Umbilical Hypersurface Of A Generalized Recurrent Kaehlerian Weyl Spaces

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Abstract: In the present paper we have studied umbilical hypersurface of a generalized recurrent Kaehlerian Weyl spaces. An 2n —dimensional generalized recurrent Kaehlerian Weyl space with generalized recurrent Weyl Concircular curvature tensor and generalized recurrent Kaehlerian Weyl space with generalized recurrent Weyl Projective curvature tensor are defined. The condition for such hypersurface to be Concircular and Projective generalized recurrent have been shown.

Index Terms: Kaehlerian Weyl space, generalized recurrent Weyl space, Concircular generalized recurrent, Projective generalized recurrent.

1. INTRODUCTION

GAUGE invariant theory was introduced by H. Weyl in 1918 to unify gravity with electromagnetic theory. This theory was not accepted as a unified theory since the electromagnetic potential does not couple to spinor which was essential for the electromagnetic theory. This theory gained importance and is still studied in particle physics [1], invariant cosmology [2], and quantum mechanics. In complex spaces [2], Weyl geometry and geodesic of space time are studied. Physicists in quantum mechanics while dealing with phenomenon such as Berry phase (geometric phase), adiabatic transition probability in two level quantum system often rely on Kaehlerian space and Kaehlerian-Weyl space [3]. An n —dimensional differentiable manifold \( W_n \) is said to be Weyl space if it has a symmetric connection \( \nabla \) and a symmetric conformal metric tensor \( g_{ij} \) preserved by \( \nabla \). Accordingly, in local coordinates there exists a covariant vector field \( P_k \) (complementary vector field) satisfying the conditions [4], [5], and [6].

\[
\nabla_k g_{ij} - 2P_k g_{ij} = 0. \tag{1.1}
\]

The above equation can be extended to

\[
\partial_k g_{ij} - g_{hi} \Gamma^h_{jk} - 2P_k g_{ij} = 0, \tag{1.2}
\]

where \( \Gamma^h_{jk} \) are the connection coefficients of the symmetric connection \( \nabla \) and are defined as

\[
\Gamma^h_{jk} = \left\{ \begin{array}{l}
\{ h \\
\{ jk \}
\end{array} \right\} = g^{hm} (g_{mj} P_k + g_{mk} P_j - g_{jk} P_m), \tag{1.3}
\]

where \( \{ h \} \) being the coefficient of the metric connection defined by

\[
\{ h \} = \frac{1}{2} g^{hm} (\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk}). \tag{1.4}
\]

Moreover, under the renormalization condition

\[
\tilde{g}_{ij} = \lambda^2 g_{ij}, \tag{1.5}
\]

of the metric tensor \( g_{ij} \), the covariant vector field \( P_k \) is transformed by the law

\[
P_k = P_k + \partial_k n\lambda, \tag{1.6}
\]

where \( \lambda \) is a scalar function defined on \( W_n \). We denote such a Weyl space by \( W_n(\Gamma^h_{jk}, g_{ij}, P_k) \) or \( W_n(g, P) \).

An \( n \) —dimensional differential manifold having an anti —symmetric connection \( \nabla \) and anti —symmetric metric tensor \( g_{ij} \) preserved by \( \nabla \) is called generalized Weyl space [7]. It is denoted by \( GW_n(g, P) \). For such a space, in local coordinate system, the compatibility condition is

\[
\nabla_k g_{ij} - 2P_k g_{ij} = 0, \tag{1.7}
\]

where \( P_k \) are the components of a covariant vector field, called the complementary vector field of the \( GW_n(g, P) \) space. Using the concept of covariant differentiation ([8], [9]), the compatibility condition of (1.6) can be written as

\[
\partial_k g_{ij} - g_{hi} M^h_{jk} - 2P_k g_{ij} = 0, \tag{1.8}
\]

where \( M^h_{jk} \) are the connection coefficient of the anti —symmetric connection \( \nabla \) and are obtained from the compatibility condition as

\[
M^h_{jk} = \Gamma^h_{jk} + \frac{1}{2} [\chi^h_{km} g_{jh} + \chi^h_{mj} g_{ih} + \chi^h_{ij} g_{hm}] g^{mi}. \tag{1.9}
\]

