Some Finite Sample Properties Of Seemingly Unrelated Unrestricted Regression Model A New Approach

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Abstract: this article, study some finite sample properties of zellner estimators, when the case of the regressors in the second equation is subset of the regressors in the first equation. A new approach is given based on the unrestricted residual, we derive the exact first and second moment of the estimator. Consider with the orthogonal condition and normality of the disturbances terms.

Keywords: Seemingly unrelated unrestricted; (SUUR) model; exact finite sample properties; exact moment

I. INTRODUCTION

In (1962) Zellner used Aitken Generalized Least Square (GLS) method to estimate the parameter of sets of system equation regression models showed that. The (GLS) estimator is more efficient than those obtained by using Ordinary Least Square (OLS) method to estimate coefficient for each equation separately. In subsequent paper (1963) Zellner suggested a new approach of estimating the coefficient of a Seemingly Unrelated Regression equation (SURE) model and developed an operational version of (GLS) to estimate sample variance and covariance parameters ,where these estimators are based on unrestricted estimate S, by replacing their unknown population by the unrestricted sample estimator, provided the finite sample properties of estimators by using two system equation ,derived the exact first and second moment of the coefficient estimator, and compared the results with the least square estimator, showed that Aitken (GLS) approach yields minimum variance unbiased linear estimators when the disturbance covariance matrix is known .The estimators are more efficient than single equation least square estimator of any sample size when disturbance are correlated. In (1967) Kakwani showed that zellner's estimators are unbiased where disturbance terms are normally distributed, Revankar (1974), Mehta & Swamy (1978) and Aiyi liu (2002) studied the properties of Zellner's estimators. The plan of this paper is as follows. In section (2) introduce the model and assumption, section (3) devoted to introduce a new approach to derive the exact first and second moment of two equation when the case of the regressors in the second equation is subset of the regressors in the first equation, section (4) some concluding remarks followed by appendix are presented.

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II. MODEL AND ASSUMPTIONS

Consider the following system of two seemingly unrelated regression (SUR) equations initially introduced by Zellner [11,12]

$$\underset{(T.1)}{Y_i} = \underset{(T.k_i)}{X_i}\,\underset{(K_i.1)}{B_i} + \underset{(T.1)}{U_i}, \quad i = 1,2 \qquad (2\text{-}1)$$

$$K = k_1 + k_2$$

Where

y_i is a $(\mathrm{T}.1)$ vector of observations on the i^{th} dependent variable (the variable to be "explained" by the i^{th} regression equation). X_i is a $(T.k_i)$ block diagonal matrix of observation on (K_i) nonstochastic independent variable, each column of which consists of T observation on a regressors in the i^{th} equation of the model, with rank (K_i) .that X_1, X_2 are the matrices of fixed elements . B_i is a $(k_i.1)$ vector of regression coefficients that unknown parameters in the $i-th$ equation of the model. u_i is the corresponding $(\mathrm{T}.1)$ vector of random disturbances term in the $i-th$ regression equation. The system of equation (2-1) can be written as

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{pmatrix} + \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \qquad (2\text{-}2)$$

Assume that the disturbance terms have a normally distribution with zero mean vector and variance covariance matrix $(\Sigma \otimes I)$

$$U_{T.1} \sim N_T(0_{T.1}, \Sigma \otimes I) \qquad (2\text{-}3)$$

$$\left.\begin{array}{l}\mathrm{E}(u_i)=0_{(T.K)}\\ \mathrm{var}(u_i)=\sigma_{ii}I\end{array}\right\} \quad , \; i=1,2$$

$$\mathrm{cov}(u_i,u_j)=\mathrm{E}(u_i u_j')=\sigma_{ij} \qquad , \; i\neq j \qquad i,j=1,2$$

Where:

σ_{ii} is scalar and Represents the variance of the random disturbance in the i^{th} equation for each observation in the sample, I_T is an identity matrix of order T , $\Sigma=(\sigma_{ij})$, σ_{ij} Represents the covariance between the disturbances of i^{th} equations and j^{th} equations for each observation in the sample, that contemporaneous element of (u_i,u_j) have a bivariate normal distribution by applying (GLS) we get

$$\hat{\beta}_{GLS}=(X'(\Sigma^{-1}\otimes I)X)^{-1}X'(\Sigma^{-1}\otimes I)Y$$

These estimators are based on the unrestricted estimate S of Σ the elements s_{ij} of S are based on the residuals u_i obtained by regressing y_i on all the regressors in the system .then from these residuals we obtain consistent estimator of σ_{ij}'s as $s_{ij}=\frac{1}{T}\hat{u}_i'\hat{u}_j$, $\hat{u}_i=py$ where $\hat{u}$ is unrestricted residuals vector obtained by regressing y_i on all of exogenous explanatory variables x_i , $P=I-x(x'x)^{-1}x'$, $(P)_{(T.T)}$ Idempotent matrix So-called as a "residual maker" Green (2002) [5] that is produces the vector of least squares residuals in the regression of y on x when it pre-multiplies any vector y.

