Characterization Of Strong And Weak Dominating \( \chi \) - Color Number In A Graph

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Abstract: Strong dominating \( \chi \) - color number of a graph \( G \) is defined as the maximum number of color classes which are strong dominating sets of \( G \), and is denoted by \( sd(G) \). Similarly, weak dominating \( \chi \) - color number of a graph \( G \) is defined as the maximum number of color classes which are weak dominating sets of \( G \), and is denoted by \( wd(G) \). In both the cases, the maximum is taken over all \( \chi \) - coloring of \( G \). In this paper, some bounds for \( sd(G) \) and \( wd(G) \) are obtained and characterized the graphs for which strong dominating \( \chi \) - color number and strong dominating \( \chi \) - color number exist. Finally, Nordh-Kadum inequalities for \( sd(G) \) and \( wd(G) \) is derived.

Index Terms: Dominating-\( \chi \)-color number, Strong dominating \( \chi \)-color number, Weak dominating \( \chi \)-color number.

1 INTRODUCTION

In this paper, we consider finite, connected, undirected and simple graph \( G = (V(G), E(G)) \) with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The number of vertices \( |V(G)| \) of a graph \( G \) is called the order of \( G \) and the number of edges \( |E(G)| \) of a graph \( G \) is called the size of \( G \). The order and size is denoted by \( n \) and \( m \) respectively. In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number \( \chi(G) \) of a graph \( G \) which is defined to be the minimum number of colors required to color the vertices of \( G \) in such a way that no two adjacent vertices receive the same color. If \( \chi(G) = k \), we say that \( G \) is \( k \)-chromatic [1]. For any vertex \( v \in V(G) \), the open neighborhood of \( v \) is the set \( N(v) = \{u \mid uv \in E(G)\} \) and the closed neighborhood is the set \( N[v] = N(v) \cup \{v\} \). Similarly, for any set \( S \subseteq V(G) \), \( N(S) = \cup_{v \in S} N(v) \) and \( N[S] = N(S) \cup S \). A set \( S \) is a dominating set if \( N[S] = V(G) \). The minimum cardinality of a dominating set of \( G \) is denoted by \( \gamma(G) \) [4]. A set \( D \subseteq V(G) \) is a dominating set of \( G \), if for every vertex \( x \in V \) - \( D \) there is a vertex \( y \in D \) with \( xy \in E \) and \( D \) is said to be strong dominating set of \( G \), if it satisfies the additional condition \( \deg(x) \leq \deg(y) \) [2]. The strong domination number \( \gamma_{sd}(G) \) is defined as the minimum cardinality of a strong dominating set. A set \( S \subseteq V \) is called weak dominating set of \( G \) if for every vertex \( u \in V - S \), there exists vertex \( v \in S \) such that \( uv \in E \) and \( \deg(u) \geq \deg(v) \). The weak domination number \( \gamma_{wd}(G) \) is defined as the minimum cardinality of a weak dominating set and was introduced by Sampathkumar and Pushpalatha [3]. The number of maximum degree vertices of \( G \) is denoted by \( n_{\Delta}(G) \) and the number of minimum degree vertices of \( G \) is denoted by \( n_{\delta}(G) \).

2 TERMINOLOGY

We start with more formal definition of dominating \( \chi \)-color number of \( G \) [5]. Let \( G \) be a graph with \( \chi(G) = k \). Let \( C = V_{1}, V_{2}, ..., V_{k} \) be a \( k \)-coloring of \( G \). Let \( d_{C} \) denotes the number of color classes in \( C \) which are dominating sets of \( G \). Then \( d_{C}(G) = \max_{C} d_{C} \) where the maximum is taken over all the \( k \)-colorings of \( G \), is called the dominating \( \chi \)-color number of \( G \). Instead of dominating set in the definition of dominating \( \chi \)-color number, if we consider strong dominating set, then it is called strong dominating-\( \chi \)-color number \( sd_{\chi}(G) \) and, if we consider weak dominating set, then it is called weak dominating-\( \chi \)-color number \( wd_{\chi}(G) \) [6]. Though substantial work has been carried out on domination and coloring parameters and related topics in graphs, there are only a few results concerning strong and weak domination in graphs. The new parameter, strong dominating-\( \chi \)-color number and weak dominating-\( \chi \)-color number defined by us in [7]. The following are some perceived propositions,

- Proposition 1: For any graph \( G \), \( 0 \leq sd_{\chi}(G) \leq d_{\chi}(G) \).
- Proposition 2: For any graph \( G \), \( 0 \leq wd_{\chi}(G) \leq d_{\chi}(G) \).
- Proposition 3: For any cycle \( C_{n} \), \( sd_{\chi}(C_{n}) = \{ \begin{array}{ll} 3 & \text{if } n \equiv 3 \pmod{6} \\ 2 & \text{otherwise} \end{array} \) if \( n > 4 \), then \( sd_{\chi}(C_{n}) = 1 \).

