Iterative Method Of Solutions Of Evolution Stochastic Differential Equations With Local Conditions

S. A. Bishop, A. A. Opanuga, K. S. Eke, O. O. Agboola

Abstract: Using the iterative method, the existence of a strong unique solution of evolution quantum stochastic differential equations (QSDEs) is studied. The evolution operator generates a family of a semigroup. The paper shows under some carefully selected conditions, that the unique solution is stable. The iterative method applied here is simpler when compared with other methods used in literature, such as the fixed point approach which has been used extensively to establish existence of solution. The technically demands of transforming a problem to a fixed point problem is taken care of by using the iterative method. The results here generalize some results in the existing literature concerning classical stochastic differential equations. This work will have applications in Ito type stochastic differential equations.

Index Terms: Evolution operator, Iterative method, Local conditions, Stability, Stochastic differential equation, Stochastic processes, Strong solution.

1. INTRODUCTION
Several results on weak forms of solutions of the following quantum stochastic differential equation have been studied in [1, 4-8]:

\[ d z(t) = U(t, z(t)) d \mathcal{H}(t) + V(t, z(t)) dA(t) + W(t, z(t)) dA_{f, \gamma}(t) + H(t, z(t)) dt, \]

where the coefficients \( U, V, W \) and \( H \) lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, annihilation processes \( \mathcal{H}, A, A_{f, \gamma} \) and the Lebesgue measure \( t \) are defined in [2], [8]. \( z \in B \). It has been well established that quantum stochastic differential equation (1.1) introduced by Hudson and Parthasarathy [9] provides an essential tool in the theoretical description of physical systems, especially those arising in quantum optics, quantum measure theory, quantum open systems and quantum dynamical systems. In [6], some properties of solutions of nonclassical stochastic differential equations were studied. Results on existence and uniqueness of solutions of this class of equations were established under the strong topology. In [7], quantum stochastic differential inclusions of hypermaximal monotone type were studied and existence of an evolution operator connected with these inclusions were established. Further studies were carried out by [1, 4] on properties of the solution sets of quantum stochastic differential inclusions (1.1) under the weak topologies. Ayoola [1, 2] studied strong unique solutions of (1.1) under the strong topologies. The results in [1]

generalized some similar results in the classical setting. As mentioned earlier, not much study has been done on equation (1.1) under the strong topologies. In this paper, some new results on strong solutions of (1.1) are established. We introduce and study solutions of an evolution equation under the strong topology. In [3, 12], by using the fixed point approach, existence of mild solutions of evolution QSDEs were studied under the weak topologies. The fixed point method used demands that the problem be transformed into a fixed point problem before solving it. This can be technical cumbersome especially when solving problems in infinite dimensional spaces. The iterative method is used without transforming the problem. Evolution problems have found practical applications in many fields of science. See [10, 11, 13, 14, 15, 16, 17] for some interesting results on stochastic evolution equations and applications. The results in this paper improve several existing results of equation (1.1) under the weak topologies.

2 PRELIMINARIES
The following evolution equation is considered:

\[ d z(t) = A(t)z(t) + U(t, z(t)) d \mathcal{H}(t) + V(t, z(t)) dA(t) + W(t, z(t)) dA_{f, \gamma}(t) + H(t, z(t)) dt, \]

where \( z(t_0) = z_0, t \in I \) (1.1)

A is the infinitesimal generator of a family of semigroup \( \{ S(t) : t \geq 0 \} \).

