Exact Analysis Of Compound Weibull-Gamma Channel Model

Amandeep Kaur, Jyoteesh Malhotra

ABSTRACT: The quality of signal received at the destination depends upon the channel conditions and the propagation environment which can't be always homogeneous. Weibull which is terrain specific distribution along with Gamma forms a versatile, flexible and multiparameter compound Weibull-Gamma heterogeneous channel model. Its closed expressions of Probability Density Function (PDF) and Cumulative Distribution Function (CDF) are obtained here. Based on these derived expressions, exact and closed form expressions of various important performance metrics such as Coefficient of Variation (CV), Amount of fading (AF) outage probability and spectral efficiency are also evaluated. The derived expressions are tractable and are easy to evaluate using standard software tools such as MATLAB.

KEYWORDS: Weibull-gamma; Mellin transformation; heterogeneous model; fading channel; Amount of fading; Coefficient of variation.

INTRODUCTION

WITH the expansion of wireless networks and due to ubiquitous access of personal communication services, wireless systems are required to operate in increasingly hostile environments. In outdoor realistic environment both multipath fading and shadowing occur simultaneously, so the wireless signals are characterized by heterogeneous environment instead of homogeneous environment. Depending upon the radio propagation environment various multipath fading models are available in literature [1].Various composite channel models are also available like Rayleigh-lognormal, Rician–lognormal, Nakagammi – lognormal ,but these composite lognormal models involve complex and complicated mathematical calculations. Weibull-Gamma model is a heterogeneous mixture of weibull and gamma model .Some work in recent past has already been done on this model .In [2] outage probability of exponentially correlated weibull-gamma distribution is studied. In [3] PDF of weibull-gamma model has been obtained in terms of generalized hypo geometric functions. But these hypo geometric functions involve complex mathematical computation. Considering the complexity of hypo geometric function an alternative and efficient method i.e. Mellin Transformation (MT) method has been used here for the derivation of closed form expressions of PDF and CDF of heterogeneous weibull- gamma model in Meijer-G function form, as this function is easily available in many scientific software packages such as maple and mathematica. This Mellin transformation method has also been used in recent past. The expressions for PDF of Gaussian, Rayleigh and other distribution were obtained in [4].

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In [5] PDF, MGF, CV of heterogeneous Nakagami-Weibull distribution were derived and in [6] moments and PDF of multiple Rayleigh variates were computed using Mellin transformation method. Attempt on weibull-gamma has also been made in [7] to evaluate the expressions of PDF and CDF, where PDF of weibull-gamma was evaluated by using the concept of conditional probability. However various performance analysis like coefficient of variation (CV), amount of fading (AF), outage probability of compound weibull-gamma channel model has never been addressed before. In this paper closed form expressions of PDF, CDF, Moment of weibull-gamma channel model are obtained using Mellin transform method along with the exact and closed form expressions of CV, AF, spectral efficiency and outage probability. The rest of the paper is organized as follows: Section 1 presents the System and channel model, closed form expressions of PDF and CDF of compound weibull-gamma channel model are evaluated in Section 2.Various performance measures like CV, AF, spectral efficiency, outage probability are evaluated in Section 3 to quantify the reliability of selected model, before the paper is finally concluded in Section 4.

1 SYSTEM AND CHANNEL MODEL

In real time propagation the signal leaving the transmitter reaches the receiver after multiple scattering and this scattering give rise to multipath fading and shadowing. These realistic fading conditions are characterized by heterogeneous environment as shown in Fig (1).

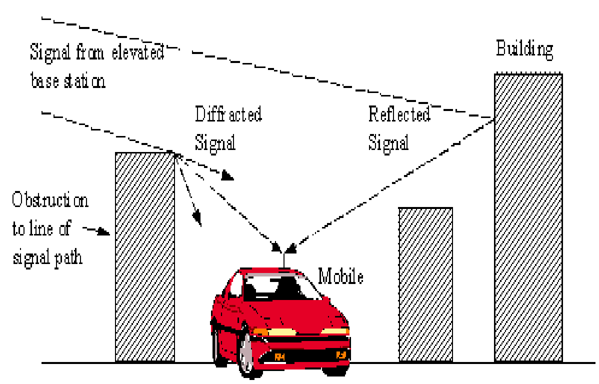


Fig (1). Realistic multipath fading and shadowing

Consider the signal transmission over a heterogeneous weibull-gamma channel in presence of AWGN (Additive White Gaussian Noise) formed by product of independent but not necessarily identically distributed weibull and gamma random variable. For this heterogeneous model consider generalized weibull PDF as:

