Utilization Of The Stability Set Of The First Kind For Solving Inverse Nonlinear Programming Problems

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Abstract: In this paper, we study inverse nonlinear programming (INLP) problem under the L_2 norm, where we adjust the cost coefficients of the given nonlinear programming (NLP) problem as less as possible such that a known feasible solution $\bar{x}$ becomes the optimal one. Here, we utilize of the basic notions of stability (the solvability set and the stability set of the first kind). A solution procedure to solve the INLP problem is given. A numerical example is presented in the sake of this paper to clarify the obtained results.

Index Terms: Inverse problem; Nonlinear programming problem; Kuhn-Tucker conditions; Stability, Optimal solution.

1 INTRODUCTION

Inverse optimization is a relatively new area of research. In recent years, many researchers have discussed some inverse combinational optimization problems, where in these problems a feasible solution $\bar{x}$ is given which may or may not to be an optimal solution with respect to the current objective function and it is required to revise (or perturb) the cost coefficient in the objective function as less as possible so that $\bar{x}$ becomes an optimal solution. Ahuja and Orlin [1] provided various references in the area of inverse optimization and compile several applications. There has already been some research devoted to inverse network flow problems. Burton and Toint [2, 3], and Burton, Pulleyblank and Toint [4] considered inverse shortest path problems (multi-source, multi-sink problems) under the L_2 norm and solved them using nonlinear programming techniques. Cai and Young [5], and Xu and Zhang [17] have considered the inverse shortest paths problem under the weighted L_1 norm; Huang and Liu [8] studied the minimum cost flow problem under the weighted L_1 norm, yang, Zhang and Ma [18], and Zhang and Cai [19] considered the minimum cut problem under the weighted L_1 norm. Each of these problems reduces to solving a minimum cost flow problem. Du and Cui [6] presented a new method in order to solve an ill-posed problem on Fredholm integral equation of the first kind. The representation of the exact solution is given and the stability of the solution on Fredholm integral equation of the first kind is discussed in the reproducing kernel space. Farnoosh and Ebrahimi [7] provided a numerical algorithm involving the combined use of the finite differences scheme and Monte Carlo method for estimating the diffusion coefficient in a one-dimensional nonlinear parabolic inverse problem. Huang et al. [9] explored a new and systematic stability analysis of advanced quadrature methods. Jiang and Cui [10] gave representation of exact solution for integral equation of third or first kind with singular kernel by the form of series in the reproducing kernel space. Levere et al. [11] established a technique for solving inverse problems for second-order linear elliptic partial differential equations. Li and Liu [12] constructed a Runge–Kutta type total variation regularization for the nonlinear ill-posed problems. Ornelas-Tellez et al. [14] presented an inverse optimal control approach in order to achieve stabilization of discrete-time nonlinear systems. In his earlier work, Osman [15- 16] introduced the basic notions of the stability (the solvability set, the stability set of the first kind and the stability set of the second kind), and analyzed these concepts for parametric convex nonlinear programming problems. In this paper, we study inverse nonlinear programming problem under the L_2 norm, where we adjust the cost coefficients of the given nonlinear programming problem as less as possible such that a known feasible solution $\bar{x}$ becomes the optimal one. Here, we utilize of the basic notions of stability (the solvability set and the stability set of the first kind). Zou et al. [20] presented a nonlinear inverse optimization approach to determine the weights for the joint displacement function in standing reach tasks. This approach can be used to determine the weights of cost function within any multi-objective optimization problems. This paper is organized as in the following sections: In section 2, a nonlinear programming problem is introduced as specific definition. In section 3, the stability set of the first kind corresponding to NLP problem is presented. In section 4, the determination of the stability set of the first kind is considered, and hence inverse nonlinear programming problem corresponding to the nonlinear programming problem considered in section 2, is formulated. In section 5, a solution procedure for obtaining the solution of INLP problem is given. In section 6, a numerical example is given in the sake of this paper for illustration. Finally, some concluding remarks are reported in section 7.

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2 PROBLEM FORMULATION AND SOLUTION CONCEPT

Consider the following nonlinear programming problem

(NLP) : $\min Z = \sum_{i=1}^{m} \lambda_i f_i(x)$

subject to

$$x \in M = \left\{ x \in R^n : g_j(x) \le 0,\ j = 1,\dots,k \right\}.$$

Where, $f_i : R^m \to R, i = 1,\dots m; g_j(x) : R^k \to R, j = 1,\dots k$, are convex function of class C^1, and $\lambda \in R^m$ is a vector of ordinary parameters. It is assumed that the set *M* is nonempty (i.e., $M \ne \phi$).

