Solution Of Partial Integro Differential Equations Using Mdtm And Comparison With Two Dimensional Dtm

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Abstract: In this paper, linear partial integro-differential equations (PIDE) with convolution kernel are solved using Modified Differential Transform Method (MDTM) and compared with Two Dimensional Differential Transform Method (DTM). The concept of two dimensional DTM and MDTM are briefly explained. The results obtained by MDTM and Two Dimensional DTM are compared. Finally, performance and accuracy of both the methods are discussed and reveals that MDTM is very effective, convenient and reduces a lot of computational work and time than two dimensional DTM.

Keywords: Maclaurin’s series, Modified differential transform method, Partial integro-differential equation, Two Dimensional Differential Transform method.

1. INTRODUCTION

PARTIAL integro differential equations are used in various fields of science and engineering. The analysis of such PIDE can be found in [5],[6],[7],[8],[9],[10],[11],[12],[13]. In last few years, various numerical schemes are proposed by “Dehghan [8]” to solve PIDE arising in viscoelasticity. “Dehghan M and Shakeri F [9]” used variational iteration technique to solve PIDE arising in heat conduction. “Bahuguna D [6]” has proposed the existence and uniqueness of a solution of PIDE by the method of lines. “J. M. Yoon [10]” has used variational iteration method (VIM) to solve PIDE arising in viscoelasticity. “Amiya K. Pani [11]” has used finite element method (FEM) for parabolic partial integro differential equations. “Hua Li [12]” has used wavelet collocation methods to solve PIDE arising in jump-diffusion models. “Hussam E. Hashim [13]” has used Taylor to solve PIDE. In this paper, we consider the following general linear PIDE with convolution kernel (see [4],[16],[22],[23],[24])

\[ \sum_{i=1}^{m} a_i \frac{\partial^{i} u}{\partial x^i} + \sum_{i=1}^{n} b_i \frac{\partial^{i} u}{\partial t^i} + cu(x,t) + \sum_{i=1}^{d} \int_{0}^{t} K_i(t-y) \frac{\partial^{i} u}{\partial x^i} dy + f(x,t) = 0 \]  

Where \( a_i, b_i \) and \( d_i \) are constants or the functions of \( x \) alone. And \( f(x,t), K_i(t-y) \) are known functions. “Jyoti Thorwe and Sachin Bhalakar [1]” have applied Laplace transform. “Ranjit Dhunde and G. L. Waghmare [2]” have used double Laplace transform. “Mohand M. Abdelrahim Mehgob and Tarig M. Elzaki [3]” have used Elzaki Transform. “Mohand M. Abdelrahim Mehgob [4]” have used double Elzaki transform to solve linear PIDE. Differential Transform Method is a semi analytical numerical technique which depends on Taylor’s series for solution of differential and integral equations. The concept of differential transform method was first introduced by “Zhou [15]” who solved linear and non-linear initial value problems in electric circuit analysis. Recently, “A. Arikoglund I. Ozkol [16]” used DTM to solve integro and integro differential equation. “A. Tari and S. Shahmorad [17]” used DTM to solve a system of two-dimensional nonlinear Volterra integro differential equations. The various types of differential equations, integro differential equations and Volterra integral equations are solved by using two dimensional DTM [18],[19],[20],[21],[22],[23],[24]. The two dimensional differential transform method is good tool to obtain solution of ODE, PDE and integro differential equations, but it is observed that it takes a lot of computational time to obtain original solution for large values of parameters to the multiple summations.

In order to avoid this difficulty, the present paper shows some properties of Modified Differential Transform method, which is modified version of two dimensional DTM “Aruna, K. and A. S. V. Ravi Kanth [25]”. It will also highlight on an analytical numerical solution of linear partial integro-differential equations (PIDE) with convolution kernel by using two dimensional DTM and modified differential transform method.

