Periodic Boundary Condition For Von Neumann CA With Radius-N

M. Rajasekar, R. Anbu

Abstract: Cellular Automata have rich computational properties and provide many models in mathematical and physical processes. In this paper, one of the most commonly used neighborhood type of two-dimensional (2D) Cellular Automata which is called the Von Neumann neighborhood in two dimensional integer lattice is considered. We study the characterization of two dimensional linear Cellular Automata defined by the Von Neumann with neighborhood radius-N of periodic boundary conditions over the field $\mathbb{Z}_3$. Transition rule matrix for periodic boundary condition for Von Neumann cellular automata with radius-N is studied.

Index Terms: Two-dimensional Cellular automata, Von-Neumann neighborhood, Ternary field, Transition rule matrix, Matrix algebra, Null boundary condition, Structure of periodic condition.

1 INTRODUCTION

CELLULAR automata are discrete dynamical systems that exhibit a variety of dynamical behaviors, although they are formed by simple basic components. Cellular automata or shortly (CA) were first used for modeling various physical, Biological process and on computer science. The concept of cellular automata was initiating the early 1950's by John Von Neumann and Stanislaw Ulam. John Von Neumann [9] showed that a cellular automaton (CA) can be universal. Cellular Automata are also called Cellular Space, Tessellation Automata, Homogeneous structures, Cellular structure, Tessellation structures and Iterative arrays [10]. The study of CA has received remarkable attention in the last few years [1], [3], [4] because CA have been widely investigated in many disciplines (For Example Mathematics, physics, computer science, chemistry, etc.) with different purposes (For Example simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, cryptography). Most of the work for CAs is done for one-dimensional (1-D) case. Recently, two dimensional (2-D) CAs have attracted much of the interest. Some basic and precise mathematical models using matrix algebra built on field with two elements $\mathbb{Z}_2$ were reported for characterizing the behavior of two dimensional nearest neighborhood linear cellular automata with null or periodic boundary conditions [3], [4], [6], [7]. A cellular automata is a model of a system of "cell" objects with the following characteristics.

- ✶ The cell live on a grid.
- ✶ Each cell has a state. The number of state possibilities is typically finite. The simplest example has the two possibilities of 1 and 0 (otherwise referred to as "ON" and "OFF" (or) "alive" and "dead"). A cellular automata is a model of a system of "cell" objects with the following characteristics.
- ✶ The cell live on a grid.
- ✶ Each cell has a state. The number of state possibilities is typically finite. The simplest example has the two possibilities of 1 and 0 (otherwise referred to as "ON" and "OFF" (or) "alive" and "dead").
- ✶ Each cell has a neighborhood. This can be defined in any number of ways,

In a cellular automaton all cells typically begin in state 0, except for a finite number that are in other states. The nonzero patterns that occur while a cellular automaton is running are called 'configurations'. At each tick of the clock, many of the cells enter a new state and a new configuration develops. It is natural to refer to the sequence of configurations that develop as 'generations'. The 2D finite cellular automata consists of (m × n) cells arranged. Where each cell takes one of the values of the field $\mathbb{Z}_3$. The relative positions of the cells is called neighbor of the cell given center cell. The state of these neighbors are used to compute the new state of the center cell.

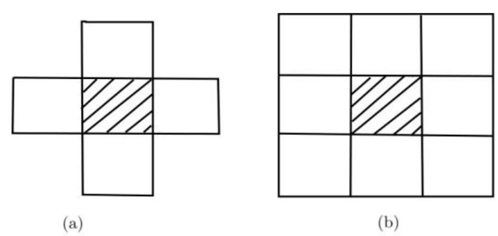

Fig. 1. (a) Von Neumann Neighborhood (b) Moore Neighborhood

Two type of neighborhood cells in Fig.1. (a) Von Neumann Neighborhood (top, right, bottom and left) (b) Moore Neighborhood (All the nearest surrounding the center cell) The paper is organized as follows. In the 2nd section, the concept used in the paper are formally defined. In 3rd section, the algebraic structure of periodic boundary condition of the 2D cellular automata is obtained by using geometric structure of 2D cellular automata. In 4th section, We deal with 2D cellular automata define by a local rule ternary field $\mathbb{Z}_3$ and we obtain the rule matrix of the Von Neumann with neighborhood radius-N for periodic boundary case.

