A New Parameter For Ramanujan's Function $\chi(q)$ Of Degree 3 And Their New Explicit Evaluations.

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Abstract: In this article, We will come to know new modular identities of Ramanujan's Remarkable product of theta-function of degree 3 and their explicit values

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1 INTRODUCTION

Ramanujan defined theta-function as

[2] $f(a,b):=\sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$, $|ab|<1$,

$$=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \quad (1.1)$$

Where $(a,q)_\infty:=\prod_{n=0}^{\infty}(1-aq^n), |q|<1$.

[2] $\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty} q^{n^2}=\frac{(-q;-q)_\infty}{(q;-q)_\infty}$ (1.2)

[2] $\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$ (1.3)

[2] $f(-q):=f(-q,q^2)=\sum_{n=-\infty}^{\infty} q^{n(3n-1)/2}=(q;q)_\infty$ (1.4)

and

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[2, Ch.16, Entry 22(iv), p.37] $\chi(q):=(-q;q^2)_\infty$. (1.5)

A new parameter $I_{m,n}$ introduced by Nipen Saikia, which is defined as

[6] $I_{m,n}:=\frac{\chi(q)}{q^{\frac{(1-m)}{24}}\chi(q^m)}$, $q=e^{-\pi\sqrt{n/m}}$. (1.6)

He has listed as many as properties of $I_{m,n}$. Recently we noticed in [3, 4] have derived some new parameters for Ramanujan's function $\chi(q)$ of degree 5, degree 9 and their explicit values.

In this article, we add as many as Modular identities and explicit evaluation of $I_{3,n}$ for n = 2, 3, 5, 7.

We now first give brief explanation on a modular equation. The ordinary hyper-geometric series is defined by

$$[2] \quad {}_2F_1(a,b;c;x):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\, n!}x^n,$$

Where, $(a)_0=1, (a)_n=a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $|x|<1$.

Let $u:=u(x):={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)$ (1.7)

[2] $q:=q(x):=\exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)}\right)$, (1.8)

Where $0 < x < 1$.

Let r to be a fixed natural number and holds good the following relation:

$$[2]\; r\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}\;. \qquad (1.9)$$

The relation between α and β of modular equation of degree r is induced by (1.9).

In the last section of my article, we have establish the modular relation between $I_{3,n}$ and I_{3,k^2n}, also given explicit evaluations of $I_{3,n}$ for n = 2,3,5 and 7.

2 Preliminary Results

Lemma 2.1. [1] If $P:=\frac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ and $Q:=\frac{\varphi(q)}{\varphi(q^3)}$ then,

$$(Q^4-1)P^4+Q^4-9=0. \qquad (2.1)$$

Lemma 2.2. [7] If $P:=\frac{\varphi(q)}{\varphi(q^3)}$ and $Q:=\frac{\varphi(q^2)}{\varphi(q^6)}$ then,

$$(Q^4+1+2Q^2)P^4+(6-12Q^2-2Q^4)P^2+Q^4+9-6Q^2=0. \qquad (2.2)$$

Lemma 2.3. [7] If $P:=\frac{\varphi(q)}{\varphi(q^3)}$ and $Q:=\frac{\varphi(q^3)}{\varphi(q^9)}$ then,

$$P^3Q^2+3P-Q^3-3P^2Q=0. \qquad (2.3)$$

Lemma 2.4. [5] If $P:=\frac{\varphi(q)}{\varphi(q^3)}$ and $Q:=\frac{\varphi(q^5)}{\varphi(q^{15})}$ then,

$$P^6+(Q^5-5Q)P^5+5P^4Q^2-5Q^4P^2+(9Q-5Q^5)P-Q^6=0. \qquad (2.4)$$

Lemma 2.5. [5] If $P:=\frac{\varphi(q)}{\varphi(q^3)}$ and $Q:=\frac{\varphi(q^7)}{\varphi(q^{21})}$ then,

$$P^8+(Q^7-7Q^3)P^7+14P^6Q^2+(7Q^5-21Q)P^5+(21Q^3-7Q^7)P^3-14P^2Q^6+(27Q-21Q^5)P-Q^8=0. \qquad (2.5)$$

Lemma 2.6.

$$[2]\; \chi(q):=\frac{\varphi(q)}{f(q)} \qquad (2.6)$$

Lemma 2.7.

$$[2]\; f^3(q)=\varphi^2(q)\psi(-q)\;. \qquad (2.7)$$

Lemma 2.8.