Definition 2.1. The solvability set *B* of problem (NLP) is defined by:

$B = \{\lambda \in R^m$: problem (NLP) has the optimal solution x^-}.

Which means that:

$$B = \{\lambda \in R^m : \min (\sum_{i=1}^{m} \lambda_i f_i(x)),\ x \in M \text{ exists}\}.$$

3 THE STABILITY SET OF THE FIRST KIND

Definition 3.1. Suppose that $\lambda \in B$ with a corresponding optimal solution $\bar{x}$, then the stability set of the first kind of NLP problem corresponding to $\bar{x}$ is defined to be: $S(\bar{x}) = \{\lambda \in B : \bar{x}(\lambda)$ is optimal solution of NLP problem }.

Which mean that:

$$S(\bar{x}) = \left\{ \lambda \in B : \lim_{x \in M} \sum_{i=1}^{m} \lambda_i f_i(x) = \sum_{i=1}^{m} \lambda_i f_i(\bar{x}) \right\}.$$

Theorem 3.1. The set $S(\bar{x})$ is convex.

Proof: Let $\lambda^1, \lambda^2 \in S(\bar{x})$, then $\sum_{i=1}^{m} \lambda_i^1 f_i(\bar{x}) \le \sum_{i=1}^{m} \lambda_i^1 f_i(x);\ \forall x \in M$, and $\sum_{i=1}^{m} \lambda_i^2 f_1(\bar{x}) \le \sum_{i=1}^{m} \lambda_i^2 f_1(x),\ \forall x \in M$, and $\sum_{i=1}^{m} (1-w)\lambda_i^2 f_i(x) \le \sum_{i=1}^{m} (1-w)\lambda_i^2 f_i(x),\ \forall x \in M$.
Therefore,
$\sum_{i=1}^{m} (w\lambda_i^1 + (1-w)\lambda_i^2) f_i(\bar{x}) \le \sum_{i=1}^{m} (w\lambda_i^1 + (1-w)\lambda_i^2) f_i(x)$, $\forall x \in M,\ 0 \le w \le 1$, i.e., $w\lambda_i^1 + (1-w)\lambda_i^2 \in S(\bar{x})$ and hence the result follows.

4 DETERMINATION OF THE STABILITY SET OF THE FIRST KIND

Let $\bar{x} \in M$ be an optimal solution of NLP problem, then there exists $u \in R^n, u \ge 0$ such that:

$$\sum_{i=1}^{m} \lambda_i \frac{\partial f_i(\bar{x})}{\partial x_\alpha} - \sum_{j=1}^{k} u_j \frac{\partial g_j(\bar{x})}{\partial x_\alpha} = 0,\ \alpha = 1, 2, \dots, n,$$

$$g_j(\bar{x}) \le 0,\ u_j g_j(\bar{x}) = 0,\ j = 1, 2, \dots, k.$$

Let the set of active constraints at *x* is defined by $G(\bar{x}) = \{ j \in \Box^k : g_j(\bar{x}) = 0\}$, so, there exists two probability as follows:

1) If $G(\bar{x}) = \phi$, then $\bar{x} \in \operatorname{int} M$, $u = 0$, and $\sum_{i=1}^{m} \lambda_i \frac{\partial f_i(\bar{x})}{\partial x_\alpha} = 0$,
2) If $G(\bar{x}) \ne \phi$, then $u_{G(\bar{x})} \ge 0$ and $\sum_{i=1}^{m} \lambda_i \frac{\partial f_i(\bar{x})}{\partial x_\alpha} + \sum_{G(\bar{x})} u_{G(\bar{x})} \frac{\partial g_{G(\bar{x})}^{(\bar{x})}}{\partial x_\alpha} = 0$, or

$$\sum_{i=1}^{m} \lambda_i \frac{\partial f_i(\bar{x})}{\partial x_\alpha} = -\sum_{G(\bar{x})} u_{G(\bar{x})} \frac{\partial g_{G(\bar{x})}^{(\bar{x})}}{\partial x_\alpha},\ \alpha = 1, 2, \dots, n.$$

Hence, the inverse nonlinear programming (INLP) problem corresponding to NLP problem illustrated in section 2 may be formulated as follows:

(INLP) : $\min \| \lambda - \bar{\lambda} \|$

subject to

$$\lambda \in S(\bar{x}) = \left\{ \lambda \in B : \min_{x \in M} \sum_{i=1}^{m} \lambda_i f_i(x) = \sum_{i=1}^{m} \lambda_i f_i(\bar{x}) \right\}.$$

Definition 4.1. The solvability set B' of problem (INLP) is defined by $B' = \{\lambda \in R^m$: problem (INLP) has the optimal solution $\bar{x}$ }. Which means that

$B' = \{\lambda \in R^m : \min \| \lambda - \lambda^- \|$ exists} , $\lambda \in S(\bar{x})$.