- *M. Rajasekar, Associate Professor, Maths Section -FEAT, Annamalai University, Annamalainagar, Chidambaram, India, 9442532038. Mdivraj1962@gmail.com*
- *R. Anbu ,Department of Mathematics, Annamalai University, Annamalainagar, Chidambaram, India, 9688705233. Anburaja2291@gmail.com*

2 PRELIMINARIES

2.1 Definition [7] A periodic Boundary CA is the one in which the extreme cells are adjacent to each other.

2.2 Definition [7] A Null Boundary CA is the one in which the extreme cells are connected to logic zero state.

2.3Definition [11] Each state of a CA is called a configuration. In particular, each configuration of a (2-D) CA(m × n) is a binary information matrix of dimension (m × n) .

2.4 Definition [2]

Cellular Automata: Cellular automata are formally defined as quadruplets
A = {d, S,N, f}
* d ∈ $\mathbb{Z}_+$ is the dimension of the cellular space, then each point $\vec{n} \in \mathbb{Z}^d$ is called a cell
.* S = {0, 1, 2, . . . , p - 1} represents the finite state set and the state of any cell at any time must be taken from S.
* $\overrightarrow{n_1}, \overrightarrow{n_2}, \overrightarrow{n_3}, \dots, \overrightarrow{n_m}$ is the neighbor vector, where $\overrightarrow{n_i} \in \mathbb{Z}^d$ and $\overrightarrow{n_i} \neq \overrightarrow{n_j}$ when $i \neq j \ (i, j = 1, 2, \dots, m)$. Thus, the neighbors of the cell $\overrightarrow{n_i}$ are the m cells, I = 1, 2, . . . , m.
* $f : S^m \to S$ is the local rule, which maps the current states of all neighbors of a cell to the next state of this cell.
A configuration is a mapping $C: \mathbb{Z}^d \to S$ which assigns each cell a state. Make C^tdenote the configuration at time t, then the state of cell $\vec{n}$ at time is $C^t(\vec{n})$ and its state at time (t+1) goes like this.

$$C^{t+1}(\vec{n}) = f(C^t(\overrightarrow{n_1}), C^t(\overrightarrow{n_2}), \dots, C^t(\overrightarrow{n_m}))$$

now we consider the case in which the local rule f is a linear function

$$C^{t+1}(\vec{n}) = f(C^t(\overrightarrow{n_1}) + C^t(\overrightarrow{n_2}) + \dots + C^t(\overrightarrow{n_m}))$$
$$= [\lambda_1 C^t(\overrightarrow{n_1}) + \lambda_2 C^t(\overrightarrow{n_2}) + \dots + \lambda_m C^t(\overrightarrow{n_m})] \pmod{S}$$

Where $\lambda_1 \in S$ is the rule co-efficient for neighbor $\overrightarrow{n_i}, i = 1, 2, \dots, m.$
In this paper, we deal with CA defined by Von Neumann rules under periodic boundary condition (PBC). For convenience of analysis, the state of each cell is an element of a finite or an infinite state set. Moreover, the state of the cell (i, j) at time t is denoted by $X_{(i,j)}^{(t)}$.

The state of the cell (i, j) at time (t+1) is denoted by $X_{(i,j)}^{(t+1)} = Y_{(i,j)}^{(t)}$.

