Automorphism Groups Of Weakly Semi-Regular Bipartite Graphs

Sreekumar . K . G, Manilal .K

Abstract : The study of automorphism of graphs plays an important role in Graph Theory. In this paper, we determine the automorphism group of some families of bipartite graphs which are weakly semiregular. Mainly two families of graphs were considered -SM sum graphs and SM Balancing graphs. These SM sum graphs are particular cases of bipartite Kneser graphs. Here we examined the automorphism groups of the bipartite Kneser type graphs too. Weakly semiregular bipartite graphs in which the neighbourhoods of the vertices in the SD part having the same degree sequence, possess non trivial automorphisms. The automorphism groups of SM sum graphs are isomorphic to the symmetric groups. Automorphism groups of SM balancing graphs are related to some classes of simple groups. SM graphs are remarkable examples because they have more automorphisms. It has been observed by using the well known algorithm Nauty, that the size of automorphism groups of SM balancing graphs are prodigious. The properties of automorphisms of the two families of graphs are also discussed. Every weakly semiregular bipartite graphs with k-NSD subparts has a matching which saturates the smaller partition. AMS subject classification: 05C99

Key words and Phrases : Weakly semi-regular bipartite graphs, Bipartite Kneser graphs, Automorphism group, Symmetric groups, SM Balancing graphs, n^{th} SM sum graphs, Simple groups

.

1. INTRODUCTION

Throughout years, algebraic graph theorists have exhibited a wide variety of approaches to the study of automorphism and isomorphism of graphs. It is a content rich field with many different applications in the information technology and related fields. Many algorithms like Nauty[3], Saucy, Trace, Bliss etc have been introduced apart from the combinatorial methods by some Mathematicians. A simple graph is usually denoted by G = (V, E), where V is the vertex set and E is its edge set. The order of G is the number of its vertices and size of G is the number of its edges. An isomorphism from a graph G to a graph H is a bijection f from the vertex set of G to that of H such that u and v are adjacent in G if and only if f(u) and f(v) are adjacent in H, for all u and v in the vertex set V(G). When two simple graphs are isomorphic, there is a one-to-one correspondence between the vertices of the two graphs that preserves the adjacency relationship. Apart from being an equivalence relation, all the isomorphisms to the same graph itself play an important role in many applied fields of Mathematics as well as in security in information technology. An automorphism of a graph G is an isomorphism with itself. On the other hand an automorphism of a graph G is a permutation $f(x)$ in V(G) with the property that $\{u_i, v_j\} \in E(G)$ if and only if $\{f(u_i), f(v_j)\} \in E(G)$. It is known that the set of all automorphisms on a graph together with the operation of composition of functions form a group. The automorphism group of a graph G is denoted by Aut(G). The automorphisms of complete graph[11], complete bipartite graph and semi-regular bipartite graphs were given in [6]. Here we study the automorphism of weakly semi-regular connected bipartite graphs. In the case of complete bipartite graph $K_{m,n}$, the automorphism group is isomorphic to $S_m \times S_n$, when $m \neq n$ [2]. The automorphism group of $K_{m,n}$, is isomorphic to $S_m \times S_n \times Z_2$, when $m = n$ [2].

- *Sreekumar. K.G : Department of Mathematics, University College, University of Kerala. India. Email Id: sreekumar3121@gmail.com*
- *Manilal K : Department of Mathematics, University College, University of Kerala. India. Email Id: manilalvarkala@gmail.com*

The isomorphism and automorphism of graphs are largely used in data structure for database retrieval and in cryptography etc. In the study of graph parameters, the graph isomorphism and graph automorphisms have a big role. A study of direct product and uniqueness of automorphism groups of graphs was done by W.Peisert [13]. A characterization of automorphism groups of generalized Hamming graphs was done by F.A Chaouche and A.Berrachedi [4]. Here we considered mainly two families of graphs - SM sum graphs and SM balancing graphs. SM sum graphs are related to the intrinsic connection between the powers of 2 and the natural numbers which is the basic logic of binary number system while the SM balancing graphs are associated with the balanced ternary number system which was used in the SETUN computer made in Russia. These graphs are vertex labelled graphs and are explained in the next section.