Then $s_{ij}=\frac{1}{T}y_i'Py_j$,

$$y_i'Py_j=B_i'x_i'Px_jB_j+B_i'x_i'P\hat{u}_j+\hat{u}_i'Px_jB_j+\hat{u}_i'P\hat{u}_j$$

$$\because x'P=Px=0$$

$$=\hat{u}_i'P\hat{u}_j$$

$$\therefore \hat{u}_i'\hat{u}_j=y_i'Py_j=\hat{u}_i'P\hat{u}_j$$

Then, from these residuals we obtain consistent estimator of σ_{ij}'s as $S_{ij}=\frac{1}{T}\hat{u}_i'\hat{u}_j=\frac{1}{T}\hat{u}_i p\hat{u}_j$ From above assumption we get the Feasible (GLS) estimator as $\hat{\beta}_{FGLS}$

$$\hat{\beta}_{FGLS}=(X'(S^{-1}\otimes I)X)^{-1}X'(S^{-1}\otimes I)Y$$

To obtain the estimators of the two equations in this case. We introduce the following notation

$$\nabla=s_{11}s_{22}-s_{12}^2$$

$$c^2=\frac{(s_{12})^2}{s_{11}s_{22}}$$

, $P_2^*=\left(x_2(x_2'x_2)^{-1}x_2'\right)$, $P_i=\left(I-P_i^*\right)$

$$H_{12}=\left(x_1'\left(I-\frac{(s_{12})^2}{s_{11}s_{22}}(P_2^*)\right)x_1\right)^{-1}=\left(x_1'\left(I-c^2(P_2^*)\right)x_1\right)^{-1}$$

Then

$$\hat{\beta}_{FGLS}=\begin{pmatrix}b_{1F}\\ b_{2F}\end{pmatrix}=\begin{pmatrix}s^{22}x_1'x_1 & -s^{12}x_1'x_2\\ -s^{12}x_2'x_1 & s^{11}x_2'x_2\end{pmatrix}^{-1}\begin{pmatrix}s^{22}x_1'y_1-s^{12}x_1'y_2\\ -s^{21}x_2'y_1+s^{11}x_2'y_2\end{pmatrix}$$

Apply the inverse of partition matrix .it is obvious that:

$$a_{11}^{-1}=[a_{11}-a_{12}a_{22}^{-1}a_{12}]^{-1}$$

$$a_{22}^{-1}=a_{22}^{-1}+a_{22}^{-1}a_{21}a_{11}^{-1}a_{12}a_{22}^{-1}$$

, $a_{12}^{-1}a_{21}^{-1}=-[a_{11}-a_{12}a_{22}^{-1}a_{21}]^{-1}a_{12}a_{22}^{-1}$

Then we get

$$a_{11}^{-1}=\frac{1}{s_{11}s_{22}-s_{12}^2}\left[\frac{1}{s_{22}}\left(x_1'(I-\frac{(s_{12})^2}{s_{11}s_{22}}(x_2(x_2'x_2)^{-1}x_2')x_1\right)^{-1}\right]$$

$$=\frac{1}{\nabla s_{22}}H_{12}=A$$

$$a_{12}^{-1}=\frac{(s_{12})^2}{s_{11}s_{22}s_{12}}\left[\left(x_1'(I-\frac{(s_{12})^2}{s_{11}s_{22}}(x_2(x_2'x_2)^{-1}x_2')x_1\right)^{-1}\right]$$

$$(x_1'x_2)(x_2'x_2)^{-1}$$

$$=\frac{(s_{12})^2}{\nabla}H_{12}(x_1'x_2)(x_2'x_2)^{-1}$$

$$a_{21}^{-1}=\frac{1}{\nabla}\left(\frac{c^2}{s_{12}}H_{12}(x_2'x_2)^{-1}(x_2'x_1)\right)$$

$$=\frac{1}{\nabla}\begin{pmatrix} s_{11}(x_2'x_2)^{-1}+(s_{11}x_2'x_2)^{-1}(-s_{12}x_2'x_1) \\ [a_{11}^{-1}](-s_{12}x_1'x_2)(s_{11}x_2'x_2)^{-1} \end{pmatrix}$$

$$=\frac{1}{\nabla}\left(\frac{1}{s_{11}}(x_2'x_2)^{-1}+\frac{c^2}{s_{11}}(x_2'x_2)^{-1}(x_2'x_1)H_{12}(x_1'x_2)(x_2'x_2)^{-1}\right)$$

$$=D$$

$$a_{22}^{-1}=a_{22}^{-1}+a_{22}^{-1}a_{21}a_{11}^{-1}a_{12}a_{22}^{-1}$$

$$=\frac{1}{\nabla}\left(\frac{1}{s_{11}}(x_2'x_2)^{-1}+\frac{(s_{12})^2}{(s_{11})^2 s_{22}}(x_2'x_2)^{-1}(x_2'x_1)H_{12}(x_1'x_2)(x_2'x_2)^{-1}\right)$$

$$a_{22}^{-1}=\frac{1}{\nabla}\begin{pmatrix}\frac{1}{s_{11}}(x_2'x_2)^{-1}+\frac{c^2}{s_{11}}(x_2'x_2)^{-1}(x_2'x_1) \\ H_{12}(x_1'x_2)(x_2'x_2)^{-1}\end{pmatrix}=Z$$