$$[2]\; f^3(q^3)=\varphi^2(q^3)\psi(-q^3)\;. \qquad (2.8)$$

Lemma 2.9. [6]

$$I_{m,1}=1. \qquad (2.9)$$

Lemma 2.10. [6]

$$I_{m,n}I_{m,1/n}=1. \qquad (2.10)$$

3 Modular Identities and Explicit Evaluations of $I_{m,n}$

Theorem 3.1. If $v:=q^{1/12}\frac{\chi(q)}{\chi(q^3)}$ and $p:=q^{1/6}\frac{\chi(q^2)}{\chi(q^6)}$ then

$$p^{20}v^{32}+(16p^{16}-p^{28}+p^4)v^{28}+(71p^{12}-25p^{24})v^{24}+(71p^8-200p^{20}+p^{32})v^{20}+(16p^4+16p^{28}-550p^{16})v^{16}+(71p^{24}-200p^{12}+1)v^{12}+(71p^{20}-25p^8)v^8+(p^{28}+16p^{16}-p^4)v^4+p^{12}=0. \qquad (3.1)$$

Proof. Substitute q by q^3 in (2.6), we arrive

$$\chi(q^2):=\frac{\varphi(q^3)}{f(q^3)}. \qquad (3.2)$$

Cubing and dividing (2.6) by (3.2), we get

$$\frac{\chi^3(q)}{\chi^3(q^3)}=\frac{\varphi^3(q)}{\varphi^3(q^3)}\left\{\frac{f^3(q^3)}{f^3(q)}\right\}\;. \qquad (3.3)$$

Using (2.7) and (2.8) in (3.3), we obtain

$$\frac{\chi^3(q)}{\chi^3(q^3)}=\frac{\varphi(q)}{\varphi(q^3)}\left\{\frac{\psi(-q^3)}{\psi(-q)}\right\}\;. \qquad (3.4)$$

Raising by four and multiplying by q on both side of (3.4), we arrive at

$$\frac{q\chi^{12}(q)}{\chi^{12}(q^3)}=\frac{\varphi^4(q)}{\varphi^4(q^3)}\left\{q\frac{\psi^4(-q^3)}{\psi^4(-q)}\right\}\;. \qquad (3.5)$$

By using lemma (2.1), we obtain

$$P^4=\frac{9-Q^4}{Q^4-1} \qquad (3.6)$$

Using the equation (3.6) in (3.5), we obtain

$$v=\frac{Q^8-Q^4}{9-Q^4} \qquad (3.7)$$

Where, $Q:=\frac{\varphi(q)}{\varphi(q^3)}$ and $v:=q\frac{\chi^{12}(q)}{\chi^{12}(q^3)}$, the equation (3.7) can be expressed as

$$Q=\frac{(1-v)+\sqrt{v^2+34v+1}}{2} \qquad (3.8)$$

Where, $v=I_{3,n}^{12}$.

Adopting the equation (3.8) in (2.2), we obtain (3.1).
This completes the proof.

Corollary 3.1. We have

$$I_{3,2}=\{-44+27\sqrt{3}+33\sqrt{2}-18\sqrt{6}\}^{1/12}, \qquad (3.9)$$

$$I_{3,1/2}=\{-44+27\sqrt{3}-33\sqrt{2}+18\sqrt{6}\}^{1/12}. \qquad (3.10)$$

$$I_{3,4}=\frac{1}{2}\{-32+24\sqrt{6}-32\sqrt{3}+24\sqrt{2}\}^{1/4}, \qquad (3.11)$$

$$I_{3,1/4}=\frac{1}{2}\{-32+24\sqrt{6}+32\sqrt{3}-24\sqrt{2}\}^{1/4}. \qquad (3.12)$$

Proof. Considering the Theorem (3.1), substituting n =1/3 to it and with the Lemma (2.10), we obtain

$$1+176I_{3,2}^{12}-1002I_{3,2}^{24}+176I_{3,2}^{36}I_{3,2}^{48}=0. \qquad (3.13)$$

By solving (3.13), we arrive at (3.9) and (3.10) equations.
Substituting n = 1 in (3.1) and using (2.9), we get

$$\{I_{3,1/2}^{16}+8I_{3,1/2}^{12}-36I_{3,1/2}^{8}+8I_{3,1/2}^{4}+1\}^2\{I_{3,1/2}^{8}+4I_{3,1/2}^{4}+1\}=0 \qquad (3.14)$$

Since one of the factor vanishes identically where as another factor does not satisfies the condition $I_{3,4}<1$, so (3.14) can be written as

$$x^2+8x-38=0, \text{ where } x=I_{3,4}^{4}+I_{3,4}^{-4}. \qquad (3.15)$$

By solving the (3.15), we arrive at (3.11) and (3.12) equations.