1.1 Preliminary

In this section we provide the basic definitions and some results from the related previous work. A bipartite graph $G = (V_1, V_2)$ is called $(q_1 + 1, q_2 + 1)$ −semiregular if the degree of the vertex v, $d(v) = q_i + 1$ for each $v \in V_i$, $i = 1,2$ [6]. Furthermore $q_1 + 1$ *and* $q_2 + 1$ are called the degrees of G. We begin with the definition of some families of graphs. First let us see the definition of SM balancing graphs [8]. Consider the set $T_n = \{3^m : m \text{ is an integer}, 0 \leq m \leq n - 1\}$ for a fixed positive integer $n \geq 2$. Let $I = \{-1, 0, 1\}$. Let $x \leq \frac{1}{2}(3^n - 1)$ be any positive integer which is not a power of 3. Then x can be expressed as

$$x = \sum_{j=1}^{n} \alpha_j y_j \quad (1)$$

where $\alpha_j \in I$ and $y_j \in T$, y'_js are distinct. Each y_j such that $\alpha_j \neq 0$ is called a balancing component of x.

Consider the simple graph G = (V, E) , where the vertex set $V = \left\{v_1, v_2, v_3, \ldots, v_{\frac{1}{2}(3^n - 1)}\right\}$ and adjacency of vertices defined by: for any two distinct vertices v_x and v_{y_j} , $\left(v_x, v_{y_j}\right) \in E$ if (1) holds and $\alpha = -1$ and the vertex $\left(v_{y_j}, v_x\right) \in E$ if (1) holds and $\alpha = 1$. This directed graph G is called the n^{th} SMD Balancing graph denoted by $SMD(B_n)$. Its underlying

undirected graph is called the n^{th} SM Balancing graph denoted by $SM(B_n)$.
Now let us see the definition of SM sum graph [7] . If p $< 2^n$, is a positive integer that are not powers of 2, then $p = \sum_1^n x_i$, with $x_i = 0$ or 2^m, for some integer m, $0 \le m \le n-1$ and x_i's are distinct. The coefficient of each x_i's is 1. Each $x_i \ne 0$ is called an additive component of p. For a fixed $n \ge 2$, define a simple graph $SM(\Sigma_n)$, called the n^{th} SM sum graph, is the graph with vertex set $\{v_1, v_2, v_3, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by: v_i and v_j are adjacent if either i is an additive component of j, or j is an additive component of i .
For a fixed integer $n \ge 2$, let $T_n = \{3^m: m\ is\ an\ integer, 0 \le m \le n-1\}$, $N = \{1,2,3,\ldots,t\}$ where $t = \frac{1}{2}(3^n - 1)$. Also let $P_n = \{2^m: m\ is\ an\ integer,\ 0 \le m \le n-1\}$, $M = \{1,2,3,\ldots,2^n - 1\}$. Then consider ${P_n}^c = M - P_n$ and ${T_n}^c = N - T_n$ throughout this paper unless otherwise specified.
The Hamming weight (by James W L, in 1899) of a string was defined as the number of 1's in the string representation using 0 and 1. Here the number of additive components is the Hamming weight of string (binary) representation of all numbers in ${P_n}^c$. The Hamming weight of string (binary) representation of numbers in P_n is always 1.
In the graph $SM(\Sigma_n)$, the degree of the vertex v_{2^n-1} is n and $\sum_{v\in V} d(v) = 2n(2^{n-1} - 1)$ [8].
In $SM(B_n)$, the number of vertices is $\frac{1}{2}(3^n - 1)$ and $\sum_{v\in V} d(v) = 2n(3^{n-1} - 1)$ [7].

A graph is **asymmetric** if its automorphism group is the identity group. It has been proved by Erdos and Renyi, that almost all graphs are asymmetric. This means that the proportion of graphs on n vertices that are asymmetric goes to zero as $n \to \infty$. All regular graphs need not be non asymmetric, for example, the Frucht graph which is a 3-regular graph with 12 vertices and has no non trivial automorphism. The problem of finding the automorphisms of a graph belongs to the class NP of computational complexity [3].

