# Common Fixed Point Theorem For Finite Mappings On Menger Space Using Semi Compatibility

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**Abstract:** In this paper we establish Common fixed point theorem for eight mappings in Menger space using the notion of semi compatibility.

**Keywords**: Menger space, Weak compatibility, Compatibility, Semi compatibility, Common fixed point.

## 1. Introduction

There have been number of generalizations of metric spaces. One such generalization is Menger space introduced in 1942 by Menger [5] who used distribution functions instead of nonnegative real numbers as values of the metric. This space was rapidly with the pioneering of Schweizer and Sklar [7, 8]. This concept plays a vital role in probabilistic functional analysis , nonlinear analysis and applications. In 1972 , V.M. Sehgal and A.T. Bharucha-Reid [9] obtained a generalization of Banach contraction principle on a complete Menger space which is mile stone in development fixed point theory in Menger space. In 1986, jungck [3] introduced the notion of compatible mappings in metric spaces, And this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [4]. Cho et. al. [2] have introduced the notion of semi-compatible maps in a d-topological space. Singh and Jain [11] have established some fixed point theorems in Menger space using semi-compatibility of the mappings. The concept of weakly compatible mappings is most general as each pair of compatible mappings is compatible but the reverse is not true. Singh and jain [10] established a common fixed point theorem in Menger space using the concept of weak compatibility and compatibility of pair of self maps. In this paper we generalize and extend the result of B.D.Pant and Sunny Chauhan [6] for eight mapping opposed to six mappings in complete menger space using the concept of compatibility.

## 2. Preliminaries

**Definition 2.1** A probabilistic metric space ( PM-space ) is an ordered pair $(X,F)$ consisting of a non empty set $X$ and a function $F: X \times X \to L$, where $L$ is the collection of all distribution functions and the value of $F$ at $(u,v) \epsilon X \times X$ is represented by $\mathrm{F}_{\mathrm{u,v}}$ . The function $F_{u,v}$ is assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$ ,for all $x > 0$ if and only if $u = v$

(PM- 2) $F_{u,v}(0) = 0$;

(PM- 3) $F_{u,v} = F_{v,u}$ ;

(PM- 4) $F_{u,v}(x) = 1$ and $F_{v,w}$ (y) = 1 then $F_{u,w}(x+y) = 1$ for all $u, v, w$ in $X$ and $x, y \geq 0$.

**Definition 2.2**: A mapping $t: [0,1] \times [0,1] \to [0,1]$ is called a t- norm if

(a) $t(a,1) = a, t(0,0) = 0$

(b) $t(a,b) = t(b,a)$ (symmetric property)

(c) $t(c,d) \geq t(a,b)$ for $c \geq a, d \geq b$

(d) $t(t(a,b),c) = t(a,t(b,c))$

**Definition 2.3**: A Menger space is a triplet $(X,F,t)$ where $(X,F)$ is a PM- space and $t$ is a t-norm such that the inequality

$F_{u,w}(x+y) \geq$ t { $F_{u,v}(x)$ , $F_{v,w}(y)$ } for all $u, v, w \; in \; X$ and $x, y > 0$

**Definition 2.4 :** The self maps $A$ and $B$ of a Menger space $(X,F,t)$ are said to be compatible if

$F_{ABx_n,BAx_n}(t) \to 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $Ax_n, Bx_n \to z$ for some $z \in X$ as $n \to \infty$.

**Definition 2.5**: Self –maps $A$ and $S$ of a Menger space $(X,F,t)$ are said to be weak compatible if they commute at their coincidence points i.e. if $Ap = Sp$ for some $p \epsilon$ N then $ASp = SAp$

**Definition 2.6 :** A pair $(A,B)$ of Self –maps of a Menger space $(X,F,t)$ are said to be semi compatible if

$F_{ABx_n,BAx}(t) \to 1$ for all $t > 0$ ,

Whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} B\, x_n = x \in X$.

It follows that if $(A,B)$ is semi compatible and $Ax = Bx$ then $= BAx$ . thus if the pair $(A,B)$ is semi –compatible then it is weakly compatible but the converse is not true .