Similarly, we find

$$X'(S^{-1}\otimes I_T)Y=\frac{1}{\nabla}\begin{pmatrix} s_{22}x_1'y_1-s_{12}x_1'y_2 \\ -s_{12}x_2'y_1+s_{11}x_2'y_2\end{pmatrix}$$

Thus by applying (FGLS) method

$$\begin{pmatrix} b_{1F} \\ \hline b_{2F}\end{pmatrix}=\nabla\left(\begin{array}{c|c} A & C \\ \hline D & Z\end{array}\right).\frac{1}{\nabla}\begin{pmatrix} s_{22}x_1'y_1-s_{12}x_1'y_2 \\ -s_{12}x_2'y_1+s_{11}x_2'y_2\end{pmatrix}$$

Then we have

$$b_{1F}=(x_1'x_1)^{-1}x_1'y_1-\frac{s_{12}}{s_{22}}(x_1'x_1)^{-1}x_1'P_2y_2 \qquad (2\text{-}4)$$

$$b_{2F}=(x_2'x_2)^{-1}x_2'y_2 \qquad (2\text{-}5)$$

Thus, it is clear that the (FGLS) estimator of β_2 is just the (OLS) estimator in this case of two-equations when the regressors in the second equation x_2 is subset of the regressors in the first equation x_1, then we show that b_{2F} estimator of β_2 is (OLS).

III. EXACT FIRST AND SECOND MOMENTS

This section is the main section of this paper. Present some lemmas employed in our proof, derive the variance covariance matrix including the variance of unrestricted estimator $b_{1(SU)}$ and $b_{2(SU)}$, derive Cross moment of $b_{1(SU)}$ and $b_{2(SU)}$

THEOREM (1)

The variance of the (SUUR) estimator of $(\tilde{\beta})$ when x_2 in second equation is subset from x_1 in the first equation is given by

$$V(\tilde{\beta}_{1(SU)})=\sigma_{11}(x_1'x_1)^{-1}-\sigma_{11}\left(\ell_{12}^2-\frac{1-\ell_{12}^2}{(n-2)}\right)$$

$$.(x_1'x_1)^{-1}x_1'P_2x_1(x_1'x_1)^{-1}$$

, Cross moment of $b_{1(SU)}$ and $b_{2(SU)}$ is given by

$$\operatorname{cov}(\tilde{\beta}_{(1)SU},\tilde{\beta}_{(2)SU})=\sigma_{12}(x_1'x_1)^{-1}(x_1'x_2)(x_2'x_2)^{-1}$$

We shall give four lemmas which conceder the basis for driving the first and second exact moment

LEMMA (1)

Let us x_1 be positive random variable, x_2 be an arbitrary random variable. A joint moment generating function of x_1 and x_2

$$\mu(t_1,t_2)=\mathrm{E}\left(e^{t_1x_1+t_2x_2}\right)$$

For $t_1<\varepsilon$ and $|t_2|<\varepsilon$ where ε is some positive constant then r-th order moment of (x_2/x_1) is given by

$$\mathrm{E}\left(\frac{X_2}{X_1}\right)^r=\frac{1}{\Gamma r}\int_{-\infty}^{0}(-t^{r-1})\frac{\partial^r\mu(t_1,t_2)}{\partial t_2^r}\Big|_{t_2=0}dt_1$$

Where (r) is a positive integer. (see sawa (1972,p658)). Proof this lemma in appendix (unpublished lecture note Ghazal, 2010)

LEMMA (2)

Let (U) is $T.1$ vector and P is idempotent matrix. The $u'pu$ is quadratic form then it can be transformed $t_1\hat{u}_2'p\hat{u}_2+t_2\hat{u}_1'p\hat{u}_2$ as

$$(\hat{u}_1' \quad \hat{u}_2')_{(1.2)}\begin{pmatrix} 0 & \frac{1}{2}pt_2 \\ \frac{1}{2}pt_2 & pt_1\end{pmatrix}_{(2.2)}\begin{pmatrix}\hat{u}_1 \\ \hat{u}_2\end{pmatrix}_{(2.1)}$$

LEMMA (3)

Let A is nonsingular matrix then $|A|=|A_{11}|.|A_{22}-A_{21}A^{-1}{}_{11}A_{12}|$

LEMMA (4)

Let (P) is symmetric matrix of order T and we have exists an orthogonal matrix (Q)