$$f_X(x) = \frac{cx^{c-1}}{\lambda^c}\exp\left(-\frac{x^c}{\lambda^c}\right) \qquad x \geq 0 \qquad (1)$$

where "c" and "λ" represents the fading and shaping parameter. The PDF of gamma channel model is given as

$$f_Y(y) = \frac{y^{m-1}}{\beta^m \Gamma(\mathrm{m})}\exp\left(-\frac{y}{\beta}\right) \qquad y \geq 0 \qquad (2)$$

where m is the fading parameter and β is scaling factor. The mellin transform of any function $p_X(x)$ is given as ([8], eq. (8.2.5))

$$\varphi_X(\mathrm{p}_X(\mathrm{x}),\mathrm{s}) = \mathrm{E}(\mathrm{X}^{s-1}) = \int_0^\infty p(\mathrm{x})\,\mathrm{x}^{s-1}\,dx$$

Using the above formula the mellin transformation of PDF of Weibull "$f_X(x)$" and Gamma channel model "$f_Y(y)$" are given as:

$$\varphi_X(\mathrm{s}) = \Gamma\left(1+\frac{s-1}{c}\right)\lambda^{S-1} \qquad (3)$$

$$\varphi_Y(\mathrm{s}) = \frac{\Gamma(\mathrm{m}+s-1)}{\Gamma(\mathrm{m})}\beta^{S-1} \qquad (4)$$

Let "R" represents the random variable of product model. Using product convolution property, the mellin transformation of product i.e. compound weibull-gamma channel model is

$$\varphi_R(\mathrm{s}) = \frac{(\lambda\beta)^{S-1}}{\Gamma(\mathrm{m})}\Gamma(\mathrm{m}+\mathrm{s}-1)\Gamma\left(1+\frac{s-1}{c}\right) \qquad (5)$$

The n^{th} order moment of weibull-gamma is obtained by replacing " s-1" by n in eq.(5)

$$E(\mathrm{R}^n) = (\lambda\beta)^n \frac{\Gamma(\mathrm{m}+n)\Gamma\left(1+\frac{n}{c}\right)}{\Gamma(\mathrm{m})} \qquad (6)$$

These moments are of important use in evaluating closed form expressions of various performance measures.

2. Closed Form Expressions Of PDF And CDF Of Weibull-Gamma Model

The PDF of heterogeneous weibull-gamma channel model is given by inverse mellin transforms as ([8], eq. (8.2.6))

$$f_R(r) = \frac{1}{2\pi i}\int_{c-j\infty}^{c+j\infty} r^{-s}\,\varphi_R(\mathrm{s})\,ds$$

$$f_R(\mathrm{r}) = \frac{1}{2\pi i}\int_L r^{-S}(\lambda\beta)^{S-1}\frac{\Gamma(\mathrm{m}+s-1)\Gamma\left(1+\frac{s-1}{c}\right)}{\Gamma(\mathrm{m})}ds \qquad (7)$$

Substituting $\frac{s-1}{c} = s'$ and $\lambda\beta = K$ in (7), the expression becomes

$$f_R(\mathrm{r}) = \frac{c}{\Gamma(\mathrm{m})\,\mathrm{r}\,2\pi i}\int_L r^{-cs'}K^{cs'}\Gamma(\mathrm{m}+cs')\Gamma(1+s')ds'$$

Applying gauss multiplication formula ([9].eq. (6.1.20))

$$f_R(\mathrm{r}) = \frac{(2\pi)^{\frac{1-c}{2}}c^{m+\frac{1}{2}}}{r\Gamma(\mathrm{m})2\pi i}\int_L \left(\frac{r^c}{K^c c^c}\right)^{-s'}\prod_{i=0}^{c-1}\Gamma\left(\frac{i}{c}+\frac{m}{c}+s'\right)\Gamma(1+s')ds' \qquad (8)$$