1 TWO DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD (DTM)

Definition: Let the function \( u(x,t) \) is an arbitrary analytic and continuously differentiable function of \( x \) and \( t \) defined on \( D = [0,X] \times [0,T] \subset R^2 \) and \( (x_0,t_0) \in D \), then differential transform of \( u(x,t) \) is defined as:

\[ U(k,h) = \frac{1}{k!h!} \left( \frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t) \right)_{x=x_0,t=t_0} \quad k,h \geq 0 \]  

Where \( u(x,t) \) is original function and \( U(k,h) \) is transformed function.

Therefore, its inverse transform define as

\[ u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h)(x-x_0)^k(t-t_0)^h \quad ; \quad k,h \geq 0 \]  

Note: If \( x_0 = 0 \) and \( t_0 = 0 \)

\[ U(k,h) = \frac{1}{k!h!} \left( \frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t) \right)_{x=0,t=0} \quad ; \quad k,h \geq 0 \]  

\[ u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h)x^k t^h \quad ; \quad k,h \geq 0 \]

In the following theorem, we summarize some fundamental
properties of two dimensional differential transforms (see [16], [17], [18], [19], [20]):

Theorem 2.1: Let \( U(k, h), Y(k, h), V(k, h) \) and \( G(h) \) are differential transforms of \( u(x, t), y(x, t), v(x, t) \) and \( g(t) \) respectively then

i) If \( u(x, t) = x^n t^m \) then
\[
U(k, h) = \delta(k - m, h - n) = \begin{cases} 1, & \text{if } k = m, h = n \\ 0, & \text{if } k \neq m, h \neq n \end{cases}
\]

ii) If \( u(x, t) = e^{ax + bt} \) then
\[
U(k, h) = e^{\frac{a}{h!} k^h b^h}
\]

iii) If \( u(x, t) = e^{ax} \) then
\[
U(k, h) = e^{\frac{a}{h!} h!} (h!)
\]

iv) If \( u(x, t) = x^m \sin \theta \) then
\[
U(k, h) = \delta(k - m) \frac{a^h}{h!} \sin \left( \frac{h!}{2} \right)
\]

v) If \( u(x, t) = x^m \cos \theta \) then
\[
U(k, h) = \delta(k - m) \frac{a^h}{h!} \cos \left( \frac{h!}{2} \right)
\]

vi) If \( u(x, t) = \frac{a^m}{h!} \) then
\[
U(k, h) = (k + 1)(k + 2) \ldots (k + m) Y(k + m, h)
\]

vii) If \( u(x, t) = \frac{a^m}{h!} \) then
\[
U(k, h) = (h + 1)(h + 2) \ldots (h + n) Y(k, h + n)
\]

viii) If \( u(x, t) = \frac{a^m}{h!} \) then
\[
U(k, h) = (k + 1)(k + 2) \ldots (k + m)(h + 1)(h + 2) \ldots (h + n) Y(k + m, h + n)
\]

ix) If \( f(x, t) = u(x, t)v(x, t) \) then
\[
U(k, h) = \sum_{m=0}^{k} \left( \sum_{l=0}^{m} U(m, h - l) V(k - m, l) \right)
\]

x) If \( f(x, t) = g(t)u(x, t) \) then
\[
F(k, h) = \sum_{l=0}^{h} G(l) U(k, h - l)
\]

xi) If \( f(x, t) = g(t)u(x, t) \) then
\[
F(k, h) = \sum_{m=0}^{k} G(m) U(k - m, h)
\]

Now we give the basic theorem of two dimensional DTM of this paper for solving linear PIDE with convolution kernel.

Theorem 2.2: Assume that \( F(k, h), U(k, h), G(h) \) and \( V(h) \) are differential transforms of \( f(x, t), u(x, t), g(t) \) and \( v(t) \) respectively, then we have

(a) If \( f(x, t) = \int_{0}^{t} u(x, y)dy \) then
\[
F(k, h) = \begin{cases} U(k, h - 1), & \text{if } h \geq 1 \\ 0, & \text{if } h = 0 \end{cases}
\]

(b) If \( f(x, t) = g(t) \int_{0}^{t} u(x, y)dy \) then
\[
F(k, h) = \begin{cases} \frac{1}{h!} \sum_{l=0}^{h-1} G(l) U(k, h - l - 1), & \text{if } h \geq 1 \\ 0, & \text{if } h = 0 \end{cases}
\]