For details on semigroup and their applications, see [6] in [12]. \( \mathcal{B} \) is a locally convex space whose topology is generated by a family of seminorms:

\[ \| \| \| \| = \{ \| \| \| \|, \xi \in D \otimes E \}, \]

where \( \| \| \| \| \) is the norm of the space \( \mathcal{R} \otimes \mathcal{R} (L_2 (\mathcal{H}), (L^2_\gamma (\mathcal{H}))). \)

The notations and structures of the following spaces are from the references [1, 2]: \( \mathcal{R} \otimes \mathcal{R} (L_2^\gamma (\mathcal{H})), A d (\mathcal{B}), A d (\mathcal{B}) ac, L_0^{\mathcal{L}_1} (\mathcal{B}), L_0^{\mathcal{L}_2} (\mathcal{B}), L_2^{\mathcal{L}_2} (\mathcal{B}), L_2^{\mathcal{L}_2} (\mathcal{B}). \)

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We prove $S_i - S_j$ by induction as follows:
For $n \geq 0$, we have

$$
\phi_{n+1}(t) = S(t) s_{t_0} + \int_{t_0}^{t} \left[ U(s, \varphi_n(s)) d \wedge_n(s) + V(s, \varphi_n(s)) dA_{f_n}(s) + H(s, \varphi_n(s)) ds \right].
$$

By the hypothesis,

$$
U(s, \varphi_n(s)), V(s, \varphi_n(s)), W(s, \varphi_n(s)), H(s, \varphi_n(s)) \in \tilde{\beta}_t, \quad s \in [t_0, T].
$$

While $U(s, \varphi_n(s)), V(s, \varphi_n(s)), W(s, \varphi_n(s)), H(s, \varphi_n(s)) \in \tilde{\beta}_t$.

Therefore, the quantum stochastic integral which defines $\varphi_n(t)$ exists for $t \in [t_0, T]$.

By theorem 2.8 in [2], $\varphi_1(t) \in L^2_t(\tilde{\beta})$. Hence, it implies that each $U(s, \varphi_n(s)), V(s, \varphi_n(s)), W(s, \varphi_n(s))$ and $H(s, \varphi_n(s))$ also lie in $L^2_t(\tilde{\beta})$. Thus, proving assumptions $S_i - S_j$ by induction.

Next, we show that the sequence of successive approximations converge. By (2.2) we have,

$$(2.2) \quad \| p_{\gamma(t)} - \varphi_\gamma(t) \| = \| S(t) \| + \int_{t_0}^{t} \left[ \left( \left( U(s, \varphi_\gamma(s)) - U(s, \varphi_n(s)) \right) d \wedge_n(s) \right) + \left( \left( V(s, \varphi_\gamma(s)) - V(s, \varphi_n(s)) \right) dA_{f_n}(s) \right) + \left( \left( H(s, \varphi_\gamma(s)) - H(s, \varphi_n(s)) \right) ds \right) \right].$$

By theorem 2.8 in [2] and hypothesis (ii) of the theorem, $\| p_{\gamma(t)} - \varphi_\gamma(t) \|$ decreases with $n$.

**3 Main Results**

**Theorem 3.1.** Suppose that

(i) the coefficients $U, V, W, H \in L^2_{loc}(\tilde{\beta})$ are Lipschitzian

(ii) there exists a constant $M > 0$ such that

$$
\| S(t) \| \leq M \quad \text{for each } t \geq 0.
$$

Then for $(t_0, z_{t_0}) \in \tilde{\beta}$, there exists a unique strong solution $\varphi$ of (2.1) satisfying $\varphi(t_0) = z_{t_0}$.

Proof. To prove the theorem, we state the following assumptions:

$S_1$. Let $(\varphi_n(t))_{n \geq 0}$ be a sequence of successive approximations of $\varphi \in \tilde{\beta}$, and

$S_2$. Let $\varphi_n(t), n \geq 1$ define an absolutely continuous process in $L^2_{loc}(\tilde{\beta})$. Let $T > 0, t \in [0, T]$ be fixed.
Hence, by (2.1) and \( \psi_{n+1}(t) \) converges uniformly to some \( \psi_0(t) \). 
Continuing the iteration, we have 
\[
\| \psi_{n+1}(t) - \psi_n(t) \| \leq M \| L_n(T) \| e^{\frac{\xi}{\Delta}} \sum_{k=0}^{n-1} \| L_k(T) \| e^{\frac{\xi}{\Delta}} \| \psi_{n+1}(t) - \psi_n(t) \| 
\]
where \( M = M^2, L_n(T) = 6K(T)^2 K^2 \).