Considering Meijer-G function of the form

$$G_{p,q}^{m,n}\left(y\middle|_{b_1,b_2,\ldots\ldots\ldots\ldots b_q}^{a_1,a_2,\ldots\ldots\ldots\ldots a_p}\right) = \int \frac{\prod_{i=1}^{m}\Gamma(b_i+s)\prod_{i=1}^{n}\Gamma(1-a_i-s)}{\prod_{i=n+1}^{p}\Gamma(a_i+s)\prod_{i=m+1}^{q}\Gamma(1-b_i-s)}\,y^{-s}\,ds \qquad (9)$$

Comparing (8) with (9) the expression for PDF in terms of Meijer G function is

$$f_R(\mathrm{r}) = \frac{\mathrm{V}}{r}G_{0,c}^{c+1,0}\left(\frac{r^c}{K^c c^c}\middle|_{\frac{m}{C},\frac{m+1}{C},\ldots\ldots\ldots\ldots\frac{m+c-1}{C},1}^{-}\right) \qquad (10)$$

Where $V = \frac{(2\Pi)^{\frac{1-c}{2}}c^{m+\frac{1}{2}}}{\Gamma(\mathrm{m})}$

Another form of PDF is generated by substituting $\frac{s}{c} = s'$ in (7)

$$f_R(\mathrm{r}) = \frac{c}{K\Gamma(\mathrm{m})2\pi i}\int_L \left(\frac{r^c}{K^c}\right)^{-s'}\Gamma(\mathrm{m}-1+cs')\Gamma\left(1+s'-\frac{1}{c}\right)ds'$$

Applying gauss multiplication formula ([9], eq. (6.1.20))

$$f_R(\mathrm{r}) = \frac{V}{cK2\pi i}\int_L \left(\frac{r^c}{K^c c^c}\right)^{-s'}\prod_{i=0}^{c-1}\Gamma\left(s'+\frac{m-1+i}{c}\right)\Gamma\left(s'+1-\frac{1}{c}\right)ds' \qquad (11)$$

Comparing (11) with (9), another form of PDF is

$$f_R(\mathrm{r}) = \frac{\mathrm{V}'}{Kc}G_{0,c}^{c+1,0}\left(\frac{r^c}{K^c c^c}\middle|_{\frac{m-1}{C},\frac{m}{C},\ldots\ldots\ldots\ldots 1-\frac{1}{C}}^{-}\right) \qquad (12)$$

The CDF of weibull-gamma fading channel model is given as

$$F_R(\mathrm{r}) = \int_0^r f_R(\mathrm{r})\,\mathrm{dr} \qquad (13)$$

Substituting eq. (11) in eq. (13) and using gamma reduction rule ([9], eq.(6.1.15)) to solve the inner integral ,the expression of CDF becomes.

$$F_R(r) = \frac{rV}{Kc^2 2\pi i}\int_L \left(\frac{r^c}{K^c c^c}\right)^{-s'} \frac{\prod_{i=0}^{c-1}\Gamma\left(s' + \frac{m-1+i}{c}\right)\Gamma\left(1+s' - \frac{1}{c}\right)\Gamma\left(\frac{1}{c} - s'\right)}{\Gamma\left(1+\frac{1}{c} - s'\right)} ds' \quad (14)$$

Comparing (14) with (9) the expression of CDF of heterogeneous compound weibull-gamma channel model in terms of Meijer-G function is.

$$F_R(r) = \frac{Vr}{Kc^2} G_{0,c}^{c+1,1}\left(\frac{r^c}{K^c c^c} \middle| \begin{matrix} 1-\frac{1}{c} \\ \frac{m-1}{c}, \frac{m}{c}, \ldots\ldots, 1-\frac{1}{c}, \frac{-1}{c} \end{matrix} \right) \quad (15)$$

3. PERFORMANCE METRICS

Four different measures have been investigated to evaluate the performance of heterogeneous weibull-gamma channel model.