In [5], they consider the information matrix

$$C^{(t)} = \begin{pmatrix} X_{11}^{(t)} & \dots & X_{1n}^{(t)} \\ \vdots & \ddots & \vdots \\ X_{m1}^{(t)} & \dots & X_{mn}^{(t)} \end{pmatrix}$$

The matrix C^t is called the configuration of the 2D finite CA at time t. We associate Von Neumann neighbors presentations with row vectors by transforming them from C^t to $([X]_{1 \times mn}) = (X_{11}^{(t)}, X_{12}^{(t)}, \dots, X_{1n}^{(t)}, \dots, X_{m1}^{(t)}, \dots, X_{mn}^{(t)})$. Where $X_{11}^{(t)} \in \mathbb{Z}_3$. Hence, we consider the transition matrix T_R that changes set of states of cellular automata from (t) to (t+1) such that $[X]_{1 \times mn} . (T_R)_{mn \times mn} = [Y]_{mn \times 1}$,
Where,

$$([Y]_{mn \times 1}) = (X_{11}^{(t+1)}, X_{12}^{(t+1)}, \dots, X_{1n}^{(t+1)}, \dots, X_{m1}^{(t+1)}, \dots, X_{mn}^{(t+1)}).$$
$$= (Y_{11}^{(t)}, Y_{12}^{(t)}, \dots, Y_{1n}^{(t)}, \dots, Y_{m1}^{(t)}, \dots, Y_{mn}^{(t)}).$$

3 2D CELLULAR AUTOMATA OVER THE FIELD Z_3

In [8], the 2D finite CA consists of m× n cells arranged in m rows and n columns, where each cell takes one of the values of the field $\mathbb{Z}_3$. From now on, we denote 2D finite CA order to $(m \times n)$ by $2DCA_{(m \times n)}$. A configuration of the system is an assignment of the states to all cells. Every configuration determines a next configuration via a linear transition rule that is local in the sense that the state of a cell at time (t+1) depends only on the states of some of its neighbor at the time 't' using modulo 3 algebra. For 2D CA nearest neighbors, there are (mn) cells arranged in a (m × n) matrix centering that particular cell.

3.1 Von Neumann with neighborhood radius-n

The John Von Neumann with neighborhood radius-N which comprises 4N cell surrounding the center cell $X_{(i,j)}$ on a 2D square lattice. The state $X_{(i,j)}^{(t+1)}$ is a function $f: \mathbb{Z}_3^{(4N+1)} \to \mathbb{Z}_3$ defined as follows.

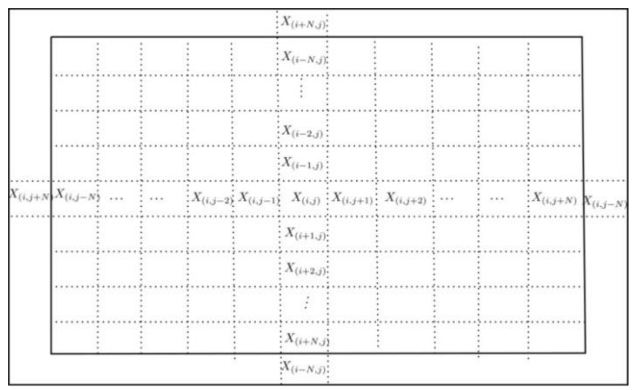

Fig. 2. Elements of Periodic boundar for Von Neumann neighborhood surround the center $X_{(i,j)}$ of $CA_{(m \times n)}$

Fig . 2, shows the John Von Neumann with neighborhood with radius-N which comprises (4N) square cell surrounding the center cell $X_{(i,j)}$. The boundary cell adjacent to each other. The state $X_{(i,j)}^{(t+1)}$ function $f: \mathbb{Z}_3^{(4N+1)} \to \mathbb{Z}_3$ as follows.