Definition 1.1 [11]: For a fixed integer $n \ge 1$, let $S = \{1,2,3, . . . , n\}$ and V be the set of all k-subsets and $(n-k)$ subsets of S. The bipartite Kneser graph H(n,k) has V as its vertex set and two vertices A, B are adjacent if and only if $A \subset B$ or $B \subset A$.
The SM sum graphs are closely related to bipartite Kneser graphs.

Definition 1.2 For a positive integer $n \ge 1$, let $Q_n = \{1,2,3, . . . , n\}$. Consider all non empty subsets of Q_n. Let V_1 be the set of 1-subsets and V_2 be the set of all non empty subsets of Q_n except the 1-subsets. Define a bipartite graph with parts V_1 and V_2 , and adjacency of vertices is defined as: a vertex $A \in V_1$ is adjacent to a vertex $B \in V_2$ if and only if $A \subset B$. This graph is called **bipartite kneser type graph**. This bipartite kneser type graph has $2^n - 1$ vertices and $n(2^n - 1)$ edges for each $n \ge 2$. Also bipartite kneser type graphs are neither vertex transitive nor edge transitive. But they are non asymmetric. The SM sum graphs are isomorphic to this bipartite kneser type graph for each $n \ge 2$.

Weakly semi-regular bipartite

Revisiting bipartite graphs, a simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 and no edge in G connects either two vertices in V_1 or two vertices in V_2. We call (V_1, V_2) the bipartition of G; V_1 and V_2 are called the parts of G. Here we consider only connected bipartite graphs.
A bipartite graph G is semi-regular of bi-degree (k, m) if every vertex in one member of the bipartition has degree k and every vertex in the other has degree m. We considered cases where one part of G has vertices of equal degree.

Definition 2.1 :
A bipartite graph with bipartition (V_1, V_2), $|V_1| > 1$ and $|V_2| > 1$ is weakly semiregular if the vertices in exactly one V_i have same degree. The part of G in which all vertices have the same degree is called a SD-part. The other part of G is called a NSD-part.
Let $G = (V, E)$ be a graph. The neighborhood of $v \in V$, written $N_G(v)$ or N(v), is the set of vertices adjacent to v.

Definition 2.2 A weakly semi-regular bipartite graph G is called a WSB_{END} graph if the vertices in the NSD-part do not have all distinct degrees and the neighborhoods of the vertices in the SD-part have same degree sequence.
For each $k \ge 1$, the set of vertices in the NSD-part of degree k is called a k-NSD subpart. Suppose there are 3 servers , 5 computers and the maximum allowed connections to each server is 4. So we have a total of $4 \times 3 = 12$ possible connections. 5 computers need to be connected. One of the arrangements is as follows: 3 computers (having only 2 ports) to two each server and other two computers (having only 3 ports) to all the 3 servers. Only one direct connection to a server can be active at any time. This connection leads to a graph which is a WSB_{END} graph. The question of interchanging the connections without altering the connection structure raising the question of automorphism. In the cases of on line examinations, these connection automorphisms may reduce the possibility of cheating. This can be solved by using a symmetric swap using an automorphism. The automorphism relationship is an equivalence relation on the vertices of a graph. Two vertices are equivalent if there exists an automorphism taking one to the other. Like all equivalence relations, this also produces a partition of the vertex set into equivalence classes . These classes are usually called automorphism classes or orbits. The orbits are the vertices of each k-NSD part and SD part. The automorphism classes of these types of graphs are yet to be studied.