3.1. Coefficient of Variation (CV)

It is very important and reliable measure for comparing any two channel models and for measuring the stability of any channel model. More the value of coefficient of variation lesser is the stability. It is defined as the ratio of standard deviation to the mean of random variable.

$$C.V = \frac{\sigma}{\mu}$$

For weibull-gamma channel model CV is given as

$$\mu = \lambda\beta \frac{\Gamma(m+1)\Gamma(1+\frac{1}{c})}{\Gamma(m)}$$

$$\sigma = \sqrt{(\lambda\beta)^2 \frac{\Gamma(m+2)\Gamma(1+\frac{2}{c})}{\Gamma(m)} - (\lambda\beta)^2 \frac{\Gamma^2(m+1)\Gamma^2(1+\frac{1}{c})}{\Gamma^2(m)}}$$

$$C.V. = \sqrt{\frac{\Gamma(m)\Gamma(m+2)\Gamma(1+\frac{2}{C})}{\Gamma^2(m+1)\Gamma^2(1+\frac{1}{C})} - 1}$$

$$C.V. = \sqrt{\frac{(m+1)\Gamma(1+\frac{2}{C})}{(m)\Gamma^2(1+\frac{1}{C})} - 1}$$

The values of CV for various values of c and m are plotted in Fig (2).

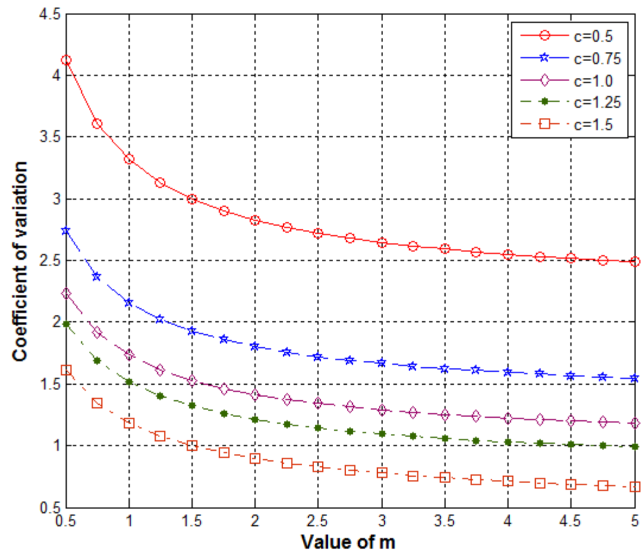


Fig (2). Plot of CV for various values of m and c.

3.2. Amount of Fading (AF)

It is a quantitative measure of studying fading present in wireless channel. It is given by the formula ([10] eq. (1.27))

$$AF = \frac{E(\gamma^2) - (E(\gamma))^2}{(E(\gamma))^2} \quad (16)$$

The instantaneous received SNR " γ " at the receiver end is given as

$$\gamma = R^2 \frac{E_s}{N_0} \quad (17)$$

Where E_s is the transmitted symbol's average energy and N_0 is one sided AWGN power spectral density. The average SNR $\bar{\gamma}$ is

$$\bar{\gamma} = E(R^2)\frac{E_s}{N_0} \quad (18)$$

Dividing (17) by (18) and using (6) the relation between γ and R is

$$R^2 = \frac{\gamma}{\bar{\gamma}}(\lambda\beta)^2 \frac{\Gamma(m+2)\Gamma(1+\frac{2}{c})}{\Gamma(m)} \quad (19)$$

Mellin transform for the PDF of instantaneous SNR " γ " is given as.