Such that $Q'PQ = QPQ' = \wedge_{(T.T)}$,

$\wedge_{(T.T)}$ is the diagonal Matrix

$$\wedge_{(T.T)} = \begin{pmatrix} 0_K & 0 \\ 0 & I_{T-K} \end{pmatrix}$$

To proof the theorem (1) we have

$$b_{1(SU)} = (x_1'x_1)^{-1}x_1'y_1 - \frac{\tilde{s}_{12}}{\tilde{s}_{22}}(x_1'x_1)^{-1}x_1'P_2y_2$$

where $b_{1(SU)}$ refer to unrestricted estimator of β_1 , $\tilde{S}$ unrestricted estimator of $\Sigma = (\sigma_{ij})$

$$b_{1(SU)} = (x_1'x_1)^{-1}x_1'x_1b_1 + (x_1'x_1)^{-1}x_1'u_1$$

$$-\frac{\tilde{s}_{12}}{\tilde{s}_{22}}(x_1'x_1)^{-1}x_1'P_2x_2b - \frac{\tilde{s}_{12}}{\tilde{s}_{22}}(x_1'x_1)^{-1}x_1'P_2u_2$$

$$P_2x_2 = 0$$

$$\tilde{S}_{IJ} = \frac{1}{T}\hat{u}_i'p\hat{u}_j$$, then we get

$$(b_{1(SU)} - b_1) = (x_1'x_1)^{-1}x_1'u_1 - \frac{\tilde{s}_{12}}{\tilde{s}_{22}}(x_1'x_1)^{-1}x_1'P_2u_2$$

Then

$$Var(b_{1(SU)}) = \mathrm{E}(b_{1(SU)} - b_1)(b_{1(SU)} - b_1)'$$

$$V(b_{1(SU)}) = (x_1'x_1)^{-1}E\begin{bmatrix} x_1'(u_1u_1')x_1 - \left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)x_1'(u_1u_2')P_2x_1 \\ -\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)x_1'P_2(u_2u_1')x_1 \\ +\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)^2 x_1'P_2(u_2u_2')P_2x_1 \end{bmatrix}(x_1'x_1)^{-1}$$

$$= (x_1'x_1)^{-1}E\begin{bmatrix} x_1'(u_1u_1')x_1 \\ -\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)x_1'(u_1u_2'P_2 + P_2u_2u_1')x_1 \\ +\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)^2 x_1'P_2(u_2u_2')P_2x_1 \end{bmatrix}(x_1'x_1)^{-1}$$

$$= (x_1'x_1)^{-1}E\begin{bmatrix} x_1'(u_1u_1')_1 x_1 \\ -\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)x_1'(u_1u_2' + u_2u_1')P_2x_1 \\ +\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)^2 x_1'P_2(u_2u_2)P_2x_1 \end{bmatrix}(x_1'x_1)^{-1}$$

$$\because E(u_iu_i') = \sigma_{ii}$$

$$,Cov(u_1,u_2) = E(u_1u_2') = \ell_{12}\sqrt{\sigma_{11}\sigma_{22}}$$

$$E(u_1u_2' + u_2u_1') = 2\ell_{12}\sqrt{\sigma_{11}\sigma_{22}}$$

Where ℓ_{12} is the coefficient of correlation between the disturbances in the two equation of the model.

$$= (x_1'x_1)^{-1}\begin{bmatrix} \sigma_{11}x_1'x_1 - 2\ell_{12}\sqrt{\sigma_{11}\sigma_{22}}E\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)x_1'P_2x_1 \\ +\sigma_{22}E\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)^2 x_1'P_2x_1 \end{bmatrix}(x_1'x_1)^{-1}$$

$$= (x_1'x_1)^{-1}\sigma_{11}x_1'x_1(x_1'x_1)^{-1}$$

$$-\left(2\ell_{12}\sqrt{\sigma_{11}\sigma_{22}}E\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right) - \sigma_{22}E\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)^2\right)$$

$$.(x_1'x_1)^{-1}x_1'P_2x_1(x_1'x_1)^{-1}$$

Then the variance of estimator is

$$V(b_{1(SU)}) = \sigma_{11}(x_1'x_1)^{-1} - \left(2\ell_{12}\sqrt{\sigma_{11}\sigma_{22}}E\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right) - \sigma_{22}E\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)^2\right)$$

$$.(x_1'x_1)^{-1}x_1'P_2x_1(x_1'x_1)^{-1}$$

(3-1)

Now we are going to find $\mathrm{E}\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)$ and $\mathrm{E}\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right)^2$ to get complete variance

$$\mathrm{E}\left(\frac{\tilde{s}_{12}}{\tilde{s}_{22}}\right) = \mathrm{E}(e^{t_1\tilde{S}_{22} + t_2\tilde{S}_{12}}) = \mathrm{E}(e^{t_1\hat{u}_2'p\hat{u}_2 + t_2\hat{u}_1'p\hat{u}_2})$$

