Qualitative Behaviour Of Differential Equations

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Abstract: Qualitative analysis of differential equations can provide valuable insight to solve complicated biological models. In this paper, a comparison is made between exponential growth and logistic growth of the population of a given species using differential equations. The application of geometrical methods to obtain important qualitative information directly from the differential equation, without solving the equation is studied in this paper. AMS SUBJECT CLASSIFICATION CODE: 34D20

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1. INTRODUCTION

A first order ordinary differential equation in which the independent variable does not appear explicitly, ie, an equation of the form

$$\frac{dy}{dt} = f(y) \rightarrow (1)$$

is called an AUTONOMOUS EQUATION.

Autonomous equations play an important role in the growth or decline of the population of a given species. Section1 deals with the basic concepts relating to exponential growth and logistic growth of the population of a given species using autonomous equations. In section 2, we have a look onto the qualitative behaviour of differential equations pertaining to logistic growth using geometrical approach. Section 3 deals with some practical applications of logistic equations. Finally, we make a comparison between exponential growth and logistic growth.

1.1. EXPONENTIAL GROWTH [1]

Let $y = \phi(t)$ be the population of the given species at time 't'. The main hypothesis concerning the variation of population is that the rate of change of y is proportional to the current value of y

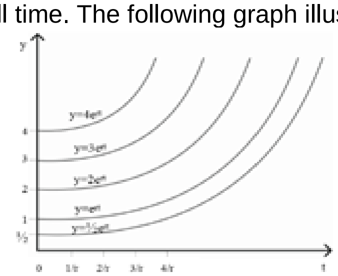
$$\text{ie,} \frac{dy}{dt} = ry \rightarrow (2)$$

where 'r' is the constant of proportionality indicating the growth or decline of the given species.

Assume $r > 0$, ie, the population is growing
Let the initial condition be $y(0) = y_0 \rightarrow (3)$
Solving equation (2) subject to (3), we get

$$y = y_0 e^{rt} \rightarrow (4)$$

Thus the mathematical model consisting of the initial value problem (2), (3) with r>0 predicts that the population will grow exponentially all time. The following graph illustrates this fact:



1.2 LOGISTIC GROWTH [1]

The growth rate 'r' actually depends on the population. So we replace 'r' in equation (2) by a function h(y)
Equation (2) becomes

$$\frac{dy}{dt} = h(y)y \rightarrow (5)$$

We now choose $h(y)\ \exists\!: h(y) \cong r > 0$ when y is small; h(y) decreases as y grows larger and h(y)<0 when y is sufficiently large. The simplest function that has these properties is h(y)=r-ay, where 'a' is a positive constant.
Using this function in equation (5),

$$\frac{dy}{dt} = (r - ay)y \rightarrow (6)$$

$$\text{ie,} \frac{dy}{dt} = r\left[1 - \frac{a}{r}y\right]y = r\left[1 - \frac{y}{K}\right]y$$

where $K = \frac{r}{a}$

Equation (6) is known as the logistic equation and the constant 'r' is called the intrinsic growth rate. This differential equation was first introduced by the Belgian biologist Verhulst [2], so it is referred to as the Verhulst equation.

2. GEOMETRIC BEHAVIOUR OF SOLUTIONS [3]

2.1 EQUILIBRIUM SOLUTIONS

Consider the solutions of the equation

$$\frac{dy}{dt} = r\left[1 - \frac{y}{K}\right]y \rightarrow (7)$$

First consider solutions of the simplest possible type,
ie, constant functions
For such a solution, $\frac{dy}{dt} = 0,\ \forall t$
So, any constant function of equation (7) must satisfy the algebraic equation

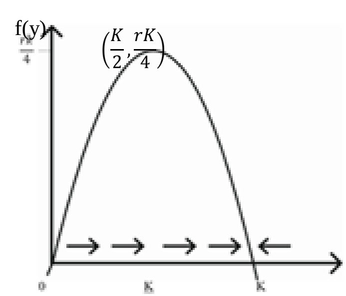
$$r\left[1 - \frac{y}{K}\right]y = 0$$

Thus, the constant solutions are

$$y = \phi_1(t) = 0 \quad \& \quad y = \phi_2(t) = K$$

These solutions are called the EQULIBRIUM SOLUTIONS of (7) To find other solutions of equation (7), we consider the graph of the equation.

$f(y) = 0$ when $y = 0$ and $y = K$.



When $y = \frac{K}{2}, f(y) = r\left(1 - \frac{1}{2}\right)\frac{K}{2} = \frac{rK}{4}$

Equation (7) gives

$$\frac{dy}{dt} = ry - \frac{r}{K}y^2$$

$$\text{ie,} \quad f(y) = ry - \frac{r}{K}y^2$$

$$\therefore \text{at } y = \frac{K}{2}, f(y) = \frac{dy}{dt} = \frac{rK}{2} - \frac{r}{K}\frac{K^2}{4} = \frac{rK}{2} - \frac{rK}{4} = \frac{rK}{4}$$

$$\therefore \frac{dy}{dt} > 0 => r\left[1 - \frac{y}{K}\right]y > 0$$

Hence, for y>0,

$$\frac{dy}{dt} > 0 => 1 - \frac{y}{K} > 0 => y < K$$

$$\therefore \frac{dy}{dt} > 0 \text{ for } 0 < y < K$$

ie, y is an increasing function of t in the interval 0<y<K This is indicated by the rightward pointing arrows near the y axis. Similarly, if y>K, $\frac{y}{K} > 1$

$$\text{ie,} 1 - \frac{y}{K} < 0$$

$$\text{ie,} \quad \frac{dy}{dt} < 0$$

ie, y is a decreasing function of t and this is indicated by the leftward pointing arrow. In this case, the y-axis is called the PHASE LINE. y=0 and y=K are the equilibrium solutions. If y is near zero or K, then f(y) is near zero. ie, the solution curves are relatively flat. They become steeper as the value of y leaves the neighborhood of 0 or K.

