Restrained Independence In Graphs

I. Sahul Hamid, M. Fatima Mary, A. Anitha

Abstract: A set $S \subseteq V(G)$ is an independent set if no two vertices of $S$ are adjacent. An independent set $S$ such that $< V - S >$ has no isolates is called Restrained independent set. A restrained independent set is maximal if it is not a proper subset of any restrained independent set. The minimum and maximum cardinalities of a maximal restrained independent set are called respectively restrained independence number and upper restrained independence number. This paper initiates a study of these parameters.

Index Terms: Covering number, Domination number, Independence number, Restrained Independence number.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a non-trivial finite graph with neither loops nor multiple edges. For graph theoretical parameters, we refer to Chartrand and Lesniak [1]. One of the fastest growing areas within graph theory is the study of domination and related subset problems such as independence, covering and matching. In fact, there are scores of graph theoretic concepts involving domination, covering and independence. The bibliography in domination maintained by Haynes et al. [2] now has over 1200 entries in which one can find out an appendix listing nearly 75 different types of domination and domination related parameters which have been studied in the literature. Hedetniemi and Laskar [6] edited an issue of Discrete Mathematics entirely devoted to domination and a survey of advanced topics in domination is given in the book by Haynes et al. [4]. This vast survey on domination and its related concepts has no bounds and gives an ample opportunity for the researchers to envisage the potentials of domination in graphs. Several research articles were published on domination in graphs in which the following chain established by Cockayne et al. [3], has become the focal point:

$$ir(G) \leq \gamma(G) \leq \gamma_r(G) \leq \Gamma(G) \leq \Gamma_r(G) \leq IR(G) \cdots (1)$$

Here $ir(G)$ and $IR(G)$ denote the lower and upper irredundance numbers, $\gamma(G)$ and $\Gamma(G)$ denote the lower and upper domination numbers, and $i(G)$ and $\beta_0(G)$ denote respectively the independent domination number and independence number of a graph $G$. More details of these parameters are found in [2]. Since then more than 100 research papers have been published in which this inequality chain is the focus of study and still a good number of researchers are carrying out research focusing on this chain. Basically the dominating chain, being the focus of study for many researchers, is built with just two fundamental concepts namely domination and independence of graphs. Nevertheless, in spite of the many variations of domination, we could identify only few variations of independence in the literature. This may be because of the fact that the structure of an independent set is clearly known (as empty graph) while that of domination is unknown and hence one might continue to define new variations of domination by imposing conditions on dominating sets. However, as the structure outside the independent sets is not predictable, variations of independence are possible by having conditions on outside the independent sets (in fact this type of domination parameters can also be seen in the literature). Motivated by this, the concept of outer connected independence was introduced and studied in [7]. In this sequence, this paper introduces another variation of independence namely restrained independence and initiates a study on this concept.

2. DEFINITION AND EXAMPLES

In this section, we define the notion of restrained independence along with the respective parameters lower restrained independence number and restrained independence number for a graph.

2.1 Definition

An independent set $S$ of a graph $G$ is said to be a restrained independent set (RI-set) if the subgraph $< V - S >$ induced by $V - S$ has no isolates. The restrained independence number $\beta_r(G)$ is the maximum cardinality of a restrained independent set and the lower restrained independence number $\beta_l(G)$ is the minimum cardinality of a maximal restrained independent set. By a $\beta_r$-set we mean a restrained independent set of cardinality $\beta_r(G)$ and a $\beta_l$-set is a restrained independent set of cardinality $\beta_l(G)$.

2.2 Remark

(i) Obviously, a restrained independent set must contain all the isolates. Further, if $v$ is a support vertex in a graph $G$, then the respective pendant vertices adjacent to $v$ in $G$ will become isolates in $G - v$. Therefore, a restrained independent set cannot have support vertices.