$$\varphi_\gamma(s) = E(\gamma^{S-1})$$

$$\varphi_\gamma(s) = \left(\frac{E_s}{N_0}\right)^{s-1}\varphi_R(2s-1)$$

Replacing s by 2s-1 in (5), the n^{th} order moment of output SNR (γ) is as

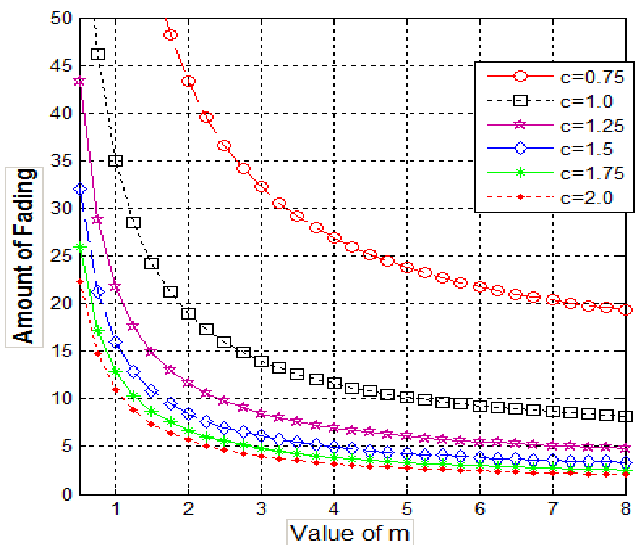
$$E(\gamma^n) = (\frac{E_s}{N_0})^n \frac{(\lambda\beta)^{2n}}{\Gamma(\mathrm{m})} \Gamma(\mathrm{m}+2n))\Gamma(1+\frac{2n}{c}) \quad (20)$$

Using (20) in (16) the expression for amount of fading of heterogeneous weibull-gamma channel model is

$$\therefore AF = \frac{\Gamma(\mathrm{m})\Gamma(\mathrm{m}+4)\Gamma(1+\frac{4}{c})}{\Gamma^2(\mathrm{m}+2)\Gamma^2(1+\frac{2}{c})} - 1$$

$$AF = \frac{(\mathrm{m}+3)(\mathrm{m}+2)\Gamma(1+\frac{4}{c})}{(\mathrm{m}+1)(\mathrm{m})\Gamma^2(1+\frac{2}{c})} - 1$$

The plot of amount of fading for various values of c and m is given in Fig (3).



Fig(3). Plot of amount of fading for various values of m and c.

3.3. Spectral Efficiency (SE)

The amount of fading is also used to study the spectral efficiency of a flat fading channel in a very noisy region. The slope of the SE in a very noisy region is given as

$$S = \frac{2}{AF+1}$$

The slope of spectral efficiency is inversely proportional to amount of fading .as the value of AF becomes greater than 1 the value of S decreases below 1.

3.4. Outage Probability

The outage probability is defined as the probability that the outage SNR falls below a certain threshold value γ^{th} . It is given by the formula ([10].eq. (1.4))

$$P_{out}(\gamma^{th}) = \int_0^{\gamma^{th}} f_\gamma(\gamma)d\gamma = F_\gamma(\gamma^{th})$$

Basically outage probability is the value of CDF at γ = γ^{th}

i.e. $$P_{out}(\gamma^{th}) = \mathrm{F}_\gamma(\gamma^{th})$$

$$F_\gamma(\gamma^{th}) = \mathrm{F}_R\left(\sqrt{\frac{\gamma}{\bar{\gamma}}(\lambda\beta)^2 \frac{\Gamma(\mathrm{m}+2)\Gamma(1+\frac{2}{c})}{\Gamma(\mathrm{m})}}\right)$$

Using (19) in (15) the expression for outage probability is

$$P_{out} = \frac{V}{c^2} DG^{c+1,1}_{0,c}\left(\frac{D^c}{c^c}\middle|\begin{matrix}1-\frac{1}{c}\\ \frac{m-1}{c},\frac{m}{c},\ldots\ldots\ldots\ldots,1-\frac{1}{c},\frac{-1}{c}\end{matrix}\right)$$ where

$$D = \left(\frac{\gamma}{\bar{\gamma}} \frac{\Gamma(\mathrm{m}+2)\Gamma(1+\frac{2}{c})}{\Gamma(\mathrm{m})}\right)^{\frac{1}{2}}$$

4. Conclusion

The realistic signal transmission is characterized by both multiple scattering and fading whether in indoor or outdoor environment. Such fading condition statistics are better studied by considering heterogeneous models. Weibull-Gamma model is selected here to model the fading conditions of heterogeneous environment. Closed and exact form expressions of PDF and CDF have been obtained here using Mellin transform. Various performance metrics of heterogeneous weibull-gamma fading channel are investigated by computing the moments, mean, Outage Probability, Spectral Efficiency and even by plotting graphs of Coefficient of Variation (CV), Amount of Fading (AF) for various values of fading parameters .These measures gave us an insight of the behavior of the selected channel under various fading conditions and hence are of great use to quantify the reliability of heterogeneous channel model.

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