Proposition 2.3 Let G be a WSB_{END} graph. Then G has a non trivial automorphism group. Proof: Let G be a semi-regular bipartite graph with parts V_1 and V_2. Let $|V_1| = m > 1$ and $|V_2| = n > 1$. Let V_1 be the SD part having degree p to each of its vertices and V_2 be the NSD part. Let $\{n_1, n_2, n_3, . . . , n_j\} = \{d(x): x \in V_2\}$ with $n_1 < n_2 < n_3 < . . . < n_j$. Let ${V_2}^i = \{x \in V_2 : d(x) = n_i\}$ be the j k-NSD sub parts having $n_1, n_2, n_3, . . . , n_j$ as the corresponding degrees of vertices in each k-NSD subpart. Obviously $m > n_1$ and also $mp = |{V_2}^1|n_1 + |{V_2}^2|n_2 + . . . + |{V_2}^j|n_j$. Now consider a permutation on $m+n$ vertices. For every graph, there exists a trivial permutation which is an automorphism. Since the vertices in the SD part are of same degree and each is having a neighborhood with same degree sequence, there exist some permutations which permute among these vertices together with elements of k-NSD subpart so that it results in automorphisms. In this way these graphs have non trivial automorphisms too. Let Aut(G) be the collection of all these automorphisms of G. Here the trivial permutation is the identity automorphism. The collection of all permutations is closed under the operation of composition. Here the permutations on the k-NSD subpart permute in itself in accordance with the permutation of elements of SD part. Consider two automorphisms α and β having cycles on the k-NSD subpart, then there will be two cases.

Case 1. When α and β belonging to the set of permutations containing cycles of the same orbit. In this case when the composition is taken, then $\alpha \circ \beta$ is a member of the permutation with product of cycles on the same k-NSD subpart. Therefore $\alpha \circ \beta \in Aut(G)$.

Case 2 When α and β belonging to the set of permutations containing cycles of the different orbits. In this case these are disjoint permutations. Therefore $\alpha \circ \beta$ is a member of the permutation on Aut(G). In both the cases $\alpha \circ \beta \in Aut(G)$. Therefore Aut(G) is closed under the operation of composition. The function composition is associative on Aut(G). Now for each of these permutations, there exists an inverse permutation. This inverse permutation acts as the inverse automorphism for each of these elements of Aut(G). Therefore G has a non trivial automorphism group. ■ The degree sequence given in this work have been already defined in [7] and [8].

Theorem 2.4: The graphs $SM(\Sigma_n)$ and $SM(B_n)$ are WSB_{END} graphs for all $n \geq 2$. Proof: Consider the graph G= $SM(\Sigma_n)$, $n \geq 2$. The graph G is a bipartite graph with parts $V_1 = \{v_i: i \in P_n\}$ and $V_2 = \{v_i: i \in {P_n}^c\}$ where $P_n = \{2^m,$ $m\ is\ an\ integer,\ 0 \leq m \leq n-1\}$. The graph G has $2^n - 1$ vertices and $n(2^{n-1} - 1)$ edges. All the vertices of V_1 are of same degree $2^n - 1$ and the vertices of V_2 are not of same degree and has a degree sequence $\left\{2_{\left(\binom{n}{2}\right)}, 3_{\left(\binom{n}{3}\right)}, \cdots, n_{\left(\binom{n}{n}\right)}\right\}$, for $n \geq 2$. There are $\binom{n}{2}$ vertices of degree 2 and $\binom{n}{3}$ vertices of degree 3 and so on. Also each vertices in V_1 has a neighborhood with degree sequence $\left\{2_{\left(\binom{n-1}{1}\right)}, 3_{\left(\binom{n-1}{2}\right)},\right.$ $\left.\cdots, n_{\left(\binom{n-1}{n-1}\right)}\right\}$. This implies that G is a WSB_{END} graph. Similarly for the graph $SM(B_n)$, is having $\frac{1}{2}(3^n - 1)$ number of vertices and $2n(3^{n-1} - 1)$ edges. It is a bipartite graph with parts $V_1 = \{v_i: i \in T_n\}$ and $V_2 = \{v_i: i \in {T_n}^c\}$. The vertices in V_1 are of same degree $(3^{n-1} - 1)$ and each vertex is having a neighbourhood with same degree sequence $\left\{2_{\left(2\binom{n-1}{1}\right)}, 3_{\left(2^2\binom{n-1}{2}\right)}, \cdots, n_{\left(2^{n-1}\binom{n-1}{n-1}\right)}\right\}$. The vertices in V_2 are of different degree and has a degree sequence $\left\{2_{\left(2\binom{n}{2}\right)}, 3_{\left(2^2\binom{n}{3}\right)},\right.$ $\left.\cdots, n_{\left(2^{n-1}\binom{n}{n}\right)}\right\}$. Therefore the graph $SM(B_n)$ is also a WSB_{END} graph. Hence proved. ■

Corollary 2.5: The graph $SM(\Sigma_n)$ is a union of sub graphs which are WSB_{END} graphs..