(ii) It is certain that for a graph with isolates, the set consisting of all the isolates is a RI-set. Suppose $G$ has no isolates. Now, if $G$ has a component $H$ of order at least 3, then the set $\{x\}$, where $x$ is a non-support vertex of $H$, is a RI-set of $G$. Besides, if each component of $G$ is $K_2$ then $G$ will not admit a RI-set. Thus a graph $G$ admits a RI-set if and only if $G$ is not isomorphic to $\cup K_2$.

(iii) It immediately follows from the definition that $\beta_l(G) \leq \beta_r(G) \leq \beta_\ell(G)$ for any graph $G$. 

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2.3 Example

(i) For the graph G of Figure 2.1, the set $S = \{v_4, v_5, v_6\}$ is a maximal RI-set. As a restrained independent set contains no support vertices, $S$ is the only maximal RI-set of $G$. Hence $\beta_r(G) = \beta_r(G) = 3$.

![Figure 2.1: A graph G with $\beta_R(G) = \beta_r(G) = 3$.](image)

(ii) Consider the graph $G$ in Figure 2.2. Here the sets $S_1 = \{u_1, u_4, u_6\}$ and $S_2 = \{u_2, u_7\}$ are maximal restrained independent sets of $G$ and so $\beta_r(G) \leq 2$ and $\beta_R(G) \geq 3$. Further, any maximal RI-set of $G$ contains exactly one vertex from each of the sets $\{u_1, u_2, u_3\}$ and $\{u_5, u_6, u_7\}$ so that $\beta_r(G) \geq 2$. Also, as $\beta_R(G) \leq \beta_R(G)$ and $\beta_R(G) = 3$, it follows that $\beta_R(G) \leq 3$. Thus $\beta_r(G) = 2$ and $\beta_R(G) = 3$.

![Figure 2.2: A graph G with $\beta_r(G) = 2$ and $\beta_R(G) = 3$.](image)

(iii) In a star $K_{1,n-1}$ with center vertex $v$, the set $S$ of all pendant vertices is a maximum independent set; but it is not a RI-set because $|V - S| = K_1$, wherein $v$ is the isolate vertex of $V - S$. Therefore, the set consisting of any $n - 2$ pendant vertices is the only possible maximal RI-set of $K_{1,n-1}$. Hence $\beta_r(G) = \beta_R(G) = n - 2$.

(iv) Let $G$ be complete $k$-partite graph that is not a star with partition $(V_1, V_2, \ldots, V_k)$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$. When $k > 2$, the set $V_1$ is a maximal RI-set of minimum cardinality and $V_k$ is that of maximum cardinality. When $k = 2$, the sets $V_1 - \{x\}$ and $V_2 - \{y\}$, for some $x \in V_1$, and $y \in V_2$, are maximal RI-sets of minimum and maximum cardinality respectively. Thus we have

$$\beta_r(G) = \begin{cases} (i(G) - 1) & \text{if } k > 2 \\ i(G) & \text{if } k = 2 \end{cases}$$

$$\beta_R(G) = \begin{cases} \beta_r(G) & \text{if } k > 2 \\ \beta_R(G) - 1 & \text{if } k = 2 \end{cases}$$

In the following proposition, we determine the values of $\beta_r$ and $\beta_R$ for paths and cycles.

2.4 Proposition

(i) For a path $P_n$ ($n \geq 4$) we have

$$\beta_r(P_n) = \begin{cases} 2 & \text{if } n = 4 \\ \frac{n}{2} & \text{if } n = 0 \pmod{5} \end{cases}$$

$$\beta_R(P_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n = 1, 2, 3, 4 \pmod{5} \\ \frac{n}{3} & \text{if } n = 0 \pmod{3} \end{cases}$$

(ii) For a cycle $C_n$ on $n$ vertices $\beta_r(C_n) = \left\lceil \frac{n}{3} \right\rceil$, for all $n$, and $\beta_R(C_n) = \left\lceil \frac{n}{3} \right\rceil + 1$ if $n = 1, 2, 3 \pmod{5}$.