$$X_{(i,j)}^{(t+1)} = f(X_{(i,j+1)}, X_{(i+1,j)}, X_{(i,j-1)}, X_{(i-1,j)}, X_{(i,j+2)}, X_{(i+2,j)}, X_{(i,j-2)}, X_{(i-2,j)}, \dots, X_{(i,j+N-1)}, X_{(i+N-1,j)}, X_{(i,j-(N-1))}, X_{(i-(N-1),j)}, X_{(i,j+N)}, X_{(i+N,j)}, X_{(i,j-N)}, X_{(i-N,j)})$$
$$= (a_1 X_{(i,j+1)} + b_1 X_{(i+1,j)} + c_1 X_{(i,j-1)} + d_1 X_{(i-1,j)} + \dots + a_N X_{(i,j+N)} + b_N X_{(i+N,j)} + c_N X_{(i,j-N)} + d_N X_{(i-N,j)}) \quad (mod\ 3)$$

Where, $X_{(i,j+1)} \in \mathbb{Z}_3$ and $a_i, b_i, c_i, d_i \in \mathbb{Z}_3^* = \mathbb{Z}_3/\{0\}$ for i = 1, 2, 3,…, N

4 TRANSITION RULE MATRIX OF THE VON NEUMANN NEIGHBORHOOD WITH RADIUS-N FOR PERIODIC BOUNDARY CONDITION

We obtain the rule matrix of 2D finite CA with Von Neumann neighborhood radius-N rule over the field $\mathbb{Z}_3$ under the periodic boundary condition. This rule matrix which takes the t^{th} finite configuration matrix C^t of order m × n to the $(t+1)^{th}$ state of $c^{(t+1)}$.

	⋮	⋮	⋮	⋮	⋮	⋮		⋮	
X_{mn}	X_{m1}	X_{m2}	X_{m3}	X_{m4}	X_{m5}	X_{m6}		X_{mn}	X_{m1}
$\cdots X_{1n}$	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	$\cdots$	X_{1n}	X_{11} $\cdots$
$\cdots X_{2n}$	X_{21}	X_{22}	X_{23}	X_{24}	X_{25}	X_{26}	$\cdots$	X_{2n}	X_{21} $\cdots$
$\cdots X_{3n}$	X_{31}	X_{32}	X_{33}	X_{34}	X_{35}	X_{36}	$\cdots$	X_{3n}	X_{31} $\cdots$
$\cdots X_{4n}$	X_{41}	X_{42}	X_{43}	X_{44}	X_{45}	X_{46}	$\cdots$	X_{4n}	X_{41} $\cdots$
$\cdots X_{5n}$	X_{51}	X_{52}	X_{53}	X_{54}	X_{55}	X_{56}	$\cdots$	X_{5n}	X_{51} $\cdots$
	⋮	⋮	⋮	⋮	⋮	⋮		⋮	
$\cdots X_{mn}$	X_{m1}	X_{m2}	X_{m3}	X_{m4}	X_{m5}	X_{m6}	$\cdots$	X_{mn}	X_{m1} $\cdots$
X_{1n}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}		X_{1n}	X_{11}
	⋮	⋮	⋮	⋮	⋮	⋮		⋮	

Fig. 3. Periodic Condition of $m \times n$ matric

4.1 Transition rule matrix for periodic boundary condition

Theorem 1. Let a_i, b_i , c_i , $d_i \in \mathbb{Z}_3^* = \mathbb{Z}_3/\{0\}$, (for i = 1, 2, 3,…, N) m ≥ (2r + 1) and n ≥ (2r + 1) . Then the rule matrix T_{R} which takes the t^{th} finite Von Neumann neighborhood 2D finite Cellular automata configuration $C^{(t)}$ of order (m × n) to the $(t+1)^{th}$ state $C^{(t+1)}$ is given by