On putting

\[
\chi^h_{jk} = \frac{1}{2} [\chi^h_{km} g_{jh} + \chi^h_{mj} g_{ih} + \chi^h_{ij} g_{hm}] g^{mi}, \tag{1.10}
\]

we obtain

\[
M^h_{jk} = \Gamma^h_{jk} + \chi^h_{jk}, \tag{1.11}
\]

where \( \Gamma^h_{jk} \) and \( \chi^h_{jk} \) are respectively the coefficient of a Weyl connection and the torsion tensor of \( GW_n(g, P) \) space and are expressed as

\[
\Gamma^h_{jk} = \frac{1}{2} [M^h_{ij} + M^k_{ji}] = M^h_{ij}, \tag{1.12}
\]

and

\[
\chi^h_{jk} = \frac{1}{2} [M^h_{ki} - M^h_{ik}] = M^h_{[ki]}, \tag{1.13}
\]

where square bracket stands for anti —symmetry.

The components of mixed curvature tensor and Ricci tensor of \( GW_n(g, P) \) are respectively

\[
M^h_{kl} = \partial_k M^h_{lj} - \partial_l M^h_{kj} + M^h_{lk} M^l_{ji} - M^l_{ki} M^h_{lj}, \tag{1.14}
\]

and

\[
L^a_{ij} = L^a_{ij}. \tag{1.15}
\]

On the other hand, the scalar curvature of \( GW_n(g, P) \) is defined by

\[
M = g^{ij} M_{ij}. \tag{1.16}
\]