Proof: (i) Let $P_n = (v_1, v_2, \ldots, v_n)$. First we shall find the value of $\beta_r(P_n)$. Suppose that $n \equiv 0 \pmod{3}$. Then the set $S_n = \left\{ v_{3i+1} : 0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1 \right\}$ is a maximal RI - set of $P_n$ and so $\beta_R(P_n) \geq \left\lceil \frac{n}{3} \right\rceil$. On the other hand, if $D$ is any $\beta_R$ - set of $P_n$, then for each $i \in \{0, 1, 2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor - 1\}$ at most one of the vertices $v_{3i+1}$, $v_{3i+2}$ and $v_{3i+3}$ can be in $D$; for otherwise either $D$ becomes a non-independent set or $< V - D >$ would contain an isolated vertex. Thus $D$ contains at most $\left\lceil \frac{n}{3} \right\rceil$ vertices of $P_n$ and so $\beta_r(P_n) \leq \left\lceil \frac{n}{3} \right\rceil$. Hence $\beta_r(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Hence $\beta_r(P_n) = \left\lceil \frac{n}{3} \right\rceil$ when $n \equiv 0 \pmod{3}$. For the remaining cases that when $n \equiv 1 \text{ or } 2 \pmod{3}$, the set $S \cup \{v_1\}$ becomes a $\beta_R$ - set of $P_n$.

Let us now find $\beta_r(P_n)$. Since a RI - set cannot have support vertices the set consisting of the two pendant vertices of $P_d$ is the only maximal RI - set of $P_d$ and so $\beta_r(P_d) = 2$. Assume $n \geq 5$. Let $n \equiv 0 \pmod{3}$. Then $S_1 = \left\{ v_{5i+3} : 0 \leq i \leq \left\lfloor \frac{n}{5} \right\rfloor - 1 \right\}$ is a maximal RI - set of $P_n$ so that $\beta_r(P_n) \leq \left\lceil \frac{n}{5} \right\rceil$. On the other hand, let $S_1$ be any $\beta_R$ - set of $P_n$. Then for each $i \in \{0, 1, 2, \ldots, \left\lfloor \frac{n}{5} \right\rfloor - 1\}$ at least one
of the vertices $v_{5i+1}$, $v_{5i+2}$, $v_{5i+3}$, $v_{5i+4}$ and $v_{5i+5}$ belongs to $S_i$; for otherwise $S_i \cup \{v_{5i+3}\}$ becomes a RI – set as $\beta_r(P_n) \geq \left\lceil \frac{n}{5} \right\rceil$. Moreover, $S_i \cup \{v_n\}$ or $S_i \cup \{v_{n-2}\}$ becomes a $\beta_r$ – set of $P_n$ according as

$$n = 1, 2 \text{ or } 3 \mod 5 \text{ or } n = 4 \mod 5.$$  

(ii) Let $C_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. Certainly, 

$$S = \left\{ v_{3i+1} : 0 \leq i \leq \left\lceil \frac{n}{3} \right\rceil - 1 \right\}$$

is a maximal RI – set of $C_n$ and so $\beta_r(C_n) \geq \left\lceil \frac{n}{3} \right\rceil$. Further any restrained independent set of $C_n$ will contain at most one vertex among the three vertices $v_{3i+1}$, $v_{3i+2}$ and $v_{3i+3}$ for each 

$$\left\{ 0, 1, 2, \ldots, \left\lceil \frac{n}{3} \right\rceil - 1 \right\}.$$ 

This means that any RI – set of $C_n$ contains at most $\left\lceil \frac{n}{3} \right\rceil$ vertices so that 