Theorem 3.2. If $v:=q^{1/12}\frac{\chi(q)}{\chi(q^3)}$ and $p:=q^{1/4}\frac{\chi(q^3)}{\chi(q^9)}$ then

$$p^6v^{18}+(p^3-p^{15})v^{15}+(1-10p^{12})v^{12}-20v^9p^9+(p^{18}-10p^6)v^6+(p^{15}-p^3)v^3+p^{12}=0. \qquad (3.16)$$

Proof. Adopting the equation (3.8) in (2.3), we obtain,

$$(v^{15}p^3-v^3p^3+p^{18}v^6-20v^9p^9+p^6v^{18}-10p^6v^6-v^{15}p^{15}-10v^{12}p^{12}+v^{12}+p^{12}+v^3p^{15})(v^3p^3-v^{15}p^3+p^{18}v^6+20v^9p^9+p^6v^{18}+p^{12}-10p^6v^6+v^{15}p^{15}-10v^{12}p^{12}+v^{12}-v^3p^{15})(v^{36}p^{12}+p^{30}v^{30}+18p^{18}v^{30}-p^6v^{30}+62v^{24}p^{24}+v^{24}+18p^{30}v^{18}+200p^{18}v^{18}+18p^6v^{18}+p^{36}v^{12}+62v^{12}p^{12}-p^{30}v^6+18p^{18}v^6+p^6v^6+p^{24})=0. \qquad (3.17)$$

Since q→0, one of the factors vanishes of the (3.17), but the remaining factor does not disappear. So we come at the equation (3.16). Thus it concludes the proof.

Corollary 3.2. We have

$$I_{3,3}=\{2-\sqrt{3}\}^{1/6}, \qquad (3.18)$$

$$I_{3,1/3}=\{2+\sqrt{3}\}^{1/6}. \qquad (3.19)$$

$$I_{3,9}=\{-1+2^{1/3}\}^{1/3}, \qquad (3.20)$$

$$I_{3,1/9}=\{2^{2/3}+2^{1/3}+1\}^{1/3}. \qquad (3.21)$$

Proof. Considering Theorem (3.2), Lemma (2.10) and (2.9), we arrive (3.18) - (3.21).

Theorem 3.3. If $v:=q^{1/12}\frac{\chi(q)}{\chi(q^3)}$ and $p:=q^{5/12}\frac{\chi(q^5)}{\chi(q^{15})}$ then

$$v^6-v^5p^5-vp-5v^3p^3+p^6=0. \qquad (3.22)$$

Proof. Adopting the equation (3.8) in (2.4), we obtain

$$\frac{-1}{2}(-v^5p^5-vp+v^6-5v^3p^3+p^6)(v^5p^5+vp+v^6+5v^3p^3+p^6)(-5p^4v^4-v^7p-v^{11}p^5+v^2p^2-vp^7+26p^6v^6+v^{12}+p^{12}+10v^3p^9-5p^8v^8+p^{10}v^{10}+10v^9p^3-v^5p^{11})(-5p^4v^4+v^7p+v^{11}p^5+v^2p^2+vp^7+26p^6v^6+v^{12}+p^{12}-10v^3p^9-5p^8v^8+p^{10}v^{10}-10v^9p^3+v^5p^{11})(-10p^4v^4+v^2p^2+25p^6v^6+v^{12}+p^{12}-10p^8v^8+p^{10}v^{10})(v^{24}-p^{10}v^{22}+p^{20}v^{20}+10p^8v^{20}+10p^{18}v^{18}+50p^6v^{18}+75p^{16}v^{16}+10v^{16}p^4+230p^{14}v^{14}-v^{14}p^2+526v^{12}p^{12}-p^2{}^2v^{10}+230p^{10}v^{10}+10p^{20}v^8+75p^8v^8+50p^{18}v^6+10p^6v^6+10p^{16}v^4+p^4v^4-p^{14}v^2+p^{24})(68884p^{36}-85p^{48}n+102p^{60}+3903p^{48}+550383p^{24}-p^{60}n+p^{72}+793585-183461p^{12}n-269297n+1615782p^{12}-34442p^{24}n-2602p^{36}n)=0. \qquad (3.23)$$

Where $n=\sqrt{b^{24}+34b^{12}+1}$.

Since q→0, one of the factors vanishes of the (3.23), but the remaining factor does not disappear. So we come at the equation (3.22). Thus it concludes the proof.

Corollary 3.3. We have,

$$I_{3,5}=\frac{\{12-4\sqrt{5}\}^{1/3}}{2}, \qquad (3.24)$$

$$I_{3,1/5} = \frac{\{12+4\sqrt{5}\}^{1/3}}{2}. \qquad (3.25)$$

$$I_{3,25} = \frac{1+10^{2/3}+10^{1/3}+\sqrt{-15+3(10^{2/3})+12(10^{1/3})}}{6}, \qquad (3.26)$$

$$I_{3,1/25} = \frac{1+10^{2/3}+10^{1/3}-\sqrt{-15+3(10^{2/3})+12(10^{1/3})}}{6}. \qquad (3.27)$$

Proof. Considering Theorem (3.3), Lemma (2.10) and (2.9), we obtain (3.24)- (3.27).