**Lemma (1)** : Let $\{x_n\}$be a sequence in a Menger space $(X,F,t)$ with continuous t-norm and $t(x,x) \geq x$ . suppose for all $x \epsilon$ [0, 1] there exists $k \epsilon$ (0, 1) such that for all $x > 0$ and $n \epsilon N$

$F_{x_n,x_{n+1}}(kx) \geq F_{x_{n-1},x_n}(x)$

Then $\{x_n\}$is a Cauchy sequence in $X$.

**Lemma (2)** : Let $(X,F,t)$ be a Menger space . If there exists $k \epsilon (0,1)$ such that for $p, q \; \epsilon \; X$

$F_{p,q}(kx) \geq F_{p,q}(x)$. Then $p = q$

In 2010, B D Pant and Sunny Chauhan [6] gave the following fixed point theorem for six mappings in Menger space through semi-compatibility.

**Theorem:** Let $A, B, S, T, L$ and $M$ be self maps on a complete Menger space $(X,F,T)$ with continuous t- norm T defined by $T(a,b) = \min\{a,b\}$ for $a, b \in [0,1]$ and satisfying the following;

(a) $AB(X) \subset M(X)$ and $ST(X) \subset L(X)$;
(b) $M(X)$ and $L(X)$ are complete subspace of $X$;
(c) Either $AB$ or $ST$ is continuous;
(d) $(AB,L)$ is semi-compatible and $(ST,M)$ is weakly compatible;
(e) For all $x, y \in X$ , $k \in (0,1), t > 0$.

$$F^3{}_{ABx,STy}(kt) \geq \min\{F^3{}_{Lx,My}(t), F^3{}_{ABx,Lx}(t), F^3{}_{STy,My}(t), \quad F_{ABx,My}(2t), F_{STy,Lx}(2t), F^2{}_{STy,My}(t)\}$$

Then $AB, ST, L$ and $M$ have a common fixed point in $X$.

## 3. Main Results:

**Theorem (3.1)**: Let $A, B, S, T, L, M, P$ and $Q$ are self mappings on a complete Menger space $(X, F, t)$ satisfying:

(3.1.1) $A(X) \subseteq ST(X) \cap L(X) \cap M(X), B(X) \subseteq PQ(X)$

(3.1.2) $PQ = QP, ST = TS, AQ = QA, BT = TB, LT = TL, MT = TM$.

(3.1.3) Either $A$ is continuous or $PQ$ is continuous.

(3.1.4) $(A, PQ)$ is Semi-compatible and $(L, ST), (B, ST), (L, M)$ are weak compatible.

(3.1.5) There exists $k \epsilon (0,1)$ such that

$$F^3{}_{Ax,By}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{PQx,Ly}(t)\ F^3{}_{STy,Ax}(t), F^3{}_{PQx,STy}(t), \\ F^3{}_{By,Ly}(t), F_{Ax,Ly}(2t), F_{PQx,By}(2t), \\ F^2{}_{By,My}(t), F_{Ly,PQx}(t) \end{array} \right\}$$

For all $x, y \in X$ and $t > 0$. Then self –maps $A, B, S, T, L, M, P$and $Q$ have a unique common fixed point in X.

**Proof** :

Let $x_0 \in X$ By condition 3.1.1 there exists $x_1, x_2 \in X$such that $Ax_0 = STx_1 = Lx_1 = Mx_1 = y_0$ and $Bx_1 = PQx_2 = y_1$. Inductively we can construct sequence $\{x_n\}$ and $\{y_n\}$ in $X$ such that $Ax_{2n} = STx_{2n+1} = Lx_{2n+1} = Mx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = PQx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots\ldots$

Now we prove $\{y_n\}$ is a Cauchy sequence in $X$.