$$\beta_r(C_n) \leq \left\lceil \frac{n}{3} \right\rceil.$$ 

Further, every $\beta_r$ – set of $C_n$ contains a vertex from the set 

$$\{ v_{3i+1}, v_{3i+2}, v_{3i+3}, v_{3i+4}, v_{3i+5} \},$$

for each

$$\left\{ 0, 1, 2, \ldots, \left\lfloor \frac{n}{5} \right\rfloor - 1 \right\};$$

for otherwise

$$S \cup \{v_{5i+3}\} \text{ becomes a RI – set, a contradiction. That is, } \beta_r \text{ – set}$$

of $C_n$ contains at least $\left\lfloor \frac{n}{5} \right\rfloor$ vertices of $C_n$ and so $\beta_r(G) \geq \left\lfloor \frac{n}{5} \right\rfloor$. Further, $S' \cup \{v_{n-2}\}$ or $S' \cup \{v_{n-3}\}$ becomes a $\beta_r$ – set of $C_n$ according as $n = 1, 2 \text{ or } 3 \mod 5 \text{ or } n = 4 \mod 5$.

Now the following theorem characterizes the graph $G$ attaining the lower bound when $G$ is connected.

### 3.1 Theorem:

Let $G$ be a connected graph of order $n \geq 3$. Then $\beta_{\text{rl}}(G) = 1$ if and only if

Proof: Let $\beta_{\text{rl}}(G) = 1$. We first prove that $\text{diam} (G) \leq 2$. Suppose on the contrary that $\text{diam} (G) \geq 3$. Let $x$ and $y$ be two vertices in $G$ such that $\text{d}(x,y) \geq 3$. Take $S = \{x, y\}$. Certainly, $S$ is an independent set in $G$. As $\beta_{\text{rl}}(G) = 1$, $S$ cannot be a RI-set of $G$ which implies that $< V -- S >$ has isolates. An isolated vertex of $< V -- S >$ is either a pendant or a vertex of degree 2 in $G$ adjacent to both $x$ and $y$. Otherwise, as $\text{d}(x, y) \geq 3$, $V -- S$ cannot have a vertex adjacent to both $x$ and $y$. Hence the isolated vertices of $< V -- S >$ are pendant vertices of $G$. But, pendant vertices of $G$ form a RI-set, and so $< V -- S >$ has exactly one pendant vertex of $G$, say $z$. Now, the set $D = (\{x, y\} \cup \{z\})$ becomes a RI-set of $G$, contradicting the assumption that $\beta_{\text{rl}}(G) = 1$. Thus $\text{diam}(G) \leq 2$. If $\text{diam}(G) = 1$ then $G$ is complete to which the value of $\beta_{\text{rl}}(G) = 1$. Suppose $\text{diam}(G) = 2$. In this case it is enough to prove that $\Delta(G) \leq 2$. Suppose not. Then there exists a vertex $x$ such that $\text{deg} x \geq 3$. Choose a vertex $y \in N(x)$, where $N(x) = V -- N[x]$. $\Delta (G)$ will be adjacent to at least one vertex of $N(G)$. Now consider the set $U = \{x, y\}$. Certainly, $U$ is an independent set of $G$. As $\beta_{\text{rl}}(G) = 1$, $U$ cannot be a RI-set of $G$ and so $< V -- U >$ has an isolated vertex. Since $\text{diam}(G) = 2$, every vertex in $\overline{N(x)}$ has a neighbor in $N(x)$ and so no vertex in $\overline{N(x)}$ can be an isolate of $< V -- U >$. So, $\text{deg} x \geq 3$. Choose a vertex $y \in N(x)$, where $N(x) = V -- N[x]$. $\Delta (G)$ will be adjacent to exactly one of $x$ and $u$ or $v$ is of degree 2 in $G$ adjacent to both $x$ and $u$. In either case, $z$ has a non-neighbor $u$ in $N(x)$. So, the set $W = \{u, z\}$ is independent but it cannot be a RI – set of $G$ and so $V -- W$ has an isolate. Certainly, $v \in N(x)$. Also, $v$ is either a pendant vertex of $G$ adjacent to exactly one of $x$ and $u$ or $v$ is of degree 2 in $G$ adjacent to both $x$ and $u$. If for instance $v$ is a pendant neighbor of $z$, then $\text{d}(x, y) \geq 3$, which is not possible. So, the only case left with us is, $v$ is of degree 2 adjacent to both $u$ and $z$. Now consider a vertex $w \in N(x)$. Then $X = \{w, v\}$ is an independent such that $< V -- X >$ has an isolate vertex $t$ lying in $\overline{N(x)}$. Also, since $\text{deg} x = 2$, the vertex $t$ must be a pendant in $G$ adjacent to $w$ and so $\text{d}(t, w) \geq 3$, a contradiction. Thus $\Delta(G) \leq 2$ as desired. This argument is illustrated in Figure 3.1.
We now proceed to obtain a bound for $\beta_R$ in terms of the number of pendant vertices of a graph. For this we prove the following lemma.