(c) Define \( v(x, t) = g(t)u(x, t) \)

Using theorem 2.1 (x),
\[
V(k, h) = \sum_{l=0}^{h} G(l) U(k, h - l)
\]

Therefore replace \( h \) by \( h - l \) in \( V(k, h) \) we get,
\[
F(k, h) = \begin{cases} \frac{1}{h!} \sum_{l=0}^{h-1} G(l) U(k, h - l - 1), & \text{if } h \geq 1 \\ 0, & \text{if } h = 0 \end{cases}
\]

2 MODIFIED DIFFERENTIAL TRANSFORM METHOD (MDTM)

Definition: The Taylor's series expansion of the function \( u(x, t) \) with respect to specific variable \( t = t_0 \) is,
\[
U(x, h) = \frac{1}{h!} \left( \frac{\partial^h}{\partial t^n} u(x, t) \right)_{t=t_0} ; \quad h \geq 0
\]
Where \( u(x, t) \) is original function and \( U(x, h) \) is transformed function
Therefore, the inverse modified differential transform of a function \( U(x,h) \) is defined by

\[
U(x,t) = \sum_{h=0}^{\infty} U(x,h)(t-t_0)^h ; \; h \geq 0
\]  

(9)

Note: \( t_0 = 0 \)

\[
U(x,h) = \frac{1}{h!} \left\{ \frac{\partial^h}{\partial t^h} u(x,t) \right\}_{t=0} ; \; h \geq 0
\]  

(10)

\[
\therefore U(x,t) = \sum_{h=0}^{\infty} U(x,h)t^h ; \; h \geq 0
\]

In the following theorem, we summarize some fundamental properties of modified differential transforms "Aruna K. and A. S. V. Ravi Kanth [25]".

Theorem 3.1: Let \( U(x,h), Y(x,h), V(x,h), F(x,h) \) and \( G(h) \) are modified differential transforms of \( u(x,t), y(x,t), v(x,t), f(x,t) \) and \( g(t) \) respectively then

i) If \( f(x,t) = u(x,t) \pm v(x,t) \) then \( F(x,h) = U(x,h) \pm V(x,h) \)

ii) If \( f(x,t) = \alpha u(x,t) \) then \( F(x,h) = \alpha U(x,h) \)

iii) If \( f(x,t) = \frac{\partial y(x,t)}{\partial x} \) then \( U(x,h) = \frac{d}{dx} Y(x,h) \)

iv) If \( f(x,t) = \frac{\partial y(x,t)}{\partial t} \) then \( U(x,h) = (h+1)Y(x,h+1) \)

v) If \( f(x,t) = \frac{\partial^m y(x,t)}{\partial x^m} \) then \( U(x,h) = \frac{d^m}{dx^m} Y(x,h) \)

vi) If \( f(x,t) = \frac{\partial^m y(x,t)}{\partial t^m} \) then \( U(x,h) = (h+1) \cdots (h+n) \frac{d^m}{dx^m} Y(x,h+n) \)

vii) If \( f(x,t) = x^n \) then \( U(x,h) = x^m \delta(h-n) = x^m \left\{ \begin{array}{ll} 1, & \text{if } h = n \\ 0, & \text{if } h \neq n \end{array} \right. \)

Now we give new basic properties of Modified Differential Transform Method are as follows:

Theorem 3.2: If \( U(x,h) \) is modified differential transforms of \( u(x,t) \) then

(a) If \( u(x,t) = t^n \) then \( U(x,h) = \delta(h-n) = \left\{ \begin{array}{ll} 1, & \text{if } h = n \\ 0, & \text{if } h \neq n \end{array} \right. \)

(b) If \( u(x,t) = x^n \) then \( U(x,h) = x^m \delta(h) = \left\{ \begin{array}{ll} 1, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0 \end{array} \right. \)