Theorem 2.6: Let G be a WSB_{END} graph and having parts V_1 and V_2, $|V_1| = p > 1$ and $|V_2| = q > 1$. Let the k-NSD subparts be $X_1, X_2, X_3, \ldots, X_m$ with $|X_n| = k_n$, n = 1, 2, ..., m. Then the number of automorphisms of G is at most $p!\,(k_1! + k_2! + \ldots + k_m!)$. Proof: Given that the graph G is a WSB_{END} graph.. The parts of G are V_1 and V_2 , $|V_1| = p > 1$ and $|V_2| = q > 1$. It has m numbers of k-NSD subparts. It has been proved in Proposition 2.3 that G has a non trivial automorphism group. Here the total number of vertices is $p + q$. Consider a permutation on $p + q$ vertices. Assume that the vertices of V_1 are of same degree n. Each vertex of V_1 have a neighbourhood of same degree sequence. Therefore we can permute the p elements in the $p + q$ element permutations in $p!$ ways. And in each of these permutations again can be permuted among each of the k_i vertices, $i = 1,2,3, . . . , m$. Each of these give raise to $k_n!$ permutations, where $n = 1,2,3, \ldots, m$; which include automorphisms of G. Therefore by the sum and product rule of permutations, the total number of automorphism is at most $p!\,(k_1! + k_2! + \ldots + k_m!)$. Hence proved. ■Let X and Y be the parts of a bipartite graph G. It is known that if a matching M saturates X, then for every $S \subseteq X$, there must be at least $|S|$ vertices that have neighbors in S. We use N(S) to denote the set of vertices having a neighbor in S.

Theorem 2.7: Every weakly semi-regular bipartite graph with k-NSD subpart has a matching which saturates the smaller partition. Proof: Let G be a weakly semi-regular bipartite graph with k-NSD subparts. Let X and Y be the parts of G. Since it has k-NSD subparts, $|X| \neq |Y|$Let $|X| < |Y|$. This implies that for all $x \in X$ and $y \in Y$, $\deg x > \deg y$. Now to prove the theorem, we use Hall's necessary and sufficient conditions. It says that an X, Y bigraph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all S ⊂ X . For proving that there exists a matching that saturates the smaller partition X, it is enough to prove that $|N(S)| \geq |S|$, for all $S \subset X$. On the contrary assume that $|N(S)| < |S|$. Let G_1 be the sub graph induced by $S \cup N(S)$. Let M be a matching in G and M does not saturate X. So we get $\deg x < \deg y$ and $|X| > |Y|$. This is a contradiction . Therefore $|N(S)| < |S|$ is wrong. So we get $|N(S)| \geq |S|$. Hence proved. ■

The graph $SM(\Sigma_n)$ has no perfect matching and but have a non trivial biclique for all $n \geq 2$. A WSB_{END} graph having odd number of edges is non-Eulerian. Since number of edges is odd, then a WSB_{END} graph must have vertices with odd degree.

Definition 2.11 A semi-regular bipartite graph G is semi-regular bipartite graph with same parity if each vertex in either of the partition has a neighborhood with same degree sequence.

Theorem 2.12 A semi-regular bipartite graph with same parity is non asymmetric.Proof: Let G be a (k, m) semi-regular bipartite graph with parttion set X and Y. Since it has same parity; it will have non trivial automorphisms. Therefore it is non asymmetric. ■

Automorphisms of the graphs $SM(\Sigma_n)$ and $SM(B_n)$ The automorphism of graphs is a degree preserving and distance preserving function. In this section we are examining the automorphism group of SM sum graphs and SM balancing graphs. The maximum simple bipartite graph is the complete bipartite graph. If $G = K_{m,n}$ is the complete bipartite graph, then G is a subgraph of $SM(\Sigma_p)$ or (B_p), $n \geq 2$ where $p = max(m,n) + 1$. There is a relation between a bipartite graph especially complete bipartite graph with the intrinsic relationship between the powers of 2 and other integers as well as powers of 3 and other integers.