### 3.2 Lemma:

Every graph $G$ other than a star admits a $\beta_R$ - set containing all the pendant vertices.

**Proof:** Let $S$ be a $\beta_R$ - set of $G$. Suppose there exists a pendant vertex $v$ lying outside $S$. Then the neighbor $u$ of $v$ must be in $V - S$. Also, $N(u) - \{v\} \subseteq S$, for if $u$ has a neighbor in $V - S$, then $S \cup \{v\}$ is a RI-set of $G$, a contradiction. As $G$ is not a star, one of the neighbors of $u$ must be non-pendant in $G$. Let it be $x$. Now $S_1 = (S - \{x\}) \cup \{v\}$ is a RI-set of $G$. If $S_1$ contains all the pendant vertices, we are done, otherwise repeat the process. The following theorem is an immediate consequence of the Lemma 3.2.

### 3.3 Theorem:

For any graph $G$, we have $\beta_R(G) \geq l$, where $l$ denotes the number of pendant vertices of $G$. The following theorem characterizes the trees attaining the bound given in Theorem 3.3

### 3.4 Theorem:

Let $T$ be a tree with $l$ pendant vertices other than a star graph. Then $\beta_R(T) = l$ if and only if every non-pendant vertex $v$ of $T$ is either a support vertex or it has a support neighbor all of whose neighbors other than $v$ are pendant. Proof: Let $\beta_R(T) = l$ and let $v$ be an arbitrary non-pendant vertex of $T$. Suppose $v$ is not a support in $T$. Now, as $\beta_R(T) = l$, by virtue of Lemma 3.2, there is a $\beta_R$ - set $S$ consisting of all the pendant vertices of $T$. Certainly $v \in V - S$ and $D = S \cup \{v\}$ is an independent set of $T$. However, the set $D$ cannot be a RI-set of $T$ and so $V - D$ has an isolated vertex, say $x$. Therefore $x$ must be adjacent to all the vertices of $D$ as $x$ is not pendant in $T$. So $x$ is a neighbor of $v$ all of whose neighbors are pendant in $T$. Thus $x$ is a required neighbor of $v$. Conversely, suppose the given condition holds. We wish to prove that $\beta_R(T) = l$. Let $S'$ be a $\beta_R$ - set of $T$ containing all the pendant vertices of $T$. Now, it is enough to prove that $S'$ contains no non-pendant vertex of $T$. Let $v$ be a non-pendant vertex of $T$. If $v$ is a support, certainly $v \notin S'$. So, assume that $v$ is not a support vertex of $T$. Then by assumption $v$ has a support neighbor $x$ of $T$ all of whose neighbors other than $v$ are pendant. By the choice of $S'$ all the pendant neighbors of $x$ are in $S'$. Also $x \notin S$, being $x$ a support vertex of $T$. So, the vertex $v$ cannot be in $S'$; for otherwise the vertex $x \in V - S'$ would be an isolated vertex of $\{V - S'\}$, a contradiction. Thus $S'$ consists of only the pendant vertices of $T$ as desired. Hence $\beta_R(T) = l$. A tree $T$ which satisfies the condition described in Theorem 3.4 is given in Figure 3.2 and therefore $\beta_R(T) = l$.