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It is easy to see that curvature tensor \( M^h_{jk} \) of \( GW_n (g,T) \) can be written as
\[
M^h_{jk} = B^h_{jk} + \chi^h_{jk},
\]
where the tensors \( B^h_{jk} \) and \( \chi^h_{jk} \) are defined respectively as
\[
B^h_{jk} = \partial_k \Gamma^h_{ij} - \partial_i \Gamma^h_{jk} + \Gamma^h_{ikj} - \Gamma^h_{ikj},
\]
\[
\chi^h_{jk} = \nabla \chi^h_{jk} - \nabla \chi^h_{jk} + \chi^h_{ikj} - \chi^h_{ikj} - 2\chi^h_{kji}.
\]
(1.16)
(1.17)
(1.18)
The curvature tensor of \( GW_n (g,T) \) satisfies the relation [6].
\[
M^h_{jk} + M^h_{kj} = 0,
\]
(1.19)
\[
M^h_{ijk} + \nabla_k M^h_{ij} + \nabla_i M^h_{jk} = 2[M^i_{jk} \chi^h_{ik} + M^j_{ik} \chi^h_{kj} + M^l_{jk} \chi^h_{il} + M^l_{ik} \chi^h_{jl}],
\]
(1.20)
(1.21)
A Kaehlerian Weyl space denoted by \( KW_n \) is an \( n \)-dimensional space with an almost complex structure \( g \) satisfying
\[
\nabla \chi = 0,
\]
(1.22)
(1.23)
(1.24)
(1.25)
the tensors \( F_{ij} \) and \( F^{ij} \) are of weight 2 and 2 respectively [7].
The mixed curvature tensor \( R^h_{ijk} \) and the covariant curvature tensor \( R_{hijk} \) of \( W_n (g,T) \) are given respectively as
\[
R^h_{ijk} = \frac{\partial}{\partial \chi} \Gamma^h_{ij} - \frac{\partial}{\partial \chi} \Gamma^h_{ikj} + \Gamma^h_{ikj} - \Gamma^h_{ikj},
\]
(2.1)
and
\[
R_{hijk} = g_{ik} R^h_{ijk}.
\]
(2.2)
The Ricci tensor and the scalar curvature of \( W_n (g,T) \) are defined by
\[
R^h_{ih} = R_{ij} + g^{ij} R_{ij},
\]
(2.3)
Also, it can be seen that the anti-symmetric part of the Ricci tensor satisfies
\[
R_{ij} = n \nabla_i T_j
\]
(2.4)
where \( R_{i} = n R^i_{kj} - R^i_{k} \). Then the following relations hold
\[
H_{ij} = \frac{n-2}{n} R_{ij} + \frac{2}{n} R_{ij} = \frac{1}{n} (R_{ij} - R_{ij})
\]
(2.5)
(2.6)
Multiplying (2.5) by \( g^{ij} \) we get
\[
\nabla_i R_{ij} = \lambda g_{ij} + R_{ij}(n-1) g_{ij},
\]
(2.7)
Transvecting (2.3) by \( g^{ij} \) we have
\[
\nabla^R = \lambda^R + \mu g_{mn}(n-1) g_{ij},
\]
(2.8)
Eliminating \( g_{mn} \) from (2.1) and (2.4) we have
\[
\nabla^R S_{hijk} = K_n S_{hijk},
\]
(2.9)
where
\[
S_{hijk} = R_{hijk} - \frac{R}{n(n-1)} g_{hijk}.
\]
(2.10)
Multiplying (2.5) by \( g^{h} \) we get
\[
S_{ij} = R_{ij} - \frac{R}{n} g_{ij}.
\]
(2.11)
Such a space is denoted by \( W_n \).
The Weyl Concircular curvature tensor \( Z_{hijk} \) and Weyl Projective curvature tensor \( W_{hijk} \) in \( W_n (g,T) \) is given by
\[
Z_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{ij} g_{hk} - g_{ik} g_{hj}),
\]
(2.12)
and
\[
W_{hijk} = R_{hijk} - \frac{1}{n+1} (R_{ij} g_{hk} - R_{ik} g_{hj}).
\]
(2.13)
Let \( W_n (g,T) \) be a hypersurface of the Weyl space \( W_{n+1} (g,T) \) defined in a local coordinate system by the system of parametric equation \( y^a = y^a(x^i) \), where \( g \) is the induced metric. Let \( \gamma^a (\alpha = 0,1,2,3, ... n + 1) \) and \( x^i (i = 1,2,3, ... n) \) be respectively the coordinates of \( W_{n+1} (g,T) \) and \( W_n (g,T) \). Let \( N^a \) be a local unit normal to \( W_{n+1} (g,T) \) and let \( B^i = \partial y^a / \partial x^i \).
Then
\[
\nabla y^a = \bar{g}_{ab} B^b + B^b
\]
(2.14)
\[
\bar{g}_{ab} N^a B^b = 0, \bar{g}_{ab} N^a N^b = \epsilon, \quad \epsilon = \pm 1
\]
(2.15)
(2.16)
The Gauss and Codazzi equations for \( W_n (g,T) \) can be written in the form [10]
\[
\bar{R}_{ab} y^a B^b + \bar{B}^a = R_{ijk} - \epsilon (h_{ij} h_{jk} - h_{ik} h_{j}),
\]
(2.17)
\[
\bar{R}_{ab} N^a B^b + \bar{B}^a = \bar{V}_{hij} - \nabla_h h_{ij}.
\]
(2.18)
Also [8]
\[
\nabla y^a B^b = \epsilon h_{ij} N^a, \quad \bar{V}_{hij} N^a = -h_{jk} \bar{g}^{ab} B^a.
\]
(2.19)
A hypersurface of a Weyl space \( W_n (g,T) \) is called quasiumbilical if
\[
\nabla y^a = \alpha g_{ij} + \beta v_{ij},
\]
(2.20)
where \( \alpha \) is a satellite of \( g_{ij} \) with weight \(-1\). If \( \beta = 0 \) then \( W_n (g,T) \) is umbilical hypersurface. We know that \( M = \frac{\partial}{\partial \chi} \) where, \( M \) is the mean curvature of the hypersurface defined by \( M = R_{ij} g^{ij} \). A hypersurface of a Weyl space is called totally geodesic if \( h_{ij} = 0 \). If there exist in \( W_n (g,T) \) two functions \( \alpha, \beta \) and a covariant vector \( v_i \) such that
\[
\nabla y^a = \alpha g_{ij} + \beta v_{ij},
\]
(2.21)
\[
\nabla y^a = \alpha g_{ij} - \alpha_k g_{ij},
\]
(2.22)
A hypersurface of a Weyl space \( W_n (g,T) \) is called quasiumbilical if
\[
\nabla y^a = \alpha g_{ij} + \beta v_{ij},
\]
(2.23)
\[
\nabla y^a = -\alpha B^a N^a.
\]
(2.24)
Differentiating (2.11) covariantly with respect to \( r \) and using (2.13), we have
\[
\nabla y^a \bar{R}_{ab} y^a B^b + \bar{B}^a = R_{ijk} - \epsilon (g_{ij} g_{hk} - g_{ik} g_{hj}),
\]
(2.25)
\[
\nabla y^a \bar{R}_{ab} N^a B^b + \bar{B}^a = \bar{V}_{hij} - \nabla_h h_{ij}.
\]
(2.26)
Using (2.12), above equation reduces to
Now, if $W_{n+1}(g,T)$ is a generalized recurrent space then (2.15) on account of (2.1) and (2.2) reduces to