Since the map \( x \mapsto \| \psi_1(x) - \psi_0(x) \| \) is continuous on \( I \), we put 
\[
R_x = \sup_{x \in I} \| \psi_1(x) - \psi_0(x) \|, \quad x \in I
\]
in (3.1) to get 
\[
\| \psi_{n+1}(t) - \psi_n(t) \| \leq M \sum_{k=0}^{n-1} \| L_k(T) \| e^{\frac{\xi}{\Delta}} \| \psi_{n+1}(t) - \psi_n(t) \| \leq \frac{R_x}{e^{\frac{\xi}{\Delta}}} \frac{1}{m!} \leq \frac{R_x}{e^{\frac{\xi}{\Delta}}} \frac{1}{m!} < \infty.
\]

Showing that \( \psi_n(t) \) is a Cauchy sequence in \( \mathcal{B} \) and converges uniformly to some \( \psi_0(t) \).

Now since \( \psi_n(t) \) is adapted and absolutely continuous, then so is \( \psi(t) \).

Next, it suffices to show that \( \psi(t) \) satisfies the quantum stochastic differential equations (2.1). Since \( \psi(t_0) = \psi_0(t) \) by using (2.2), one gets
\[
\| \psi_{n+1}(t) - \psi_n(t) \| \leq 6M^2 K(T) K e^{\frac{\xi}{\Delta}} \int_{t_0}^{T} \| \psi_{n+1}(t) - \psi_n(t) \| \, ds
\]
\[
\leq M L_n(T) e^{\frac{\xi}{\Delta}} \sum_{k=0}^{n-1} \| L_k(T) \| e^{\frac{\xi}{\Delta}} \| \psi_{n+1}(t) - \psi_n(t) \| \, ds
\]
\[
\leq M L_n(T) e^{\frac{\xi}{\Delta}} \sum_{k=0}^{n-1} \| L_k(T) \| e^{\frac{\xi}{\Delta}} \| \psi_{n+1}(t) - \psi_n(t) \| \, ds < \infty.
\]

Since \( \psi_n(t) \to \psi(t) \) in \( \mathcal{B} \) uniformly on \( [t_0,T] \).

Therefore, 
\[
\psi(t) = \lim_{n \to \infty} \psi_{n+1}(t) = \lim_{n \to \infty} \psi_n(t)
\]
\[
= S(t)z_0 + \lim_{n \to \infty} \psi_{n+1}(t) \int_{t_0}^{T} V(x, \psi_n(x)) dA^x(s)
\]
\[
+ W(x, \psi_n(x)) dA^x(s) + H(x, \psi_n(x)) \, ds
\]
\[
= S(t)z_0 + \lim_{n \to \infty} \psi_{n+1}(t) \int_{t_0}^{T} V(x, \psi_n(x)) dA^x(s)
\]
\[
+ W(x, \psi_n(x)) dA^x(s) + H(x, \psi_n(x)) \, ds.
\]

That is \( \psi(t) \) is a solution of (2.1).

Remark 3.1. If \( M < 1 \), the results in [2] is obtained.

3.1 Uniqueness

Suppose that \( y(t), t \in [t_0,T] \) is another adapted and absolutely continuous solution with \( y(t_0) = z_0 \).

Then, continuing in the same way existence of solution is established above, yields 
\[
\| \psi(t) - y(t) \| \leq M e^{\frac{\xi}{\Delta}} \| \psi(t) - y(t) \| \leq \frac{R_x}{e^{\frac{\xi}{\Delta}}} \frac{1}{m!} \leq \frac{R_x}{e^{\frac{\xi}{\Delta}}} \frac{1}{m!} < \infty
\]

We then conclude that for 
\[
\psi_n(t) \to \psi(t) \quad \text{as} \quad n \to \infty,
\]