Putting $= x_{2n}$ , $y = x_{2n+1}$ for $x > 0$ in 3.1.5 then we have

$$F^3{}_{Ax_{2n},Bx_{2n+1}}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{PQx_{2n},Lx_{2n+1}}(t), F^3{}_{STx_{2n+1},Ax_{2n}}(t), F^3{}_{PQx_{2n},STx_{2n+1}}(t), \\ F^3{}_{Bx_{2n+1},Lx_{2n+1}}(t), F_{Ax_{2n},Lx_{2n+1}}(2t), \\ F_{PQx_{2n},Bx_{2n+1}}(2t), F^2{}_{Bx_{2n+1},Mx_{2n+1}}(t), F_{Lx_{2n+1},PQx_{2n}}(t) \end{array} \right\}$$

$$F^3{}_{y_{2n},y_{2n+1}}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{y_{2n-1},y_{2n}}(t), F^3{}_{y_{2n},y_{2n}}(t), F^3{}_{y_{2n-1},y_{2n}}(t), \\ F^3{}_{y_{2n+1},y_{2n}}(t), F_{y_{2n},y_{2n}}(2t), \\ F_{y_{2n-1},y_{2n+1}}(2t), F^2{}_{y_{2n+1},y_{2n}}(t), F_{y_{2n},y_{2n-1}}(t) \end{array} \right\}$$

Hence

$$F^3{}_{y_{2n},y_{2n+1}}(kt) \geq \min\{F^3{}_{y_{2n-1},y_{2n}}(t), F_{y_{2n-1},y_{2n+1}}(2t), F^2{}_{y_{2n+1},y_{2n}}(t)\}$$

$$F^3{}_{y_{2n},y_{2n+1}}(kt) \geq \min\{F^3{}_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t), F^2{}_{y_{2n+1},y_{2n}}(t)\}$$

$$F_{y_{2n},y_{2n+1}}(kt) \geq \min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t)\}$$

Similarly, we can have

$$F_{y_{2n+1},y_{2n+2}}(kt) \geq \min\{F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+2}}(t)\}$$

Therefore for all n even or odd we have

$$F_{y_n,y_{n+1}}(kt) \geq \min\{F_{y_{n-1},y_n}(t), F_{y_n,y_{n+1}}(t)\}$$

Consequently, it follows that for p = 1,2,3, …

$$F_{y_n,y_{n+1}}(kt) \geq \min\left\{F_{y_{n-1},y_n}(t), F_{y_n,y_{n+1}}\left(\frac{t}{k^p}\right)\right\}$$

By noting that $F_{y_n,y_{n+1}}\left(\frac{t}{k^p}\right) \to 1$ as $n \to \infty$ it follows that

$F_{y_n,y_{n+1}}(kt) \geq \min\{F_{y_{n-1},y_n}(t)\}$ for all $n \in N$ and $t > 0$.

Hence by Lemma (1) , $\{y_n\}$ is a Cauchy sequence in X which is complete. Hence $\{y_n\} \to z \epsilon X$ . we shall use the fact that subsequence $\{y_{2n}\}$ also converges to z.

$Ax_{2n} \to STx_{2n+1} \to z, Lx_{2n+1} \to z, Mx_{2n+1} \to z$ , $Bx_{2n+1} \to z, PQx_{2n} \to z$

Case I : When A is continuous, then $AAx_{2n} \to Az$ and $A(PQx_{2n}) \to Az$.

As $(A, PQ)$ is semi compatible then we have $A(PQx_{2n}) \to PQz$

As the limit of sequence in Menger space is unique then we have $PQz = Az$

Step 1 By putting $x = z$ and $y = x_{2n+1}$ in 3.1.5 we get

$$F^3{}_{Az,Bx_{2n+1}}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{PQz,Lx_{2n+1}}(t), F^3{}_{STx_{2n+1},Az}(t), F^3{}_{PQz,STx_{2n+1}}(t), \\ F^3{}_{Bx_{2n+1},Lx_{2n+1}}(t), F_{Az,Lx_{2n+1}}(2t), F_{PQz,Bx_{2n+1}}(2t), \\ F^2{}_{z,z}(t), F_{Lx_{2n+1},PQz}(t) \end{array} \right\}$$

Taking the limit as $n \to \infty$,we get $F^3{}_{Az,z}(kt) \geq \min\{F^3{}_{Az,z}(t)\}$

$$F_{Az,z}(kt) \geq F_{Az,z}(t)$$

Therefore by Lemma 2 we have $Az = z$. since $PQz = Az$ thus we have $Az = PQz = z$