2.1.1 Note

It may seem that other solutions intersect the equilibrium solution y=K. But, this is not possible. For, the fundamental existence and uniqueness theorem [2] states that only one solution can pass through a given point in the t-y plane. Thus, although other solutions may be asymptotic to the equilibrium solution as t → ∞, they cannot intersect it at any finite time.

2.2. CONCAVITY OF THE SOLUTION AND INFLECTION POINTS

In this section, we study some geometric properties relating to logistic growth.

We have

$$\frac{d^2y}{dt^2} = \frac{d}{dt}[f(y)] = f^{|}(y)\frac{dy}{dt} = f^{|}(y)f(y) \rightarrow (8)$$

The y-t graph is concave up when $y^{||} > 0$, ie when f and $f^{|}$ have same sign; and concave down when $y^{||} < 0$, ie when f and $f^{|}$ have opposite signs.

In the case of equation (7), for $0 < y < \frac{K}{2}, f$ is positive and increasing, so that both f and $f^{|}$ are positive

∴ the solutions are concave up for $0 < y < \frac{K}{2}$

When $\frac{K}{2} < y < K, f$ is positive and decreasing

∴ f is positive and $f^{|}$ is negative

∴ $f^{||}$is negative

Hence, the solutions are concave down for $\frac{K}{2} < y < K$

When $y > K,$ both f and $f^{|}$ are negative

∴ $f^{||}$ is positive

∴ the solutions are concave up for $y > K$

The inflection point occurs when the y-t graph crosses the line $y = \frac{K}{2}$

The inflection point is got by putting $f^{|}(y)$=0

Now,

$$f^{|}(y) = 0 => r\left\{1 - \frac{y}{K}\right\} + ry\left(\frac{-1}{K}\right) = 0$$

$$=> 1 - \frac{2y}{K} = 0 \quad (\because r \neq 0)$$

$$=> y = \frac{K}{2}$$

Now, K is the upper bound that is approached by growing populations starting below this value. K is called the SATURATION LEVEL or the ENVIRONMENTAL CARRYING CAPACITY for the given species.

This is demonstrated in the next section.

3. APPLICATIONS OF LOGISTIC EQUATIONS ([3])

In this section, we explore the applications of logistic equations in population dynamics. Suppose that there are two different species that do not prey on each other, but compete for the available food. Let x and y be the populations of the two species at time 't'. The population of each of the species, in the absence of the other, is governed by the following logistic equations:

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x) \rightarrow (9)$$

$$\frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y) \rightarrow (10)$$

Here ϵ_1 and ϵ_2 are the growth rates of the two populations, and $\frac{\epsilon_1}{\sigma_1}$ and $\frac{\epsilon_2}{\sigma_2}$ represent their saturation levels.

When both species are present, each will tend to diminish the available food supply for the other. This reduces each other's growth rates and saturation levels. If the growth rate of species x is reduced due to the presence of species y, then we replace the growth rate factor $\epsilon_1 - \sigma_1 x$ in equation (9) by $\epsilon_1 - \sigma_1 x - \alpha_1 y$, where $\alpha_1$is a measure of the degree to which species y interferes with species x.

So, equation (9) becomes,

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y) \rightarrow (11)$$

Similarly, if the growth rate of y is reduced due to the presence of species x, then equation (10) becomes

$$\frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x) \rightarrow (12)$$

where α_2 is a measure of the degree to which species x interferes with species y. The values of the positive constants $\epsilon_1, \alpha_1, \sigma_1, \epsilon_2, \alpha_2,$ and σ_2 depend on the particular species under consideration and are determined from observations. The result whether co-existence occurs depends on whether $\sigma_1\sigma_2 - \alpha_1\alpha_2$ is positive or negative. If $\sigma_1\sigma_2 > \alpha_1\alpha_2$, the interaction is weak and the species can co-exist. If $\sigma_1\sigma_2 < \alpha_1\alpha_2$, then the interaction is strong and so, the species cannot co-exist, ie, one must die out.

4. CONCLUSION

A comparison of the exponential growth and logistic growth reveals the fact that the solutions of the non-linear equation (7) are different from those of the linear equation (1), at least for large values of t. Whatever be the value of K, ie no matter how small the non-linear term in equation (7) may be, solutions of that equation approach a finite value as t → ∞.

But, solutions of exponential equation (1) grow unboundedly as $t \to \infty$. Thus, even a very small non-linear term in the differential equation (7) has a precise effect on the solution for large values of 't'. Also, we observe that logistic equations are more powerful in solving practical problems.

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