Theorem 4.1. The set B' is convex.

Proof: Let $\lambda^1 \in B'$, then $\min_{\lambda \in S(\bar{x})} \| \lambda - \bar{\lambda} \| = \| \lambda^1 - d \|$, and let $\lambda^2 \in B'$, then $\min_{\lambda \in S(\bar{x})} \| \lambda - \bar{\lambda} \| = \| \lambda^2 - d \|$. Then, $\min_{\lambda \in S(\bar{x})} \| \lambda - d \| = (1-w) \| \lambda' - d \| + w \| \lambda^2 - d \|$

$\geq \| (1-w)\lambda^1 + w\lambda^2 - d \|$, $0 \leq w \leq 1$. Therefore, $(1-w)\lambda^1 + w\lambda^2 \in S(\bar{x})$, $0 \leq w \leq 1$, and hence the result follows..

5 SOLUTION PROCEDURE

Now, we can construct a solution procedure for obtaining the solution of INLP problem. The steps of the procedure are as follow:

Step 1: For a contain λ_i, $i = 1, 2, ..., m$, solve the NLP problem.

Step 2: Choose any feasible point $\bar{x}$.

Step 3: Substituting in Kuhn-Tucker conditions (see [13]) with $\bar{x}$ as the optimal solution, we have

$$\left.\begin{array}{l} \sum_{i=1}^{m} \lambda_i \frac{\partial f_i(\bar{x})}{\partial x_\alpha} + \sum_{G(\bar{x})} u_{G(\bar{x})} \frac{\partial g_{G(\bar{x})}^{(\bar{x})}}{\partial x_\alpha}, \ \alpha = 1, 2, ..., n \\ u_j \geq 0, \ j \in G(\bar{x}) \end{array}\right\}$$

Using any algorithm to solve this linear system.

Step 4: Determine the stability set of the first kind $S(\bar{x})$.

Step 5: Formulate the INLP problem corresponding to NLP problem.

Step 6: Solve the INLP problem to obtain the parameter λ.

6 NUMERICAL EXAMPLE

Because IJSTR staff will do the final formatting of your paper, Let us consider the following nonlinear programming problem

$$Z = \min(x_1^2 + x_2^2 + \lambda_1 x_1 + \lambda_2 x_2)$$

subject to

$$x_1^2 + x_2^2 \leq 1, \qquad x_1 + x_2 \leq 1.$$

To determine the parameter λ, let us follow the steps:

Step 1: Let $\lambda_1 = -2, \ \lambda_2 = 0$.

Step 2: Take $x^* = (1, 0), \ z^* = -1$.

Step 3: applying the Kuhn-Tucker condition with $\bar{x} = (0, 1)$ is an optimal solution and $\bar{z} = z^* = -1$, we get:

$$\left.\begin{array}{l} 2\bar{x}_1 + \lambda_1 + u_1(2\bar{x}_1) + u_2 = 0 \\ 2x_2 + \lambda_2 + u_1(2\bar{x}_2) + u_2 = 0 \\ u_1(\bar{x}_1^2 + \bar{x}_2^2 - 1) = 0 \\ u_2(\bar{x}_1 + \bar{x}_2 - 1) = 0. \end{array}\right\} \text{(I)}$$

From (I), and with $\bar{x} = (0, 1)$, we obtain

$$u_2 = -\lambda \geq 0, \qquad 2 + \lambda_2 - \lambda_1 \leq 0.$$

Step 4: $S(x^-) = \{\lambda_1, \lambda_2 \in R^2 : \lambda_1 \leq 0,$ and $2 + \lambda_2 - \lambda_1 \leq 0\}$.

Step 5: The INLP problem is formulated as follow:

$$\min \| \lambda - \bar{\lambda} \|$$

subject to

$$\lambda \in S(\bar{x}), \qquad z(\bar{x}, \lambda) = z^*.$$

The above problem take the form:

$$\min \sqrt{(\lambda_1 + 2)^2 + \lambda_2^2}$$

subject to

$$\lambda_1 \leq 0,$$

$$2 + \lambda_2 - \lambda_1 \leq 0,$$

$$1 + \lambda_2 = -1.$$

Step 6: The solution is: $\lambda = (0, -2)$.

7 CONCLUDING REMARKS

In this paper, we have studied inverse nonlinear programming problem under the L_2 norm, where we have utilized of the notions of the stability (the solvability set and the stability set of the first kind). An algorithm for obtaining the solution has been considered. Also, a numerical example has been given to clarify the developed theory and the proposed solution procedure.

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