Step 2 : As $A(X) \subseteq ST(X) \cap L(X) \cap M(X)$ then there exist $w \in X$ such that $z = Az = STw = Lw = Mw$. Putting $x = x_{2n,}$ $y = w$ in 3.1.5 we get

$$F^3{}_{Ax_{2n},Bw}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{PQx_{2n},Lw}(t), F^3{}_{STw,Ax_{2n}}(t), F^3{}_{PQx_{2n},STw}(t), \\ F^3{}_{Bw,Lw}(t), F_{Ax_{2n},Lw}(2t), F_{PQx_{2n},Bw}(2t), \\ F^2{}_{Bw,Mw}(t), F_{Lw,PQx_{2n}}(t) \end{array} \right\}$$

Taking the limit as $n \to \infty$,we get

$$F^3{}_{z,Bw}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{z,z}(t), F^3{}_{z,z}(t), F^3{}_{z,z}(t), F^3{}_{Bw,z}(t), \\ F_{z,z}(2t), F_{z,Bw}(2t), F^2{}_{Bw,z}(t), F_{z,z}(t) \end{array} \right\}$$

$F^3{}_{z,Bw}(kt) \geq \min\{F^3{}_{Bw,z}(t)\}$

$F_{z,Bw}(kt) \geq F_{Bw,z}(t)$

Therefore by Lemma 2 we have $Bw = z$

Hence $z = Az = STw = Lw = Mw = Bw$

As $(L, ST), (B, ST), (L, M)$ are weak compatible we have $STLw = LSTw, STBw = BSTw$ and $LMw = MLw$

$STz = Lz, STz = Bz, Lz = Mz$

i.e. $STz = Bz = Lz = Mz$

Step 3: Putting $x = x_{2n,}$ $y = z$ in 3.1.5 we get

$F^3{}_{Ax_{2n},Bz}(kt) \geq$

$\min\{F^3{}_{PQx_{2n},Lz}(t), F^3{}_{STz,Ax_{2n}}(t), F^3{}_{PQx_{2n},STz}(t), \quad F^3{}_{Bz,Lz}(t), F_{Ax_{2n},Lz}$

Taking the limit as $n \to \infty$,we get and using $STz = Bz = Lz = Mz$.

$$F^3{}_{z,Bz}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{z,Bz}(t), F^3{}_{Bz,z}(t), F^3{}_{z,Bz}(t), F^3{}_{Bw,Bz}(t), \\ F_{z,Bz}(2t), F_{z,Bz}(2t), F^2{}_{Bz,z}(t), F_{Bz,z}(t) \end{array} \right\}$$

$F^3{}_{z,Bz}(kt) \geq \min\{F^3{}_{z,Bz}(t)\}$

$F_{z,Bz}(kt) \geq F_{z,Bz}(t)$

Therefore by Lemma 2 we have $Bz = z$

Hence $z = Bz = Lz = Mz = STz$

Step 4: Putting $x = x_{2n,}$ $y = Tz$ in 3.1.5 we get

$$F^3{}_{Ax_{2n},BTz}(kt) \geq \min \left\{ \begin{array}{c} F^3{}_{PQx_{2n},LTz}(t), F^3{}_{STTz,Ax_{2n}}(t), F^3{}_{PQx_{2n},STTz}(t), \\ F^3{}_{BTz,LTz}(t), F_{Ax_{2n},LTz}(2t), F_{PQx_{2n},BTz}(2t), \\ F^2{}_{BTz,MTz}(t), F_{LTz,PQx_{2n}}(t) \end{array} \right\}$$

As $BT = TB, ST = TS, LT = TL$ and $MT = TM$ we have

$$BTz = TBz = Tz, LTz = TLz = Tz, MTz = TMz = Tz, ST(Tz) = TS(Tz) = T(STz) = Tz$$