Theorem 3.1 Any bipartite graph is isomorphic to a sub graph of SM sum graph or SM balancing graph.Proof: Consider the graph $SM(\Sigma_n)$ with vertex set $V_S = \{v_i : 1 \leq i \leq 2^n - 1\}$ for an integer $n \geq 2$. Also $SM(B_m)$ is a graph with the vertex set $V_B = \left\{v_1, v_2, v_3, \dots, v_{\frac{1}{2}(3^m-1)}\right\}$, where m is an integer ≥ 2. Both are bipartite graphs. Let G be a bipartite graph with parts V_1 and V_2, $|V_1| = r \geq 1$ and $|V_2| = s \geq 1$. Let $p = max(r,s) + 1$. From the definition of SM graphs, it is clear that any bipartite graph is isomorphic to a sub graph of $SM(\Sigma_p)$ or $SM(B_p)$. ■

Theorem 3.2 The graph $SM(\Sigma_n)$ has $n!$ automorphisms for all $n \geq 2$.
Proof: Let $G = SM(\Sigma_n)$, $n \geq 2$. From the results obtained earlier from Theorem 3.2, we have that G is a WSB_{END} graph. The graph G has a degree sequence $\left\{2_{\left(\binom{n}{2}\right)}, 3_{\left(\binom{n}{3}\right)}, \cdots, n_{\left(\binom{n}{n}\right)}, (2^{n-1}-1)_{(n)}\right\}$, for $n \geq 2$. Since the orbits are the vertices of each k-NSD part and SD part, as we permute the elements of SD part, this fixes how the elements of k-NSD subpart must be permuted to give automorphisms. The automorphisms of G depends completely on the permutations of the elements of SD part in a unique way. So we get that the number of automorphisms is $n!$. ■

Corollary 3.3 The graph $SM(B_n)$ has less than $n!\,(2\binom{n}{2})! + (2^2\binom{n}{3})! + \dots + (2^{n-1}\binom{n}{n})!)$ automorphisms.
Proof: Let G = $SM(B_n)$, $n \geq 2$. It can be easily seen that G is a WSB_{END} graph. We know that G has a degree sequence $\left\{2_{\left(2\binom{n}{2}\right)}, 3_{\left(2^2\binom{n}{3}\right)}, \cdots, n_{\left(2^{n-1}\binom{n}{n}\right)}, (3^{n-1}-1)_{(n)}\right\}$ for $n \geq 3$. The automorphisms of G depends on the permutations of the elements of SD part but not in a unique way. From Theorems 2.4 and 2.6, it follows that the total number of automorphisms is less than $n!\,(2\binom{n}{2})! + (2^2\binom{n}{3})! + \dots + (2^{n-1}\binom{n}{n})!)$. ■
More precisely we get the following result for the automorphism group of the SM balancing graphs.

Theorem 3.4 The automorphism group $Aut(SM(B_n))$ is isomorphic to $\left[\prod_{k=2}^{n}\left(S_{2^{k-1}}\right)^{\binom{n}{k}}\right] \times S_n$, where S_n is the symmetric group, $n \geq 3$.
Proof: Let G = $SM(B_n)$, $n \geq 2$. We have that G has a degree sequence $\left\{2_{\left(2\binom{n}{2}\right)}, 3_{\left(2^2\binom{n}{3}\right)}, \cdots, n_{\left(2^{n-1}\binom{n}{n}\right)}, (3^{n-1}-1)_{(n)}\right\}$ for $n \geq 3$. Also G is a WSB_{END} graph . The automorphisms of G depends on the permutations of the elements of SD part but not in a unique way. From the degree sequence , it is clear that $Aut(SM(B_n))$ is isomorphic to $\left[\prod_{k=2}^{n}\left(S_{2^{k-1}}\right)^{\binom{n}{k}}\right] \times S_n$, where $n \geq 3$. Hence proved. ■

For $n = 2$, the the automorphism group of the graph $G = SM(B_n)$ is isomorphic to the dihedral group D_4. For $n = 3$, the size of the automorphism group is 1152. For $n = 4$, the size of the automorphism group is 20547391979520 (obtained by using Nauty algorithm). Also it has been observed that the automorphism groups of SM balancing graphs are related to the conjugacy classes of some of the sporadic simple groups. But the relation is yet to be studied furthermore.