(c) If \( u(x,t) = e^{ax+bt} \) then \( U(x,h) = e^{ax+bh} \frac{h!}{h!} \)

(d) If \( u(x,t) = e^{ax} \) then \( U(x,h) = e^{ax} \frac{h!}{h!} \)

(e) If \( u(x,t) = e^{bt} \) then \( U(x,h) = \frac{b^h}{h!} \)

(f) If \( u(x,t) = x^m \sin(at) \) then \( U(x,h) = \frac{x^m}{h!} a^h \sin \left( \frac{h\pi}{2} \right) \)

(f) If \( u(x,t) = x^m \cos(at) \) then \( U(x,h) = \frac{x^m}{h!} a^h \cos \left( \frac{h\pi}{2} \right) \)

Proof: Using the definition of MDTM (10),

(a) If \( u(x,t) = t^n \)

\[
\therefore U(x,h) = \frac{1}{h!} \left\{ \frac{\partial^h}{\partial t^h} t^n \right\}_{t=0} = \left\{ \begin{array}{ll} 1, & \text{if } h = n \\ 0, & \text{if } h \neq n \end{array} \right. \]

(b) If \( u(x,t) = x^m \)

\[
\therefore U(x,h) = \frac{1}{h!} \left\{ \frac{\partial^h}{\partial t^h} x^m \right\}_{t=0} = \left\{ \begin{array}{ll} x^m, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0 \end{array} \right. \]

(c) If \( u(x,t) = e^{ax+bt} \)

\[
\therefore U(x,h) = \frac{1}{h!} \left\{ \frac{\partial^h}{\partial t^h} e^{ax+bt} \right\}_{t=0} = e^{ax+bh} \frac{h!}{h!} \]

(d) The proof is obviously proving by using theorem 2.1(ii).

(e) The proof is obviously proving by using theorem 3.2(c).

(f) If \( u(x,t) = x^m \sin(at) \)

\[
\therefore U(x,h) = \frac{1}{h!} \left\{ \frac{\partial^h}{\partial t^h} x^m \sinat \right\}_{t=0} = \frac{x^m}{h!} a^h \sin \left( \frac{h\pi}{2} \right) \]

(g) The proof is similar to the proof of (f)

Now we give the basic theorem of MDTM of this paper for solving linear PIDE with convolution kernel.

Theorem 3.3: Let \( U(x,h), F(x,h), V(x,h) \) and \( G(h) \) are modified differential transforms of \( u(x,t), f(x,t), v(x,t) \) and \( g(t) \) respectively then

(a) If \( f(x,t) = \int_0^t u(x,y)dy \) then

\[
F(x,h) = \left\{ \begin{array}{ll} U(x,h-1) + \frac{h}{0} & \text{if } h \geq 1 \\ 0 & \text{if } h = 0 \end{array} \right. \]

(11)

(b) If \( f(x,t) = g(t)u(x,t) \) then

\[
F(x,h) = \sum_{i=0}^{h} \frac{G(i)U(x,h-l)}{h} \]

(12)

(c) If \( f(x,t) = \int_0^t g(y) u(x,y)dy \) then

\[
F(x,h) = \sum_{i=0}^{h} \frac{G(i)U(x,h-l-1)}{h-l} \]

(13)

(d) If \( f(x,t) = g(t) \int_0^t u(x,y)dy \) then

\[
F(k,h) = \sum_{i=0}^{h} \frac{G(i)U(x,h-l-1)}{h-l} \]

(14)

Proof: (a) Consider,

\[
\left\{ \frac{\partial}{\partial t^h} f(x,t) \right\}_{t=0} = \left\{ \frac{\partial}{\partial t^h} \int_0^t u(x,y)dy \right\}_{t=0}
\]

Using definition of MDTM (10),

\[
\therefore h! F(x,h) = \frac{\partial}{\partial t^h} \int_0^t u(x,y)dy \right\}_{t=0}
\]

If \( h = 0 \)

\[
\therefore F(0,x) = \left\{ \int_0^t u(x,y)dy \right\}_{t=0} = \int_0^t u(x,y)dy = 0
\]