Taking the limit as $n \to \infty$,we get

$$F^3{}_{z,Tz}(kt) \geq \min \begin{Bmatrix} F^3{}_{z,Tz}(t), F^3{}_{Tz,z}(t), F^3{}_{z,Tz}(t), F^3{}_{Tz,Tz}(t), \\ F_{z,Tz}(2t), F_{z,Tz}(2t), F^2{}_{Tz,Tz}(t), F_{Tz,z}(t) \end{Bmatrix}$$

$$F^3{}_{z,Tz}(kt) \geq \min \{F^3{}_{z,Tz}(t)\}$$

$$F_{z,Tz}(kt) \geq F_{z,Tz}(t)$$

Therefore by Lemma 2 we get $Tz = z$

Now $STz = Tz = z$ implies $Sz = z$

Hence $Sz = Tz = Bz = Lz = Az = Mz = z$

Step 5: Putting $x = Qz$ and $y = x_{2n+1}$ in 3.1.5 we get

$$F^3{}_{AQz,Bx_{2n+1}}(kt) \geq$$

$$\min \begin{Bmatrix} F^3{}_{PQQz,Lx_{2n+1}}(t), F^3{}_{STx_{2n+1},AQz}(t), F^3{}_{PQQz,STx_{2n+1}}(t), \\ F^3{}_{Bx_{2n+1},Lx_{2n+1}}(t), F_{AQz,Lx_{2n+1}}(2t), F_{PQQz,Bx_{2n+1}}(2t), \\ F^2{}_{Qz,Qz}(t), F_{Lx_{2n+1},PQQz}(t) \end{Bmatrix}$$

As $AQ = QA$ and $AB = BA$ we have

$$PQ(Qz) = QP(Qz) = Q(PQz) = Qz, AQz = QAz = Qz$$

Taking the limit as $n \to \infty$,we get

$$F^3{}_{Qz,z}(kt) \geq \min \begin{Bmatrix} F^3{}_{Qz,z}(t), F^3{}_{z,Qz}(t), F^3{}_{Qz,z}(t), F^3{}_{z,z}(t), \\ F_{Qz,z}(2t), F_{Qz,z}(2t), F^2{}_{Qz,Qz}(t), F_{z,Qz}(t) \end{Bmatrix}$$

$$F^3{}_{Qz,z}(kt) \geq \min \{F^3{}_{Qz,z}(t)\}$$

$$F_{Qz,z}(kt) \geq F_{Qz,z}(t)$$

Therefore by Lemma 2 we get $Qz = z$

Since $PQz = z$ implies $Pz = z$

Therefore $Pz = Qz = Az = z$

Now combine all the results we get $Az = Bz = Sz = Tz = Lz = Mz = Pz = Qz = z$ .Thus $z$ is a common fixed point of the mappings $A, B, S, T, L, M, P$and $Q$.

Case II: If $PQ$ is continuous , the proof follows by case I

**Uniqueness**: Let $z'$ $(z \neq z')$ be another common fixed point of $A, B, S, T, L, M, P$and $Q$ then $Az' = Bz' = Sz' = Tz' = Lz' = Mz' = Pz' = Qz' = z'$

By putting $x = z$ and $y = z'$ in 3.1.5 we get

$$F^3{}_{Az,Bz'}(kt) \geq$$

$$\min \begin{Bmatrix} F^3{}_{PQz,Lz'}(t), F^3{}_{STz',Az}(t), F^3{}_{PQz,STz'}(t), F^3{}_{Bz',Lz'}(t), \\ F_{Az,Lz'}(2t), F_{PQz,Bz'}(2t), F^2{}_{z,z}(t), F_{Lz',PQz}(t) \end{Bmatrix}$$

$$F^3{}_{z,z'}(kt) \geq \min \begin{Bmatrix} F^3{}_{z,z'}(t), F^3{}_{z',z}(t), F^3{}_{z,z'}(t), F^3{}_{z',z'}(t), \\ F_{z,z'}(2t), F_{z,z'}(2t), F^2{}_{z,z}(t), F_{z',z}(t) \end{Bmatrix}$$

$$F^3{}_{z,z'}(kt) \geq \min \{F^3{}_{z,z'}(t)\}$$

$$F_{z,z'}(kt) \geq F_{z,z'}(t)$$

by Lemma 2 we get $z = z'$ for all $x, y \in X$ and $t > 0$. Therefore $z$ is the unique common fixed point of $A, B, S, T, L, M, P and\ Q$.

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