### 3.5 Theorem:

For any connected graph $G$ of order $n \geq 3$, $\beta_p(G) \geq \left[\frac{\text{diam}(G) + 1}{3}\right]$ and the bound is sharp.

**Proof:** The inequality is trivial when $\text{diam} (G) = 2$. Let $\text{diam} (G) = k \geq 3$. Let $P = (v_1, v_2, ..., v_{k+1})$ be a diametrical path in $G$. Consider the set $S = \{v_i | 1 \leq i \leq \left\lfloor\frac{k}{3}\right\rfloor\}$, set those vertices $v_i$ of $S$ that are supports in $G$, choose exactly one of its pendant neighbor, say $v'_i$. Now, let $D$ be the set consisting of the non cut-vertices of $G$ belonging to $S$ together with the chosen pendant vertices $v'_i$’s corresponding to the support vertices $v_i$’s of $G$ lying in $S$. We now claim that $D$ is a RI-set of $G$. On the contrary, suppose $D$ is not a RI-set of $G$. That is, $<V - D>$ has an isolated vertex, say $x$. Certainly, $\deg_{G}\chi \geq 2$. As $x$ is an isolate in $<V - D>$, all its neighbors are in $D$. Therefore, there exist at least two vertices $v_i$ and $v_j$ with $|i - j| = 3$, say for example $v_i$ and $v_{i+3}$, such that $x$ is adjacent to both $v_i$ and $v_{i+3}$. Now, $(v_1, v_2, ..., v_i, x, v_{i+3}, v_{i+4}, ..., v_{i+1})$ is a $v_i - v_{i+1}$ path of length $k - 1$, a contradiction to the fact that $P$ is a diametrical path. Hence $V - D$ has no isolates and thus $D$ is a RI-set of $G$ as desired.

### 4. RELATIONSHIP WITH EXISTING PARAMETERS

In Section 2, we have observed that $\beta_p(G) \leq \beta_R(G) \leq \beta_s(G)$ for any graph $G$. These inequalities are strict in the sense that they all can be distinct or they all can be equal as shown in the following example.
4.1 Example:

(i) Consider the graph \( G = K_r \circ K_1 \) which is obtained from the complete graph \( K_r \) by attaching exactly one pendant vertex at each vertex of \( K_1 \). As support vertices are not elements of a RI - set, the set of all pendant vertices of \( G \) is the only maximal RI - set of \( G \) so that \( \beta_r(G) = \beta_\infty(G) = r \). Obviously \( \beta_0(G) = r \).

(ii) For the graph \( G \) of Figure 4.1, the sets \( S_1 = \{v\} \) and \( S_2 = \{v_2, v_3\} \) are respectively maximal RI - sets of minimum and maximum cardinality so that \( \beta(G) = 1 \) and \( \beta_0(G) = 2 \). It is obvious that \( \beta_0(G) = 3 \).

We prove in the following theorem that the parameters \( \beta_r \) and \( \beta_0 \) can assume arbitrary values.

4.2 Theorem:

For given integers \( a \) and \( b \) with \( 1 < a \leq b \), there exist a graph \( G \) for which \( \beta_0(G) = a \) and \( \beta_\infty(G) = b \). Proof: Suppose that \( a \) and \( b \) are given integers with \( 1 < a \leq b \). We construct a graph \( G \) such that \( \beta_r(G) = a \) and \( \beta_\infty(G) = b \) as follows. Consider the star \( K_{1,b-a+2} \) with the center vertex \( u_0 \). Now subdivide each edge of the star exactly once and let \( u_1, u_2, ..., u_{b-a+2} \) be the pendant vertices in the resultant graph (the resultant graph thus obtained is referred to as the subdivision graph of the star). Finally attach a \( -1 \) pendant vertices \( v_1, v_2, ..., v_{b-1} \) at any of the vertices \( u_1, u_2, ..., u_{b-a+2} \), say for instance at the vertex \( u_{b-a+2} \).