To apply lemma (1) we need to find the MGF of the numerator and denominator Then by using lemma (2) to transform the power into matrix we get

$$V'^*\left[\begin{pmatrix} 0 & \frac{1}{2}t_2 \\ \frac{1}{2}t_2 & t_1 \end{pmatrix} \otimes p\right]V^* = V'^*(T \otimes P)V^*$$

Where $V'^{*} = \begin{pmatrix} \hat{u}'_1 & \hat{u}'_2 \end{pmatrix}$, $V^{*} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$

Transforming to standard normal

$$V = \begin{pmatrix} \tilde{S}_{12} \\ \tilde{S}_{22} \end{pmatrix} = \frac{\sqrt{\sigma_{11}}\left(\frac{\hat{u}'_1}{\sqrt{\sigma_{11}}}\right)p\left(\frac{\hat{u}_2}{\sqrt{\sigma_{22}}}\right)\sqrt{\sigma_{22}}}{\sqrt{\sigma_{22}}\left(\frac{\hat{u}'_2}{\sqrt{\sigma_{22}}}\right)p\left(\frac{\hat{u}_2}{\sqrt{\sigma_{22}}}\right)\sqrt{\sigma_{22}}}$$

$$= \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}\frac{V_1'PV_2}{V_2'PV_2} = \sqrt{\frac{\sigma_{11}}{\sigma_{22}}}\hat{\theta}$$

Where $V_1 = \left(\frac{\hat{u}'_1}{\sqrt{\sigma_{11}}}\right)$, $V_2 = \left(\frac{\hat{u}_2}{\sqrt{\sigma_{22}}}\right)$

$$\mathrm{E}(V_i) = \mathrm{E}\left(\frac{\hat{u}'_i}{\sqrt{\sigma_{ii}}}\right) = \frac{1}{\sqrt{\sigma_{ii}}}\mathrm{E}(\hat{u}'_i) = 0_{(T.1)}$$

$$Var(V_i) = \mathrm{E}(V_iV_i') = \mathrm{E}\left(\frac{1}{\sqrt{\sigma_{ii}}}\hat{u}_i\right)\left(\frac{1}{\sqrt{\sigma_{ii}}}\hat{u}'_i\right)$$

$$= \frac{1}{\sigma_{ii}}\mathrm{E}(\hat{u}_i\hat{u}'_i) = I_T$$

$$Cov(V_1,V_2) = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} = \ell_{12}$$

(V) is following the normal distribution with zero mean vector $(0)_{(T.1)}$ and variance covariance matrix $\begin{pmatrix} 1 & \ell \\ \ell & 1 \end{pmatrix} \otimes I$,then

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, (R \otimes I)\right)$$

Where $(R \otimes I) = \begin{pmatrix} 1 & \ell \\ \ell & 1 \end{pmatrix} \otimes I$

Then

$$\mathrm{E}\begin{pmatrix} \frac{\tilde{s}_{12}}{\tilde{s}_{22}} \end{pmatrix} = \mu(t_1,t_2) = \mathrm{E}(e^{t_1v_2'pv_2 + t_2v_1'pv_2})$$

By definition, the moment generating functions of $V_1'PV_2$ and $V_2'PV_2$ is given by

$$\mu(t_1,t_2) = \int_{-\infty}^{\infty} e^{t_1v_2'pv_2 + t_2v_1'pv_2}.f(x_1,x_2)dx_1dx_2$$

Where $f(x_1,x_2)$ stands for density function of a multivariate normal distribution by mean vector $(0)_{(T.1)}$ and variance covariance matrix $(R \otimes I)$ then we have

$$f(x_1,x_2) = \frac{1}{(2\pi)^{T/2}|R \otimes I|^{1/2}}$$

$$.\int \cdots \int e^{\frac{-1}{2}V'(R^{-1} \otimes I)V} dv$$

$$\mu(t_1,t_2) = \frac{1}{(2\pi)^{T/2}|R \otimes I|^{1/2}}$$

$$.\int \cdots \int e^{-V'(T \otimes P)V} e^{\frac{-1}{2}V'(R^{-1} \otimes I)V} dv$$

Multiply by determination to get probability density function

$$\mu(t_1,t_2) = \frac{1}{|R \otimes I|^{1/2}|(R^{-1} \otimes I) - 2(T \otimes P)|^{\frac{1}{2}}}$$

$$.\int \cdots \int \frac{|(R \otimes I) - 2(T \otimes P)|^{\frac{1}{2}}}{(2\pi)^{T/2}} e^{\frac{-1}{2}V'(R^{-1} \otimes I) - 2(T \otimes P)V} dv$$

The integral part is Probability density function equal one in view of the multiple of the a multivariate normal density