Theorem 3.5 Let $SM(\Sigma_n)$ be an SM sum graph with vertex set V. The $Aut(SM(\Sigma_n))$ form a non-abelian automorphism group for $n > 2$.
Proof : From the Theorem 2.4, we get that the graph G = $SM(\Sigma_n)$ is a WSB_{END} graph. So the automorphisms of G forms a group. When $n = 2$, the $Aut(G)$ is an abelian group. When $n > 2$, the graph G has more than one non-trivial orbit. Now we have to prove that Aut(G) is a non-abelian group for all $n > 2$. Let α and β be two non trivial distinct automorphisms of G. Since the orbits are the vertices of each k-NSD part and SD part, these permutations α and β can be written as a product of disjoint cycles as follows: $\alpha = T_1 T_2 \dots T_k$, where elements of one of the $T_i's$ are from the SD part. Le it be T_s. Similarly $\beta = T'_1 T'_2 \dots T_k{}'$, where elements of one of the $T_i's$ are from the SD part. Le it be T'_s which is different from T_s . When we find the composition of $\alpha \circ \beta$ and $\beta \circ \alpha$, then as $T_s \circ T_s{}' \neq T'_s \circ T_s$ we get $\alpha \circ \beta \neq \beta \circ \alpha$. Hence proved. ■

Lemma3.6 [9] The graph $SM(\Sigma_n)$ is isomorphic to an edge induced sub graph of $SM(B_n)$.

Corollary 3.7 Let $SM(\Sigma_n)$ be an n^{th} SM sum graph. Then $G = \overline{SM(\Sigma_n)}$ has n! automorphisms.
Proof: Since a graph and its complementary graph have the same number of automorphisms, the result follows. ■

Theorem 3.8 Suppose $G = SM(\Sigma_n), n \geq 2$ be an n^{th} SM sum graph. Then the number of isomorphisms of G = $SM(\Sigma_n)$ to edge induced subgraph of $G = SM(B_n)$ is $n! \times n!$.
Proof : The proof follows from Lemma 3.6 and above Theorem 3.2. ■

Theorem 3.9 The automorphism group of the graph $SM(\Sigma_n)$ is isomorphic to the symmetric group S_n for all $n > 2$.
Proof : It has been observed that the graph $SM(\Sigma_n)$ is non asymmetric for all $n \geq 2$. By Theorem 3.2, it is clear that G has $n!$ automorphisms for all $n \geq 2$. Also the automorphism group of $SM(\Sigma_n)$ is non abelian for all $n > 2$. Therefore the automorphism group of $SM(\Sigma_n)$ is isomorphic to the symmetric group S_n for all $n \geq 3$. Hence the theorem. ■

Example 3.10 The automorphism group of $G = SM(\Sigma_n)$, $n = 3$ is isomorphic to the symmetric group S_3.
When $n = 3$, the graph G has 7 vertices and 9 edges. Also we can see G is a WSB_{END} graph. And G has the partition sets $V_1 = \{v_1, v_2, v_4\}$ and $V_2 = \{v_3, v_5, v_6, v_7\}$. Also the vertices in V_1 are of degree 3 each and the vertices of V_2 are having degree 2 each except v_7. The vertex v_7 has degree 3 and is adjacent to vertices of degree 3. The permutations $(v_1, v_2)(v_5, v_6)$, $(v_1, v_4)(v_3, v_6)$ and $(v_2, v_4)(v_3, v_5)$ produce 3 automorphisms. Furthermore if we fix any

permutation of the vertices of v_1, v_2, v_4 , this fixes how v_3, v_5, v_6 must be permuted to give an automorphism. So we get two more automorphisms which adds to a total of 6 automorphisms including the trivial automorphism. On the other hand, no automorphisms can result from swapping the vertex from the first bipartite sets and second bipartite sets because unless such a swap is done in its entirety, the adjacency will be lost. A swap can be done in entirety only if $|V_1| = |V_2|$ which is not the case here as G is not a complete bipartite graph as well. Therefore, finally we can see that $Aut(G)$ is isomorphic to S_3. The orbits are v_1, v_2, v_4 ; v_3, v_5, v_6 ; and v_7 . ■