Let \( G \) be the graph thus constructed (See Figure 4.2.) Obviously \( \{u_0, v_1, v_2, ..., v_{b-1}\} \) is a maximal RI - set with minimum cardinality and \( \{v_1, v_2, ..., v_{b-1}, u_1, u_2, ..., u_{b-a+1}\} \) forms a maximal RI - set with maximum cardinality so that \( \beta_r(G) = a \) and \( \beta_\infty(G) = b \).

As every outer connected independent set of a connected graph \( G \) other than a star is also a RI - set of \( G \), it follows that \( I_{oc}(G) \leq \beta_0(G) \leq \beta_\infty(G) \). There are graphs \( G \) for which all these three parameters are equal and also there are graphs to which all these parameters are distinct as seen in the following example.

4.3 Example:

(i) Let \( G \) be a graph on \( n \geq 3 \) vertices with \( \Delta(G) = n - 1 \) that is not a star. If \( G \) is a complete graph on \( n \geq 3 \) vertices, then we have \( \beta_\infty(G) = I_{oc}(G) = \beta_0(G) = 1 \). Suppose that \( \beta_0(G) \geq 2 \). Let \( S \) be a maximum independent set of \( G \) and let \( u \) be a vertex with \( \deg u = n - 1 \). Certainly \( u \notin S \). Therefore \( \{V - S\} \) is connected which has at least two vertices. Hence \( S \) becomes an oci-set as well as an RI - set so that \( I_{oc}(G) \geq \beta_0(G) \) and \( \beta_\infty(G) \geq \beta_0(G) \). Thus we get \( \beta_0(G) = I_{oc}(G) = \beta_0(G) \).

(ii) For the path \( P_8 \), we have \( I_{oc}(P_8) = 2 \), \( \beta_\infty(P_8) = 3 \) and \( \beta_0(P_8) = 4 \).

We prove in the following theorem that the parameters \( I_{oc}(G) \) and \( \beta_\infty(G) \) can assume arbitrary values.

4.4 Theorem:

Given integers \( a \) and \( b \) with \( 1 < a \leq b \), there exists a graph \( G \) for which \( I_{oc}(G) = a \) and \( \beta_\infty(G) = b \).

Proof: Let \( a \) and \( b \) be the given integers with \( 1 < a \leq b \). The required graph \( G \) is obtained from the subdivision of the star \( K_{1,a} \) with each vertex \( u_0 \) and with the pendant vertices \( u_1, u_2, ..., u_a \), by attaching a path \( P = (v_1, v_2, ..., v_{b-a}) \) at any one of the vertices \( u_1, u_2, ..., u_a \), say at \( u_1 \), with \( u_1 = v_1 \). Let \( v \in N(u_0) \) be the vertex adjacent to \( u_1 \)(the graph is given in Figure 4.3). Then obviously \( \{u_2, ..., u_a, v_{b-a}\} \) is an I_{oc} - set of \( G \) and \( S_1 = \{v_1, 1 \leq i \leq b - a\} \cup \{v\} \), where \( S_1 = \{u_2, ..., u_a\} \), is a \( \beta_\infty \) - set of \( G \). Hence \( I_{oc}(G) = a \) and \( \beta_\infty(G) = b \).

4.5 Remark:

There is no relation between the parameters \( \beta_r \) and \( I_{oc} \). For example, (i) for the path \( P_{5k} \), where \( k \geq 3 \), we have \( \beta_r(P_{5k}) = k \) whereas \( \text{loc}(P_{5k}) = 2 \). Also, if \( G = K_{1,n-1} \), then \( I_{oc}(G) = n - 1 \); but \( \beta_r(G) = n - 2 \). Further, for the complete graphs both \( I_{oc} \) and \( \beta_r \).
are same as 1. In the following theorems we obtain some relationships of $\beta_R$ with some existing graph theoretic parameters such as vertex covering number, total domination number and connected domination number.

