^2 - 4\psi + 9} \]

\[ = (2\psi - 3) + \sqrt{4\psi^2 - 4\psi + 9}. \]

**Theorem: (4.3)** For \( \psi \geq 2 \), MDE energy of \( K_{1,\psi-1} \) is identical to
\[ \sqrt{\psi^2 - 5\psi^4 + 12\psi^2 - 15\psi - 10 - 3}. \]

**Proof:** Consider the Star graph \( K_{1,\psi-1} \) with vertex collection \( V = \{ v_0, v_1, v_2, \ldots, v_{\psi-1} \} \). MD collection is \( D = \{ v_0 \} \). Then

MDE adjacency matrix is

Characteristic equation is
\[ (\kappa - 1)^{2^{\psi-2}}(\kappa + 1)^{2^{\psi-2}}(\kappa^2 - (\psi - 1)\kappa - 1)(\kappa^2 + (\psi - 3)\kappa - (2\psi - 3)) = 0 \]

MDE spectrum is
\[ \text{Spec}^D(S_{2\psi}) = \]

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Theorem 4.5. For $\psi \geq 2$, the minimum extended energy of Windmill graph $W_\psi^v$ is
\[
2\phi_\psi - 4\phi + 2 + \frac{\sqrt{(\phi - 1)\psi^3 + (2\phi - 2)\psi^2 + (\phi^2 - 6\phi + 9)\psi + (\phi - 1)\psi}}{\psi}
\]

Proof: Contemplate the Windmill graph $W_\psi^v$ with vertex collection $V = \{v_0, v_1, v_2, ..., v_{\psi - 1}\}$. MD collection is $D = \{v_0\}$. Then MDE adjacency matrix is
\[
\text{Spec}^v(W_\psi^v) = \begin{bmatrix}
\phi - 4 & -1 \\
\phi - 1 & 0 \\
\phi - 2 & 0 \\
\psi - 1 & 0 \\
\phi - 2 & 0 \\
\end{bmatrix}
\]

MDE energy
\[
E^v(W_\psi^v) = \frac{1}{2} \left( \psi - 2 \psi - 1 + 1 \psi - 2 \psi + \frac{\sqrt{(\phi - 1)\psi^3 + (2\phi - 2)\psi^2 + (\phi^2 - 6\phi + 9)\psi + (\phi - 1)\psi}}{\psi} \right)
\]

If $\psi$ is even then Characteristic polynomial is
\[
(-1)^{\psi/2} [x - (x - 2)]^{\psi/2} (x + 1)^{\psi/2} \left( \left[ \frac{\phi - 3}{2} \right] + (\phi - 3\phi + 2)\psi^2 - (2\phi^2 - 8\phi + 8)\psi + \left( \frac{\phi - 3}{2} \right) \right]
\]

Then energy is
\[
E^v(W_\psi) = \frac{1}{2} \left( \psi - 2 \psi - 1 + 1 \psi - 2 \psi + \frac{\sqrt{(\phi - 1)\psi^3 + (2\phi - 2)\psi^2 + (\phi^2 - 6\phi + 9)\psi + (\phi - 1)\psi}}{\psi} \right)
\]

Theorem 4.6. For $\phi > \psi$, the minimum dominating energy of Bipartite graph $K_{\phi,\psi}$ is
\[
\phi^\psi \left( \phi - 1 \right) + \sqrt{\phi^\psi \phi^2 + 2 \phi^\psi \phi^2 + \phi^2 \phi^2}.
\]

Proof. Vertex collection of $K_{\phi,\psi}$, $(\phi > \psi)$ be
MDE energy is,

\[
E_{\text{MDE}}(K_{\phi-1}) = \frac{1}{2\phi \psi - (\phi - 1) + \sqrt{\phi \psi - \phi \psi}} + \frac{\phi \psi - \phi \psi + 2 \phi \psi - \phi \psi}{2\phi \psi}
\]

CONCLUSIONS

In this article we defined Minimum dominating extended energy of a graph and established upper and lower bounds. This energy was computed for few typical graphs.

REFERENCES