Then

$$\mu(t_1,t_2) = \frac{1}{|I_T - 2(RT \otimes P)|^{\frac{1}{2}}}$$

$$RT \otimes P = \begin{pmatrix} \frac{1}{2}\ell t_2 p & \frac{1}{2}t_2 + \ell t_1 p \\ \frac{1}{2}t_2 p & \frac{1}{2}\ell t_2 + t_1 p \end{pmatrix}$$

$$\mu(t_1,t_2) = \begin{vmatrix} (I_T - \ell t_2 p) & -(t_2 + 2\ell t_1)p \\ -(t_2 p) & I - (\ell t_2 + 2t_1)p \end{vmatrix}^{\frac{-1}{2}}$$

By using lemma (3) we get

$$\mu(t_1,t_2) = |I_T - \ell t_2 p|^{\frac{-1}{2}}$$

$$.\begin{vmatrix} (I - (\ell t_2 + 2t_1)p) - (-t_2 p) \\ .(I_T - \ell t_2 p)^{-1}(-(t_2 + 2\ell t_1))p \end{vmatrix}^{\frac{-1}{2}}$$

$$\mu(t_1,t_2)=\left|I_T-\ell t_2 p\right|^{\frac{-1}{2}}$$

$$\cdot\begin{vmatrix}(I-(\ell t_2+2t_1)p)-t_2 \\ (t_2+2\ell t_1)p(I_T-\ell t_2 p)^{-1}p\end{vmatrix}^{\frac{-1}{2}} \quad (3\text{-}2)$$

Using lemma (4) to get orthogonal transformation

$$|Q|^{\frac{-1}{2}} \quad |I-\ell t_2\wedge|^{\frac{-1}{2}}|Q'|^{\frac{-1}{2}}.|Q|^{\frac{-1}{2}}\begin{vmatrix}I-(\ell t_2+2t_1)\wedge \\ -t_2(t_2+2\ell t_1)\wedge \\ (I-\ell t_2\wedge)^{-1}\wedge\end{vmatrix}^{\frac{-1}{2}}|Q'|^{\frac{-1}{2}}$$

$$|Q|^{-\frac{1}{2}}|Q'|^{-\frac{1}{2}}=\pm 1$$, then

$$=\left|\begin{bmatrix}I & 0 \\ 0 & (1-\ell t_2)I_{T-K}\end{bmatrix}\right|^{\frac{-1}{2}}\left|\begin{bmatrix}I & 0 \\ 0 & (1-\ell t_2+2t_1)I_{T-K} - \left(\dfrac{t_2^2+2\ell t_1 t_2}{1-\ell t_2}\right)I_{T-K}\end{bmatrix}\right|^{\frac{-1}{2}}$$

$$\mu(t_1,t_2)=(1-\ell t_2)^{-(\frac{T-K}{2})}\begin{pmatrix}(1-\ell t_2+2t_1) \\ -(\dfrac{t_2^2+2\ell t_1 t_2}{1-\ell t_2})\end{pmatrix}^{-(\frac{T-K}{2})}$$

Then we have

$$\mu(t_1,t_2)=(1-2t_1-2\ell t_2+\ell^2 t_2^2-t_2^2)^{-(\frac{T-K}{2})} \quad (3\text{-}3)$$

In order to apply lemma (1), we differentiate the $\mu(t,t)$ in (3-3) with respect to t_2 , evaluate the derivatives at $t_2=0$ .and using lemma (1) when r =1

$$E\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)^r=\frac{1}{\Gamma r}\int_{-\infty}^{0}-t_1^{r-1}.\frac{\partial^1\mu(t_1,t_2)}{dt_2^r}dt_1 ,$$

$$\frac{\partial'\mu(t_1,t_2)}{dt_2}=\frac{\partial'}{dt_2}(1-2t_1-2\ell t_2+\ell^2 t_2^2-t_2^2)^{-(\frac{T-K}{2})}$$

$$=(T-K)(1-2t_1-2\ell t_2+\ell^2 t_2^2-t_2^2)^{-(\frac{T-K}{2})-1}$$
$$.(\ell-\ell^2 t_2+t_2) \quad (3\text{-}4)$$

When t2=0

$$\frac{\partial'\mu(t_1,t_2)}{dt_2}=\ell(T-K)(1-2t_1)^{-(\frac{T-K}{2})-1}$$

By using lemma (1) we get

$$E\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)=\ell(T-K)\int_{-\infty}^{0}(1-2t_1)^{-(\frac{T-K}{2})-1}dt_1$$

BY using Beta function we get

$$=\frac{\ell(T-K)}{2}\int_0^\infty\frac{1}{(1+\theta)^{(\frac{T-K}{2})+1}}d\theta$$

$$\beta(\frac{T-K}{2},1)$$

$$E\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)=\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}E(\hat{\theta})=\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}\ell \qquad \ell=\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$$

Then $$E\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)=\frac{\sigma_{12}}{\sigma_{22}} \quad (3\text{-}5)$$

Now we are going to find second moment

$$E\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)^2=\frac{1}{\Gamma 2}\int_{-\infty}^{0}-t_1^{2-1} \quad .\frac{\partial^2\mu(t_1,t_2)}{dt_2^2} \quad dt_1$$