$$T_R = \begin{pmatrix} A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ \ b_1I \\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \ldots 0\ b_NI \ldots b_3I\ \ b_2I \\ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \ldots 0\ b_NI \ldots b_3I \\ b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \ldots 0\ b_NI \ldots \\ \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \ldots 0\ b_NI \\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \ldots 0 \\ 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \ldots \\ \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI\ 0 \\ 0\ \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots d_NI \\ d_NI\ 0\ \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \ldots \\ \ldots d_NI\ 0\ \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I\ d_3I \\ d_3I\ \ldots d_NI\ 0\ \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I\ d_2I \\ d_2I\ \ d_3I\ \ldots d_NI\ 0\ \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A\ d_1I \\ d_1I\ \ d_2I\ \ d_3I\ \ldots d_NI\ 0\ \ldots 0\ b_NI \ldots b_3I\ \ b_2I\ b_1I\ \ A \end{pmatrix}_{mn\times mn}$$

Where each sub matrix is of order n × n and 0 is zero matrix of order (n × n)

$$A = \begin{pmatrix} 0\ c_1\ c_2\ c_3 \ldots\ c_{N-1}c_N\ 0 \ldots\ 0\ a_N\ a_{N-1} \ldots\ a_3\ a_2\ a_1 \\ a_1\ 0\ c_1\ c_2\ c_3 \ldots\ c_{N-1}c_N\ 0 \ldots\ 0\ a_N\ a_{N-1} \ldots\ a_3a_2 \\ a_2\ a_1\ 0\ c_1\ c_2\ c_3 \ldots\ c_{N-1}c_N\ 0 \ldots\ 0\ a_N\ a_{N-1} \ldots\ a_3 \\ a_3\ a_2\ a_1\ 0\ c_1\ c_2\ c_3 \ldots\ c_{N-1}c_N\ 0 \ldots\ 0\ a_N\ a_{N-1} \ldots \\ \ldots\ a_3\ a_2\ a_1\ 0\ c_1c_2\ c_3 \ldots\ c_{N-1}c_N\ 0 \ldots\ 0\ a_N\ a_{N-1} \\ a_{N-1}\ \ldots a_3\ a_2a_1\ 0\ c_1c_2\ c_3 \ldots\ c_{N-1}c_N 0 \ldots\ 0\ a_N \\ a_N\ a_{N-1}\ \ldots a_3a_2\ a_1\ 0\ c_1c_2c_3 \ldots\ c_{N-1}c_N 0 \ldots\ 0 \\ 0\ a_N\ a_{N-1}\ \ldots a_3\ a_2\ a_1\ 0\ c_1c_2c_3 \ldots\ c_{N-1}c_N 0\ \ldots \\ \ldots 0\ a_N\ a_{N-1}\ \ldots a_3a_2a_1\ 0\ c_1c_2\ c_3 \ldots\ c_{N-1}\ c_N 0 \\ 0\ \ldots 0\ a_N\ a_{N-1}\ \ldots a_3a_2a_1\ 0\ c_1c_2c_3 \ldots\ c_{N-1}c_N \\ c_N 0\ \ldots 0\ a_N\ a_{N-1}\ \ldots a_3\ a_2a_1\ 0\ c_1\ c_2c_3 \ldots\ c_{N-1} \\ c_{N-1}\ c_N 0\ \ldots 0\ a_N\ a_{N-1}\ \ldots\ a_3\ a_2\ a_1\ 0\ c_1c_2c_3 \ldots \\ \ldots\ c_{N-1}\ c_N 0\ \ldots 0\ a_N\ a_{N-1}\ \ldots a_3a_2\ a_1\ 0\ c_1\ c_2c_3 \\ c_3\ \ldots\ c_{N-1}c_N 0\ \ldots 0\ a_N\ a_{N-1}\ \ldots a_3a_2a_1\ 0\ c_1c_2 \\ c_2c_3\ \ldots\ c_{N-1}c_N 0\ \ldots 0\ a_N\ a_{N-1}\ \ldots a_3a_2a_1\ 0\ c_1 \\ c_1\ c_2\ c_3\ \ldots\ c_{N-1}c_N 0\ \ldots 0\ a_N\ a_{N-1}\ \ldots\ a_3\ a_2\ a_1\ 0 \end{pmatrix}$$

where,
I is an identity matrix.