4 Stability

Let the coefficients \( U, V, W, H \) satisfy the conditions of Theorem 3.1 and \( z(t), y(t), t \in [t_0,T] \) be solutions to equation (2.1) such that \( z(t_0) = z_0 \) and \( y(t_0) = y_0, z_0, y_0 \in \mathcal{B} \). The solution \( z(t) \) is stable under the changes in the initial condition as follows:

Theorem 4.1 Let hypothesis (ii) of Theorem 3.1 hold. Let \( T > t_0 \) be given and for \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if
Let \( z_n(t), y_n(t), n = 0,1, \ldots \) for the iterates corresponding to \( z_0, y_0 \) respectively. Let \( z_0(t) = z_0 \) and \( y_0(t) = y_0 \) for all \( t_0 \leq t \leq T \). Then the following estimate is obtained by employing (2.8) in [2] and (2.2).

\[
\|z_n(t) - y_n(t)\|_2 \leq \|z(t) - y(t)\|_2 < \varepsilon.
\]

Proof:

So that

\[
\|z_n(t) - y_n(t)\|_2 \leq \|z(t) - y(t)\|_2 < \varepsilon.
\]

Therefore, by continuous iteration, the result below is obtained for \( t_0 \leq t \leq T \).

By taking the square root of both sides and letting \( n \to \infty \), the following is obtained

\[
\|z(t) - y(t)\|_2 \leq \|z_0 - y_0\|_2 \left[ 2 e^{(MT_{t,t})} + \right]^\frac{1}{2}.
\]

For all \( t \in [t_0, T] \), and the desired result is obtained.

Next, the following example is considered.

Proposition 4.1. Let \( \eta \) be the usual non-anticipating Brownian functional so that

\[
\int_0^t E[\eta(s)]^2 \, ds < \infty.
\]

Then the operator-valued process \( U = (\eta(t) : t \geq 0) \) satisfies

\[
\int_0^t \|S(t-s)\|_{0,0}^2 \, ds < \infty.
\]

Where \( U = L^2_{t \to \alpha}(0, \bar{h}) \), \( h = L^2_{t \to \alpha}(\mathbb{R}_+^n) \) is bounded, \( S(t-s) \) is a family of semigroups generated by the evolution operator \( A(t) \) defined in section 2 and \( e(h) \) is defined by

\[
e(h) = \exp \left( \int_0^\infty \alpha(s) \, dw(s) - \frac{1}{2} \alpha^2(s) \, ds \right)
\]

Considering the following classical Itô evolution integral equation:

\[
Z(t,w) = \int_{t_0}^t S(t-s) \left[ U(s, Z(s,w)) \, dw(s) + \right] S(t-s) Z_{t_0} + \int_{t_0}^t H(s, Z(s,w)) \, ds \bigg|_{s = t}.
\]

Where the quantum equivalent form is given by

\[
Z(t,w) = \int_{t_0}^t \left[ U(s, Z(s,w)) \, dw(s) + \right] H(s, Z(s,w)) \, ds \bigg|_{s = t}.
\]
\( \phi(t_0) = \varphi_0, t \in I \)

Take \( \xi \) to be arbitrary and \( M < \infty, t \in [0, 1] \) then the following solution is obtained.

\[
\|Z(t)\xi\| = (M e)^{\frac{1}{2}} \left\{ E \left( Z(t)^2 z(w) \right) \right\}^{\frac{1}{2}} < \infty
\]

Where \( z(w) = \exp\left(-\int_a^w \frac{1}{2} a^2(s)ds\right) \).

**CONCLUSION**

The result obtained shows that existence and stability of a unique solution of quantum stochastic evolution equation driven by an additional operator under the strong topology is achievable using the iterative method. This result is a generalization of similar results on nonclassical stochastic differential equations under the weak topology hence, this will further enrich the theory of nonclassical stochastic differential equations within the framework of the Hudson and Parthasarathy's formulation of quantum stochastic calculus.

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**CONFLICT OF INTEREST**

The authors declare that there is no conflict of interest.

**REFERENCES**