By using the first differentiation in (3-4)

$$\frac{\partial'\mu(t_1,t_2)}{dt_2}=(T-K)$$

$$.(1-2t_1-2\ell t_2+\ell^2 t_2^2-t_2^2)^{-(\frac{T-K}{2})-1}.(\ell-\ell^2 t_2+t_2)$$

$$\frac{\partial''\mu(t_1,t_2)}{dt_2}=\frac{\partial''}{dt_2}(T-K)$$

$$.(1-2t_1-2\ell t_2+\ell^2 t_2^2-t_2^2)^{-(\frac{T-K}{2})-1}(\ell-\ell^2 t_2+t_2)$$

$$=(T-K)\begin{pmatrix}(1-2t_1-2\ell t_2+\ell^2 t_2^2-t_2^2)^{-(\frac{T-K}{2})-1}.(1-\ell^2)+ \\ \begin{pmatrix}(\ell-\ell^2 t_2+t_2)^2(T-K+2) \\ (1-2t_1-2\ell t_2+\ell^2 t_2^2-t_2^2)^{-(\frac{T-K}{2})-2}\end{pmatrix}\end{pmatrix}$$

When t2=0 we get

$$\frac{\partial''\mu(t_1,t_2)}{dt_2}|_{t_2=0}=(T-K)\begin{pmatrix}(1-2t_1)^{-(\frac{T-K}{2})-1} \\ .(1-\ell^2)+\ell^2(T-K+2)(1-2t_1)^{-(\frac{T-K}{2})-2}\end{pmatrix}$$

Applying lemma (1) when r = 2

$$\mathrm{E}\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)^2 = \frac{1}{\Gamma 2}\int_{-\infty}^{0} -t_1 \quad .\frac{\partial'' \mu(t_1,t_2)}{dt_2^2}\,dt_1$$

$$= \int_{-\infty}^{0} -t_1(T-K)\begin{pmatrix}(1-2t_1)^{-\left(\frac{T-K}{2}\right)-1}.(1-\ell^2) \\ +\ell^2(T-K+2)(1-2t_1)^{-\left(\frac{T-K}{2}\right)-2}\end{pmatrix} dt_1$$

Let $n=(T-K)$, $\left(\frac{T-K}{2}\right)>1$ then we have

$$= \frac{(n)}{2}\left(\frac{(1-\ell^2)}{(n-2)} + \frac{[\ell^2(n+2)-(1-\ell^2)]}{(n)} - \frac{\ell^2(n+2)}{(n+2)}\right)$$

$$= \left(\frac{(n)(1-\ell^2)+(n)(3\ell^2-1)-2(3\ell^2-1)}{2(n-2)}\right)$$

Take (n) as a common factor

Then

$$\mathrm{E}(\theta^2) = \ell^2 + \left(\frac{(1-\ell^2)}{(n-2)}\right)$$

$$E\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)^2 = \left(\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}\right)^2 .\mathrm{E}(\theta^2)$$

$$E\left(\frac{\tilde{S}_{12}}{\tilde{S}_{22}}\right)^2 = \ell^2\frac{\sigma_{11}}{\sigma_{22}} + \left(\frac{1-\ell^2}{(n-2)}\right)\frac{\sigma_{11}}{\sigma_{22}} \qquad (3\text{-}6)$$

Substitute (3-5),(3-6) in(3-1) we get

$$V(\tilde{\beta}_{1(SU)}) = \sigma_{11}(x_1'x_1)^{-1} - \sigma_{11}\left(\ell_{12}^2 - \frac{1-\ell_{12}^2}{(n-2)}\right)$$
$$.(x_1'x_1)^{-1}x_1'P_2x_1(x_1'x_1)^{-1} \qquad (3\text{-}7)$$

Now we are going to derive the variance covariance matrix of the (SUUR) estimator .thus we derive the variance of $\tilde{\beta}_{(2)SU}$ estimator and derive covariance between $\tilde{\beta}_{(1)SU}$, $\tilde{\beta}_{(2)SU}$

The variance of $\tilde{\beta}_{(2)SU}$ In the (SUUR) estimator of $(\tilde{\beta}_{(2)SU})$ being identical to the (OLS) estimator then the variance of $(\tilde{\beta}_{(2)SU})$ as follows

$$\mathrm{var}(\tilde{\beta}_{(2)SU}) = \sigma_{22}(x_2'x_2)^{-1} \qquad (3\text{-}8)$$

The covariance between $\tilde{\beta}_{(1)SU}$, $\tilde{\beta}_{(2)SU}$

Noted that $x_2'P_1 = 0$, $P_2x_2 = 0$, $x_1'P_2 \neq 0$

$$\mathrm{cov}(\tilde{\beta}_{(1)SU}, \tilde{\beta}_{(2)SU}) = \sigma_{12}(x_1'x_1)^{-1}(x_1'x_2)(x_2'x_2)^{-1} \qquad (3\text{-}9)$$