Proof.

Let $T_R X_{(i,j)} = Y_{(i,j)}$. $Y_{(i,j)} = X_{(i,j)}^{(t+1)}$is a equal to the linear combination ofthe (4N) neighbors in the following equations. The co-efficient of$X_{(i,j)} = 0$ if $i \leq 0$ or j ≤ 0 . By using the local rule of the cellular automata we have obtainthe following,

$$X_{(1,1)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(1,r+1)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+1,1)}\right) + \sum_{r=1}^{N}\left(c_r X_{(1,n-(r-1))}\right) + \sum_{r=1}^{N}\left(d_r X_{(m-(r-1),1)}\right)$$

$$X_{(1,2)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(1,r+2)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+1,2)}\right) + c_1 X_{(1,1)} + \sum_{r=2}^{N}\left(c_r X_{(1,n-(r-2))}\right) + \sum_{r=1}^{N}\left(d_r X_{(m-(r-1),2)}\right)$$

$$X_{(1,3)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(1,r+3)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+1,3)}\right) + c_1 X_{(1,2)} + c_2 X_{(1,1)} + \sum_{r=3}^{N}\left(c_r X_{(1,n-(r-3))}\right) + \sum_{r=1}^{N}\left(d_r X_{(m-(r-1),3)}\right)$$

⋮

$$X_{(1,n)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(1,r)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+1,n)}\right) + \sum_{r=1}^{N}\left(c_r X_{(1,n-r)}\right) + \sum_{r=1}^{N}\left(d_r X_{(m-(r-1),n)}\right)$$

$$X_{(2,1)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(2,r+1)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+2,1)}\right) + \sum_{r=1}^{N}\left(c_r X_{(2,n-(r-1))}\right) + d_1 X_{(1,1)} + \sum_{r=2}^{N}\left(d_r X_{(m-(r-2),1)}\right)$$

$$X_{(2,2)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(2,r+2)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+2,2)}\right) + c_1 X_{(1,1)} + \sum_{r=2}^{N}\left(c_r X_{(2,n-(r-1))}\right) + d_1 X_{(1,2)} + \sum_{r=2}^{N}\left(d_r X_{(m-(r-2),2)}\right)$$

$$X_{(2,3)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(2,r+3)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+2,3)}\right) + c_1 X_{(1,2)} + c_2 X_{(1,1)} + \sum_{r=3}^{N}\left(c_r X_{(2,n-(r-3))}\right) + d_1 X_{(1,3)} + \sum_{r=2}^{N}\left(d_r X_{(m-(r-2),3)}\right)$$

⋮

$$X_{(2,n)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(2,r)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+2,n)}\right) + \sum_{r=1}^{N}\left(c_r X_{(2,n-r)}\right) + d_1 X_{(1,n)} + \sum_{r=2}^{N}\left(d_r X_{(m-(r-2),n)}\right)$$

$$X_{(3,1)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(3,r+1)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+3,1)}\right) + \sum_{r=1}^{N}\left(c_r X_{(3,n-(r-1))}\right) + d_1 X_{(2,1)} + d_2 X_{(1,1)} + \sum_{r=3}^{N}\left(d_r X_{(m-(r-3),1)}\right)$$

$$X_{(3,2)}^{(t+1)} = \sum_{r=1}^{N}(a_r X_{(3,r+2)}) + \sum_{r=1}^{N}(b_r X_{(r+3,2)}) + c_1 X_{(3,1)} + \sum_{r=2}^{N}(c_r X_{(3,n-(r-2))}) + d_1 X_{(2,2)} + d_2 X_{(1,2)} + \sum_{r=3}^{N}(d_r X_{(m-(r-3),2)})$$