$$\nabla RV_{ijkl} = \nabla R_{ijkl} + \frac{\mu}{n(n-1)}\left(g_{ij}g_{hk} - g_{ik}g_{hj}\right) \quad \text{where} \quad \lambda = \lambda P^\eta_{_{\tau}},$$

and

$$\mu = \lambda P^\eta_{_{\tau}}.$$  

From (2.16) we have

$$\nabla RV_{ijkl} = \lambda R_{ijkl} + \frac{1}{n(n-1)}a_{ij}g_{hk} - \frac{1}{n}a_{ij}g_{hk}.$$  

If $\alpha = 0$, then above equation reduces to

$$\nabla RV_{ijkl} = \lambda R_{ijkl} + \frac{1}{n(n-1)}a_{ij}g_{hk} - \frac{1}{n}a_{ij}g_{hk}.$$  

Above equation in view of (2.8) reduces to

$$\nabla RV_{ijkl} = \lambda R_{ijkl} + \frac{1}{n(n-1)}a_{ij}g_{hk} - \frac{1}{n}a_{ij}g_{hk}.$$  

Therefore hyperspace is Projective generalized recurrent. Conversely, if (2.26) holds then (2.25) reduces to

$$\nabla RV_{ijkl} = \lambda R_{ijkl} + \frac{1}{n(n-1)}a_{ij}g_{hk} - \frac{1}{n}a_{ij}g_{hk}.$$  

which reduces to $\alpha = 0$.

3. RESULTS AND DISCUSSION

The necessary and sufficient condition for an umbilical hypersurface of a generalized Kaehlerian Weyl space to be concircular generalized recurrent is that its mean curvature must be zero. This also turns out to be the necessary and sufficient condition for the umbilical hypersurface to be projective generalized recurrent. Hence, every umbilical hypersurface of a generalized Kaehlerian Weyl space is totally geodesic whenever its mean curvature is zero.

4. CONCLUSION

This paper derives the results pertaining to a specific class of quasiumbilical hypersurfaces where $\beta = 0$ is imposed. However, there is always a scope for the more general result hitting a superclass of umbilical hypersurfaces without this restriction on $\beta$.

5 REFERENCES