Since for \( h \geq 1 \)

\[
\therefore h! F(x,h) = \left( h-1 \right)! U(x,h-1) \]

(15)

(16)

(b) Consider,

\[
\left\{ \frac{\partial}{\partial t^h} f(x,t) \right\}_{t=0} = \left\{ \frac{\partial}{\partial t^h} g(t)u(x,t) \right\}_{t=0}
\]

Applying Leibnitz theorem,

\[
\therefore h! F(x,h) = \sum_{l=0}^{h} \frac{h!}{l!} \frac{\partial^l g(t)}{\partial t^l} \frac{\partial^{h-l} u(x,t)}{\partial t^{h-l}}
\]

(17)

Using definition of MDTM (10),

\[
\therefore U(x,h) = \frac{1}{h!} \left\{ \frac{\partial^h}{\partial t^h} e^{ax+bt} \right\}_{t=0} = e^{ax+bh} \frac{h!}{h!} \]
3 APPLICATION OF TWO DIMENSIONAL
DIFFERENTIAL TRANSFORM METHOD (DTM)

In this section, we describe application of the two dimensional DTM for equation (1) and obtain

\[ \sum_{i=1}^{m} a_i \frac{d^i}{dx^i} U(x, h) + \sum_{i=1}^{l} b_i U(x, h + j) \left( \prod_{j=1}^{i} (h + j) \right) + \sum_{i=0}^{r} c_i \int_{0}^{t} K_i(t - y) \left( \frac{\partial^i U}{\partial x^i} \right) dy + F(x, h) = 0 \]

After expanding kernel function \( k(t - y) \) in the form as \( \Phi(t)\psi(y) \) and using theorem 3.1-3.3, a transformed function \( U(x, h) \) is obtained then we use definition (10) to obtain \( u(x, t) \).

5 RESULTS AND DISCUSSION

In this section, we solve an example "[1],[2],[3],[4]" of linear PIDE with convolution kernel using both Two Dimensional DTM and Modified Differential transform method.

Example "[1],[2],[3],[4]":

Consider PIDE, \( u_{tt} = u_x + 2 \int_{0}^{t} (t - y) u(x, y) dy - 2e^x \)

Initial condition \( u(x, 0) = e^x, u_t(x, 0) \)

and boundary condition \( u(0, t) = 2t \)

Solution: Given,

\[ u_{tt} = u_x + 2 \int_{0}^{t} u(x, y) dy - 2e^x \]

\[ u_{tt} = u_x + 2 \int_{0}^{t} u(x, y) dy - 2e^x \] (15) with initial condition \( u(x, 0) = e^x, u_t(x, 0) = 0 \) (16) and boundary condition \( u(0, t) = \text{cost} \) (17)

(a) Using two dimensional DTM:

Applying Two Dimensional DTM on both sides of (11) to (13),

\[ U(k, h + 2) = \frac{1}{(h + 1)(h + 2)} \left( (k + 1)U(k + 1, h) + \frac{2}{k!} \delta(h) \right) (18) \]

U(0, 0) = 1, U(1, 0) = 0, \forall k \text{ and } U(0, h) = \frac{1}{k!} \cos \left( \frac{h \pi}{2} \right), \forall h \]

(b) Using MDTM:

Now using (18) to (21),
If $h = 0$,  
$$U(k, 2) = \frac{1}{(1)(2)}(k + 1)U(k + 1, 0) + 2(0) - 2(0)$$
$$= \frac{1}{2} \delta(0); \quad U(k, 2)$$
$$= \frac{1}{2} \frac{(k + 1)}{(k + 1)!} - \frac{2}{k!} \delta(1)$$
$$= \frac{1}{2} \frac{1}{k!}; \quad \forall k$$

$\therefore U(1,2) = -\frac{1}{240}, U(2,2) = -\frac{1}{4}, U(3,2) = -\frac{1}{12}$