$$X_{(3,3)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(3,r+3)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+3,3)}\right) + c_1 X_{(3,2)} + c_2 X_{(3,1)} + \sum_{r=3}^{N}\left(c_r X_{(3,n-(r-3))}\right) + d_1 X_{(2,3)} + d_2 X_{(1,3)} + \sum_{r=3}^{N}(d_r X_{(m-(r-3),3)})$$

$$\vdots$$

$$X_{(3,n)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(3,r)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r+3,n)}\right) + \sum_{r=1}^{N}\left(c_r X_{(3,n-r)}\right) + d_1 X_{(2,n)} + d_2 X_{(1,n)} + \sum_{r=3}^{N}(d_r X_{(m-(r-3),n)})$$

Similarly,

$$X_{(m,1)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(m,r+1)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r,1)}\right) + \sum_{r=1}^{N}(c_r X_{(m,n-(r-1))}) + \sum_{r=1}^{N}(d_r X_{(m-r,1)})$$

$$X_{(m,2)}^{(t+1)} = \sum_{r=1}^{N}(a_r X_{(m,r+2)}) + \sum_{r=1}^{N}(b_r X_{(r,2)}) + c_1 X_{(m,1)} + \sum_{r=2}^{N}(c_r X_{(m,n-(r-2))}) + \sum_{r=1}^{N}(d_r X_{(m-r,2)})$$

$$X_{(m,3)}^{(t+1)} = \sum_{r=1}^{N}(a_r X_{(m,r+3)}) + \sum_{r=1}^{N}(b_r X_{(r,3)}) + c_1 X_{(m,2)} + c_2 X_{(m,1)} + \sum_{r=3}^{N}(c_r X_{(m,n-(r-3))}) + \sum_{r=1}^{N}(d_r X_{(m-r,3)})$$

$$\vdots$$

$$X_{(m,n)}^{(t+1)} = \sum_{r=1}^{N}\left(a_r X_{(m,r)}\right) + \sum_{r=1}^{N}\left(b_r X_{(r,n)}\right) + \sum_{r=1}^{N}(c_r X_{(m,n-r)}) + \sum_{r=1}^{N}(d_r X_{(m-r,n)})$$

4.2 Example

If we take m = 10 and n = 9, then we get rule matrix $T_R$of 2D finite CA with Von Neumann with neighborhood radius 4 rule over the field $\mathbb{Z}_3$be as follows:

$$T_R = \begin{pmatrix} A & d_1I & d_2I & d_3I & d_4I & 0 & b_4I & b_3I & b_2I & b_1I \\ b_1I & A & d_1I & d_2I & d_3I & d_4I & 0 & b_4I & b_3I & b_2I \\ b_2I & b_1I & A & d_1I & d_2I & d_3I & d_4I & 0 & b_4I & b_3I \\ b_3I & b_2I & b_1I & A & d_1I & d_2I & d_3I & d_4I & 0 & b_4I \\ b_4I & b_3I & b_2I & b_1I & A & d_1I & d_2I & d_3I & d_4I & 0 \\ 0 & b_4I & b_3I & b_2I & b_1I & A & d_1I & d_2I & d_3I & d_4I \\ d_4I & 0 & b_4I & b_3I & b_2I & b_1I & A & d_1I & d_2I & d_3I \\ d_3I & d_4I & 0 & b_4I & b_3I & b_2I & b_1I & A & d_1I & d_2I \\ d_2I & d_3I & d_4I & 0 & b_4I & b_3I & b_2I & b_1I & A & d_1I \\ d_1I & d_2I & d_3I & d_4I & 0 & b_4I & b_3I & b_2I & b_1I & A \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & c_1 & c_2 & c_3 & c_4 & 0 & a_4 & a_3 & a_2 & a_1 \\ a_1 & 0 & c_1 & c_2 & c_3 & c_4 & 0 & a_4 & a_3 & a_2 \\ a_2 & a_1 & 0 & c_1 & c_2 & c_3 & c_4 & 0 & a_4 & a_3 \\ a_3 & a_2 & a_1 & 0 & c_1 & c_2 & c_3 & c_4 & 0 & a_4 \\ a_4 & a_3 & a_2 & a_1 & 0 & c_1 & c_2 & c_3 & c_4 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & 0 & c_1 & c_2 & c_3 & c_4 \\ c_4 & 0 & a_4 & a_3 & a_2 & a_1 & 0 & c_1 & c_2 & c_3 \\ c_3 & c_4 & 0 & a_4 & a_3 & a_2 & a_1 & 0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & 0 & a_4 & a_3 & a_2 & a_1 & 0 & c_1 \\ c_1 & c_2 & c_3 & c_4 & 0 & a_4 & a_3 & a_2 & a_1 & 0 \end{pmatrix}$$