4.6 Theorem:
If $G$ is a connected graph on $n \geq 3$ vertices with $\Delta(G) = n - 1$ that is not a star, then $\beta_R(G) \leq n - \gamma_c(G)$.

Proof: If $\beta_R(G) = 1$, then by Theorem 3.1, $G = K_n$ for which $\gamma_c = 1$ so the inequality is true. Let $\beta_R(G) > 1$. Let $S$ be a $\beta_R$ - set of $G$. Then $< V - S >$ has no isolates. Also, a vertex of degree $n - 1$ cannot be in $S$ and so $< V - S >$ is connected. Certainly, $V - S$ is a dominating set in $G$ which implies that $V - S$ becomes a connected dominating set of $G$ and so $|V - S| \geq \gamma_c(G)$. That is $\beta_R(G) \leq n - \gamma_c(G)$.

4.7 Remark:
Obviously $\beta_R(G) \leq n - \alpha_0(G)$ as $\beta_R(G) \leq \beta_r(G)$ and $\alpha_0(G) + \beta_r(G) = n$. Further if $\Delta(G) = n - 1$, then by Example 4.3(i) that $\beta_R(G) = \beta_r(G)$ and so $\beta_R(G) = n - \alpha_0(G)$. But it is not necessary that $\Delta(G) = n - 1$ when $\beta_R(G) = n - \alpha_0(G)$. For example, for the graph $G$ in Figure 4.4, $\beta_R(G) = 5$, $\alpha_0(G) = 4$ and so $\beta_R(G) = n - \alpha_0(G)$, but $\Delta(G) = 4 \neq n - 1$.

![Figure 4.4: A graph $G$ with $\Delta(G) \neq n - 1$ and $\beta_R(G) = n - \alpha_0(G)$](image)

4.8 Theorem:
For any connected graph $G$ of order $n \geq 3$ with no isolates $\beta_R(G) \leq n - \gamma_c(G)$ and the bound is sharp. Proof: Let $S$ be a $\beta_R$ - set of $G$. Then $< V - S >$ has no isolates and hence $V - S$ becomes a total dominating set of $G$. Thus $|V - S| \geq \gamma_c(G)$. That is $\beta_R(G) \leq n - \gamma_c(G)$. For the sharpness of the bound, consider the graph $G = K_r \circ K_t$ on $2r$ vertices to which $\beta_R(G) = r = \gamma_c(G)$. That is $\beta_R(G) = n - \gamma_c(G)$.

5. CONCLUSION AND SCOPE
In this chapter, we have introduced a new variation of independence namely restrained independence along with the parameters $\beta$ and $\beta_R$ and initiated the study on these parameters. Also some lower and upper bounds are attained. Further we have investigated the relationships of $\beta$ with some existing graph theoretic parameters such as independence number, outer connected independence number, vertex covering number, total domination number and connected domination number. However there is a wide scope for further research on this concept. Some problems are listed below.

(A) For given integers $a$, $b$ and $c$ with $a \leq b \leq c$, find a graph $G$ with $\beta_r(G) = a$, $\beta_R(G) = b$ and $\beta_c(G) = c$.

(B) For given integers $a$, $b$ and $c$ with $a \leq b \leq c$, find a graph $G$ with $I_c(G) = a$, $\beta_R(G) = b$ and $\beta_0(G) = c$.

(C) Characterize the graphs $G$ for which $\beta_R(G) = \beta_0(G)$.

(D) Characterize the graphs $G$ for which $\beta_R(G) = 1$.

(E) One can initiate the study on vertex and edge criticality associated with the restrained independence parameters.

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