$U(4,2) = -\frac{1}{48}, U(5,2) = -\frac{1}{240} \ldots$

If $h = 1$, 

$\therefore U(k, 3) = \frac{1}{6} \frac{(k + 1)}{U(k + 1, 1)}$

$$+ 2 \sum_{i = 1}^{\infty} \frac{\delta(i - 1)}{(i - 1)!} \frac{U(k, 1 - i - 1)}{(0 - 1) - \frac{2}{k!} \delta(1)$$

$\therefore U(1,3) = U(2,3) = U(3,3) = U(4,3) = U(5,3) = \ldots$  

If $h = 2$, 

$\therefore U(k, 4) = \frac{1}{12} \frac{(k + 1)}{U(k + 1, 2)}$

$$+ 2 \sum_{i = 1}^{\infty} \frac{\delta(i - 1)}{(i - 1)!} \frac{U(k, 2 - i - 1)}{(2 - 1) - \frac{2}{k!} \delta(2)}$$

$\therefore U(1,4) = \frac{1}{12} \frac{1}{12} \frac{1}{(k + 1)!} + 2(0) + U(k, 0))$

$$- 2 \left\{ \frac{0 + U(k, 0)}{2} \right\} = \frac{1}{24} \frac{1}{k!}; \quad \forall k$$

$\therefore U(2,4) = \frac{1}{48}, U(3,4) = \frac{1}{144}, U(4,4) = \frac{1}{576}$

$U(5,4) = \frac{1}{2880}, U(6,4) = \frac{1}{17280}$ \ldots

If $h = 3$, 

$\therefore U(k, 5) = \frac{1}{20} \frac{(k + 1)}{U(k + 1, 3)}$

$$+ 2 \sum_{i = 1}^{\infty} \frac{\delta(i - 1)}{(i - 1)!} \frac{U(k, 3 - i - 1)}{(3 - 1) - \frac{2}{k!} \delta(3)}$$

$\therefore U(1,5) = U(2,5) = U(3,5) = U(4,5) = U(5,5) = \ldots$  

If $h = 4$, 

$\therefore U(k, 6) = \frac{1}{30} \frac{(k + 1)}{U(k + 1, 4)}$

$$+ 2 \sum_{i = 1}^{\infty} \frac{\delta(i - 1)}{(i - 1)!} \frac{U(k, 4 - i - 1)}{(4 - 1) - \frac{2}{k!} \delta(4)}$$

$\therefore U(1,6) = -\frac{1}{17280}, U(2,6) = -\frac{1}{1440}, U(3,6) = -\frac{1}{4320}$

$U(4,6) = -\frac{1}{86400}$ \ldots

$\therefore U(x, t) = \sum_{k = 0}^{\infty} \frac{U(k, 0)x^k}{k!} + \sum_{k = 0}^{\infty} \frac{U(k, 1)x^k t^2}{k!} + \sum_{k = 0}^{\infty} \frac{U(k, 2)x^k t^4}{k!} + \ldots$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \ldots$  

If $h = 0$ : 

$\therefore U(x, 2) = \frac{1}{2} \frac{dU(x, 0)}{dx} + 2(0) - 2(0) = -2e^x \delta(0)$

$$= \frac{1}{2} \frac{d}{dx} \left( e^x \right) - e^x \delta(0)$$

$\therefore U(x, 3) = \frac{1}{6} \frac{dU(x, 1)}{dx} + 2 \sum_{i = 0}^{0} \frac{\delta(i - 1)}{(i - 1)!} \frac{U(x, 1 - l - 1)}{(1 - l)}$

$$- 2 \sum_{i = 0}^{0} \delta(i - 1) \frac{U(x, 1 - l - 1)}{(1 - l)} - 2e^x \delta(1)$$

$$= \frac{1}{6} \left( 0 + 2(0) - 2(0) - 2e^x(0) \right) = 0$$

If $h = 2$
\[ U(x, 4) = \frac{1}{12} \left( \frac{dU(x, 2)}{dx} + 2 \sum_{i=0}^{1} \delta(l-1) \frac{U(x, 2 - l - 1)}{(2 - l)} \right) \\
- 2 \sum_{i=0}^{1} \delta(l-1) \frac{U(x, 2 - l - 1)}{(2 - l)} - 2e^x \delta(2) \]
\[ = \frac{1}{12} \left( \frac{d}{dx} \left( \frac{-e^x}{2} \right) + 2(0) + 2(1)U(x, 0) - 2(0) \right) \\
- \frac{2(1)U(x, 0)}{2} - 2e^x(0) \right) = \frac{e^x}{4!} \]