Theorem 3.4. If $v := q^{1/12}\frac{\chi(q)}{\chi(q^3)}$ and $p := q^{7/12}\frac{\chi(q^7)}{\chi(q^{21})}$ then

$$v^8-v^7p^7-vp-7v^6p^2-7p^6v^2+14v^4p^4+p^8=0. \qquad (3.28)$$

Proof. Adopting the equation (3.8) in (2.5), we obtain,

$$\frac{-1}{2}(v^7p^7+vp+v^8-7v^6p^2-7p^6v^2+14v^4p^4+p^8)(-v^7p^7-vp +v^8-7v^6p^2-7p^6v^2+14v^4p^4+p^8)(84p^{10}v^6+28v^5p^5-vp^9 +v^{16}+7v^9p^{13}+148p^8v^8+v^2p^2-v^{15}p^7+7v^{14}p^2+84p^6v^{10} +35p^{12}v^4+7p^{14}v^2+p^{14}v^{14}-v^9p+7v^{13}p^9+28v^{11}p^{11}-v^7 p^{15}+p^{16}+35v^{12}p^4+7v^3p^7+7v^7p^3)(84p^{10}v^6-28v^5p^5+v p^9+v^{16}-7v^9p^{13}+148p^8v^8+v^2p^2+v^{15}p^7+7v^{14}p^2+84p^6 v^{10}+35p^{12}v^4+7p^{14}v^2+p^{14}v^{14}+v^9p-7v^{13}p^9-28v^{11}p^{11} +v^7p^{15}+p^{16}+35v^{12}p^4-7v^3p^7-7v^7p^3)(210p^{10}v^6+v^{16} +294p^8v^8+v^2p^2+14v^{14}p^2+210p^6v^{10}+77p^{12}v^4+14p^{14}v^2 +p^{14}v^{14}+p^{16}+77v^{12}p^4)(-658p^{26}v^6-77p^{26}v^{18}-77p^{18}v^{26} -30492p^{18}v^{14}-p^{18}v^2+588p^{10}v^{10}+p^{28}v^{28}+28p^{28}v^{16}-210 p^{20}v^{24}+18676p^{20}v^{12}+119p^{28}v^4+2695p^8v^{24}-210p^8v^{12}- v^{18}p^2+v^{32}+18676p^{12}v^{20}-210p^{24}v^{20}+2695p^{24}v^8+p^{32} -210p^{12}v^8-14v^{30}p^2-658v^{26}p^6-p^{14}v^{30}+36800p^{16}v^{16}+28 p^{16}v^{28}-77p^{14}v^6-30492p^{14}v^{18}-14p^{30}v^2-p^{30}v^{14}-77v^{14} p^6+28v^{16}p^4+119v^{28}p^4+v^4p^4-8148p^{10}v^{22}+588p^{22}v^{22} +28p^{16}v^4-8148p^{22}v^{10})(128039644p^{24}+3453190p^{48} +59823937+210156040p^{12}+29964472p^{36}+136p^{84}+7516 p^{72}+p^{96}+216376p^{60}-26269505n-5637p^{60}n-p^{84}n-1123 6677np^{24}-119np^{72}-135235np^{48}-32009911p^{12}n-1726595 p^{36}n) = 0. \qquad (3.29)$$

Where $n=\sqrt{b^{24}+34b^{12}+1}$.

Since q→0, one of the factors vanishes of the (3.29), but the remaining factor does not disappear. So we come at the equation (3.28). Thus it concludes the proof.

Corollary 3.4.We have,

$$I_{3,7} = \frac{\sqrt{2\sqrt{7}-2\sqrt{3}}}{2}, \qquad (3.30)$$

$$I_{3,1/7} = \frac{\sqrt{2\sqrt{7}+2\sqrt{3}}}{2}. \qquad (3.31)$$

$$I_{3,49} = \frac{-2+4(28^{1/3})+28^{2/3}-2\sqrt{21+3(28^{1/3})+3(28^{2/3})}}{12}, \qquad (3.32)$$

$$I_{3,1/49} = \frac{-2+4(28^{1/3})+28^{2/3}+2\sqrt{21+3(28^{1/3})+3(28^{2/3})}}{12}. \qquad (3.33)$$

Proof. Considering Theorem (3.4), Lemma (2.10) and (2.9), we obtain (3.30) - (3.33).

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