A graph G is vertex-transitive [1] if for every vertex pair $u, v \in V(G)$, there is an automorphism that maps u to v. A graph G is edge transitive if for every edge pair $(d, e) \in E(G)$, there is an automorphism that maps d to e. Also it is observed that graph $SM(\Sigma_n)$ or $SM(B_n)$ is neither bi-regular, nor edge transitive nor vertex transitive.

The graph $SM(\Sigma_n)$ contains vertex transitive subgraphs for all $n > 2$ because it contains a biclique for each $n > 2$.

CONCLUSION

The automorphisms given in this paper are worthwhile as the non trivial symmetry of graphs is concerned. We have proved that the WSB_{END} graph is having a non trivial automorphism group. Also as n increases, the number of automorphisms of SM sum graph as well as SM balancing graphs is increasing. The number of automorphisms was calculated for each n. As almost all graphs are asymmetric, the automorphism groups of SM family of graphs are noteworthy. Also the automorphism classes of SM family of graphs are yet to be studied in detail. The automorphism classes of SM family of graphs may lead to a decomposition of these graphs. In which all cases the binary number system or balanced ternary number system is being used, in those cases these automorphisms will make significant effects. Further scope of edge automorphisms of these WSB_{END} graphs is to be examined.

CONFLICT OF INTEREST

I hereby declare that I have no potential conflict of interest.

Ethical Approval

This manuscript does not contain any studies with human participants or animals.

REFERENCES

[1]. Ademir Hujdurovic, Klavdija kutnar and Dragan Marusic, " Odd automorphisms in vertex transitive graphs", ARS Mathematica Contemporanea, Vol: 10, (2016), P 427-437.

[2]. Beeler, Robert A. "Automorphism Group of Graphs (Supplemental Material for Intro to Graph Theory)." (2018).

[3]. Brendan D Mckay and Adolfo piperno, "Practical graph isomorphism II", Journal of symbolic computation, Vol: 60, (2014), P 94-112.

[4]. F.Affif Chaouche and A.Berrachedi, "Automorphism groups of generalized Hamming graphs", Electronic Notes in Discrete Mathematics, Vol: 24, (2006), P 9-15.

[5]. G.H Fath Tabar, "The automorphism groups of finite graphs ", Iranian Journal of Mathematical sciences and informatics, Vol: 2, No.2 (2007), P 29-33.

[6]. Hirobumi Mizuno and Iwao sato, " Semicircle law for semiregular bipartite graphs", Journal of combinatorial theory, Vol: Series A 101, (2003), P 174-190.

[7]. K.G.Sreekumar , K Manilal " n^{th} SM Sum graphs and some parameters", International Journal of Mathematical Analysis, Vol: 11, No.3 (2017), P 105-113.

[8]. K.G.Sreekumar and Dr.Manilal.K, "Hosoya polynomial and Harary index of SM family of graphs ", Journal of information and optimization Sciences, Vol: 39,Issue 2, 2018, P 581-590.

[9]. K.G.Sreekumar,"Two-S-Three transfor-mation function and its properties", International Journal of Mathematical Archive,Vol-9, No.4 April (2018), P 83 - 88.

[10] Mirafzal, S. Morteza. "The automorphism groups of the bipartite Kneser graphs", Proc.Indian Acad.Sci (Math Sci) 129:34 (2019) P 1-8.

[11]. Laszlo Babai , " Automorphism groups of graphs and edge- contraction", Discrete Mathematics, Vol: 306, (2006), P 918-922.

[12]. Peter J Cameron and Pablo Spiga , "Most switching classes with primitive automorphism groups contain graph with trivial groups", Australian Journal of Combinatorics, Vol: 62, (2015), P 76-90.

[13]. Wojciech Peisert, "Direct product and uniqueness of automorphism groups of graphs", Discrete Mathematics, Vol: 207, (1999), P 189-197.