If \( h = 3 \)
\[ \therefore U(x, 5) = \frac{1}{20} \left( \frac{dU(x, 3)}{dx} + 2 \sum_{i=0}^{2} \delta(l-1) \frac{U(x, 3 - l - 1)}{(3 - l)} \right) \\
- \frac{2 \sum_{i=0}^{2} \delta(l-1) \frac{U(x, 3 - l - 1)}{(3 - l)} - 2e^x \delta(3)}{2} \]
\[ = \frac{1}{20} \left( \frac{d}{dx} \left( 0 \right) + 2(0) + \frac{2(1)U(x, 1)}{2} + 2(0) - 2(0) \right) \\
- \frac{2(1)U(x, 1)}{2} \right) = 0 \]

If \( h = 4 \)
\[ \therefore U(x, 6) = \frac{1}{30} \left( \frac{dU(x, 4)}{dx} + 2 \sum_{i=0}^{3} \delta(l-1) \frac{U(x, 4 - l - 1)}{(4 - l)} \right) \\
- \frac{2 \sum_{i=0}^{3} \delta(l-1) \frac{U(x, 4 - l - 1)}{(4 - l)} - 2e^x \delta(4)}{3} \]
\[ = \frac{1}{30} \left( \frac{d}{dx} \left( \frac{e^x}{24} \right) + 2(0) + \frac{2(1)U(x, 2)}{3} + 2(0) + 2(0) \right) \\
- \frac{2(1)U(x, 2)}{2} \right) = \frac{e^x}{4!} \]
\[ = \frac{1}{30} \left( e^x \right) \frac{e^x}{3} + \frac{e^x}{4} = \frac{e^x}{6!} \]

In general,
\[ \therefore U(x, 2m) = (-1)^m \frac{e^x}{(2m)!} ; \forall m \text{ and } U(x, 2m + 1) = 0 \text{ ; } \forall m \text{ ; } u(x, t) = \sum_{n=0}^{\infty} U(x, n)h^t \]
\[ U(x, 0) + U(x, 1) + U(x, 2) + U(x, 3) + U(x, 4) + \ldots \]
\[ = e^x + 0 + \left( \frac{e^x}{2!} \right) t^2 + 0 + \left( \frac{e^x}{4!} \right) t^4 + 0 \]
\[ + \left( \frac{e^x}{6!} \right) t^6 + 0 + \left( \frac{e^x}{8!} \right) t^8 + \ldots \]
\[ = e^x \left( 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \ldots \right) \]
\[ \therefore u(x, t) = e^x \text{ cost} \]

The result of the example shows that the properties of both Two Dimensional DTM and MDTM are easily applied on linear partial integro differential equation with convolution kernel to obtain series solution. But it is easy to see that Two Dimensional DTM depends on both initial and boundary conditions. If any one of the initial and boundary condition is not known then this method is not applicable for linear PIDE and this method takes a lot of computational time to obtain transformed function \( U(k, h) \) for large values of parameters \( h \) and \( k \) to multiple summations. Whereas MDTM is independent on boundary conditions and reduces computational efforts and time.

6 CONCLUSION

In this way, the present paper defines and shows the properties of both two dimensional Differential Transform Method and Modified Differential Transform Method. It also clears that both the methods can successfully be applied to obtain analytical numerical solution of linear Partial Integro Differential Equations with convolution kernel. It is also observed that MDTM is independent on boundary conditions and reduces a lot of computational time and efforts than two dimensional DTM. Therefore, it is concluded that Modified Differential Transform Method is more powerful and efficient technique to reduce a computational work and time than two dimensional DTM.

REFERENCES