If we take m = 10 and n = 9, then we get rule matrix T_R of 2D finite CA with Von Neumann with neighborhood radius 4 rule over the field $\mathbb{Z}_3$ be as follows:
where $a_1, a_2, a_3, a_4, c_1, c_2, c_3$ and $c_4 \in \{1,2\}$
.$d_1I, d_2I, d_3I, d_4I, b_1I, b_2I, b_3I$ and b_4I are the sub matrices of order (9 × 9) , 0 is the zero matrices

Similarly,
I is an identity matrix.

5 CONCLUSION

In this paper, 2D cellular automata of Von Neumann with neighborhood radius-N are studied over the field $\mathbb{Z}_3$. The rule matrix of the transition $T_R$matrix of 2D cellular automata is computed.

REFERENCES

[1] H. Akin, "On the directorial entropy of Z_2– actions generated by additive cellular automata," Applied Mathematics and Computation, pp. 339-346, 2005.
[2] Bin yang, Chao wang and Aiyun Xiang, ''Reversibility of general 1D linear cellular automata over the binary field Z_2under null boundary conditions,"Information Sciences, 324, pp. 23-31, 2015.
[3] P. Chattopadhyay, P.P. choudhury and K. Dihidar, "Characterization of a particular hybrid transformation of two- dimensional cellular automata,"Comp.Math. Appl., 38, pp. 207-216, 1998.
[4] K. Dihidar and P.P. Choudhury, "Matrix algebraic formulae concerning some special rules of two-dimensional cellular automata,"Information Sciences, Vol.165, pp.91-101, 2004.
[5] Irfan Siap, Hasan Akin and Selman Uguz, "structure and irreversibility of 2D hexagonal cellular automata," Computers and mathematics with Applications, Vol.62, pp.4161-4160, 2011..
[6] A.R. Khan, S. Mitra and P. Sarkar, "VLSI Architecture of a cellular automata machine," Computers Math. Applic. Vol. 33, No. 5, 1997.
[7] A.R. Khan. P.P. Choudhory, K. Dihidar and R. Verama,'' Text Compression Using Two-Dimensional Cellular Automata,'' Computers and Mathematics with applications, Vol . 37, pp. 115 – 127, 1999.
[8] Selman uguz, Hasan Akin, IrfanSiap and UgurSahin, "On the irreversibility of Moore cellular automata over the ternary field and image application," Applied Mathematical Modelling, 000, pp. 1-16, 2016.
[9] J. Von Neumann, "The theory of self – reproducting automata, in : A.W. Burks (Ed.)," Urbana, 1966. Wolfram and Stephen, "Statistical mechanics of cellular automata,''Reviews of modern physics, Vol. 55, No. 3, pp. 601- 644, 1983.
[10] Z. Ying, Y. Zhong, D. Pei-min,'' On behavior of two-dimensional cellular automata with an exceptional rule,'' Information Sciences, Vol. 179, No. 5, pp. 613- 622, 2009.