Similarly The covariance between $\tilde{\beta}_{(2)SU}$ *and* $\tilde{\beta}_{(1)SU}$

$$\mathrm{cov}(\tilde{\beta}_{(2)SU}, \tilde{\beta}_{(1)SU}) = \sigma_{12}(x_2'x_2)^{-1}(x_2'x_1)(x_1'x_1)^{-1} \qquad (3\text{-}10)$$

From (3-7),(3-8),(3-9) and (3-10) we get the complete variance covariance matrix of the (SUUR) estimator of $(\tilde{\beta})$ when x_2 in second equation is subset from x_1 in the first equation

$$\left(\begin{array}{c|c} \sigma_{11}(x_1'x_1)^{-1} - \sigma_{11}\left(\ell_{12}^2 - \frac{1-\ell_{12}^2}{(n-2)}\right).(x_1'x_1)^{-1}x_1'P_2x_1(x_1'x_1)^{-1} & \sigma_{12}(x_1'x_1)^{-1}(x_1'x_2)(x_2'x_2)^{-1} \\ \hline \sigma_{21}(x_2'x_2)^{-1}(x_2'x_1)(x_1'x_1)^{-1} & \sigma_{22}(x_2'x_2)^{-1} \end{array}\right)$$

This results as Revankar (1974; P.189),Srivastava (1987;P. 106), Aiyi Liu(2002)

IV. CONCLUDING REMARK

In this paper we concerned with zellner's estimators of (SURE) model in the case of the system of two equations that the regressors in the second equation are subset of the regressors in the first equation, so x_2 is a subset matrix of x_1. We have derived the exact first and second moment of coefficient estimator for seemingly unrelated unrestricted regression (SUUR).Assume that the regressors in the two equations are orthogonal and disturbance terms have a normal distribution, by applying a new approach we get the variance covariance matrix of unrestricted estimator that based on unrestricted estimate of variance of disturbance term S ,then we have two estimators ,both of them are unbiased estimator . Compare the results by applying (GLS) method .We show that the (GLS) estimator of B_2 in second equation is just (OLS) estimator under the above assumption. And it has smaller variance than the least squares estimator. It is clear that regression coefficient estimator which obtained by applying (GLS) is more efficient than those obtained by using Least Square method used to estimate coefficient for each equation separately as single equation.

APPENDIX

Lemma (1)

$$\mu(t_1,t_2)=\mathrm{E}\left(e^{t_1x_1+t_2x_2}\right)$$

$$\mathrm{E}\left(\frac{X_2}{X_1}\right)^r=\frac{1}{\Gamma r}\int_{-\infty}^{0}(-t^{r-1})\frac{\partial^r\mu(t_1,t_2)}{\partial t_2^r}\Big|_{t_2=0}dt_1$$

Where (r) is a positive integer.

Proof (unpublished lecture note Ghazal, 2010)

$$\mathrm{E}\left(\frac{X_2}{X_1}\right)^r=\int_{x_2=-\infty}^{\infty}\int_{x_1=0}^{\infty}X_2^r\cdot\frac{1}{X_1^r}\;.f(X_1,X_2)\quad dX_1dX_2$$

$$\mathrm{E}\left(\frac{X_2}{X_1}\right)^r=\int_{x_2=-\infty}^{\infty}\int_{x_1=0}^{\infty}\frac{\partial^r e^{t_2x_2}}{\partial t_2^r}\Big|_{t_2=0}\left[\frac{1}{\Gamma r}\int_0^{\infty}\theta^{r-1}e^{-x\theta}d\theta\right].f(X_1,X_2)\quad dX_1dX_2$$

$$let\,\theta=-t\quad,d\theta=-dt$$

$$\mathrm{E}\left(\frac{X_2}{X_1}\right)^r=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\partial^r e^{t_2x_2}}{\partial t_2^r}\Big|_{t_2=0}\left[\frac{1}{\Gamma r}\int_0^{-\infty}-t^{r-1}e^{xt}-dt\right].f(X_1,X_2)\quad dX_1dX_2$$

$$\mathrm{E}\left(\frac{X_2}{X_1}\right)^r=\frac{1}{\Gamma r}\int_{-\infty}^{0}(-t^{r-1})\frac{\partial^r}{\partial t_2^r}\left[\int_{-\infty}^{\infty}\int_0^{\infty}e^{xt+t_2x}.f(X_1,X_2)\quad dX_1dX_2\right]dt_1$$

$$\left[\int_{-\infty}^{\infty}\int_0^{\infty}e^{xt+t_2x}.f(X_1,X_2)\quad dX_1dX_2\right]=\mu(t_1,t_2)$$

Then

$$\mathrm{E}\left(\frac{X_2}{X_1}\right)^r=\frac{1}{\Gamma r}\int_{-\infty}^{0}(-t^{r-1})\frac{\partial^r\mu(t_1,t_2)}{\partial t_2^r}\Big|_{t_2=0}dt_1$$

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