3 STRONG (WEAK) DOMINATING \( \chi \)-COLOR NUMBER EQUALS TO ZERO

Arumugam et al. [8] observed that “Every graph contains a \( \chi \) -coloring with the property that at least one color class is a dominating set in \( G \).” Strong dominating \( \chi \)-color number and weak dominating \( \chi \)-color number may not exist for all the graphs, even though every graph has dominating \( \chi \)-color set and independent strong dominating set and weak dominating set. Because, in the definition of strong dominating \( \chi \)-color number and weak dominating \( \chi \)-color number, first priority goes to \( \chi \)-coloring of a graph. To examine the necessary condition for \( sd_{\chi}(G) \) and \( wd_{\chi}(G) \) equals zero, we proved the following theorem.
Theorem 1: If there exist a pair of non-adjacent vertices \((u,v)\) such that for all \(x \in N(u), d(u) > d(x)\), and for all \(y \in N(v), d(v) > d(y)\), has distinct color in all possible \(\chi\) -coloring of a graph \(G\), then \(sd_\chi(G) = 0\).

Proof:
Let \(D\) be a color class with vertex \(u\) of \(G\). Let there exist a pair of non-adjacent vertices \((u,v)\) such that for all \(x \in N(u), d(u) > d(x)\), and for all \(y \in N(v), d(v) > d(y)\), has distinct color in all possible \(\chi\) -coloring of a graph \(G\). Since the vertex \(u\) can not be strongly dominated by any vertices of \(G\), the only one color class \(D\) may be a strong dominating \(\chi\) -color set of \(G\). But there is vertex \(v \notin D\), that also cannot be strongly dominated by any other vertices. Therefore, \(sd_\chi(G) = 0\). The following theorem is immediate for weak dominating \(\chi\) -color number.

Theorem 2: If there exist a pair of non-adjacent vertices \((u,v)\) such that for all \(x \in N(u), d(u) > d(x)\), and for all \(y \in N(v), d(v) > d(y)\), has distinct color in all possible \(\chi\) -coloring of a graph \(G\), then \(wd_\chi(G) = 0\).

4 STRONG (WEAK) DOMINATING \(\chi\) - COLOR NUMBER EQUALS TO ONE

Negation of theorem 6, gives the following theorem 3 that characterize the graphs for which \(sd_\chi(G) \geq 1\) and \(wd_\chi(G) \geq 1\).

Theorem 3: If there is no pair of non-adjacent vertices \((u,v)\) has distinct color in \(\chi\) -coloring of a graph \(G\), then \(sd_\chi(G) \geq 1\) and \(wd_\chi(G) \geq 1\).

Theorem 4: If there is no pair of non-adjacent vertices \((u,v)\) such that for all \(x \in N(u), d(u) > d(x)\), and for all \(y \in N(v), d(v) > d(y)\), has distinct color in all possible \(\chi\) -coloring of a graph \(G\), then \(sd_\chi(G) \geq 1\).

Theorem 5: If there is no pair of non-adjacent vertices \((u,v)\) such that for all \(x \in N(u), d(u) < d(x)\), and for all \(y \in N(v), d(v) < d(y)\), has distinct color in all possible \(\chi\) -coloring of a graph \(G\), then \(wd_\chi(G) \geq 1\).

Theorem 6: For any regular graph \(G\), \(sd_\chi(G) = wd_\chi(G) = d_\chi(G)\).

Proof: Since all the degree of the regular graph is same, all the dominating color classes are strong as well weak. Consequently, \(sd_\chi(G) = wd_\chi(G) = d_\chi(G)\).

Corollary 1: For any complete graph \(K_n\), \(sd_\chi(K_n) = wd_\chi(K_n) = n\).

Theorem 7: For any bi-regular graph \(G\),
(a) If \(\Delta(G) \neq \delta(G)\), then \(sd_\chi(G) = wd_\chi(G) = 1\).
(b) If \(\Delta(G) = \delta(G)\), then \(sd_\chi(G) = wd_\chi(G) = 2\).

Proof:
(a) Since \(\Delta(G) \neq \delta(G)\), there are two partitions with different degree. If the maximum degree vertex partition and minimum degree vertex partition associated with color class \(C_1\) and \(C_2\) respectively, then \(C_1\) is strong dominating color set and \(C_2\) is weak dominating color set. Therefore, \(sd_\chi(G) = wd_\chi(G) = 1\).
(b) Since \(\Delta(G) = \delta(G)\), all the degree of the bi-regular graph is same, both the partitions are dominating color classes as well as strong and weak. Consequently, \(sd_\chi(G) = wd_\chi(G) = 2\).

Corollary 2: For any star graph, \(K_{1,n}\), \(sd_\chi(K_{1,n}) = wd_\chi(K_{1,n}) = 1\).

Theorem 8: Let \(G\) be a path \(P_n\), then \(sd_\chi(P_n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } 3 \\ 2 & \text{otherwise} \end{cases}\)

Proof:
For \(n = 1\) or \(3\), it is clear that \(sd_\chi(P_n) = 1\).
For \(n = 2\), the path \(P_2 = K_2\), \(sd_\chi(P_n) = 2\).
Now our claim is \(sd_\chi(P_n) = 2\), if \(n > 3\). Since \(P_n\) is a bipartite graph, \(d_\chi(P_n) = \chi(P_n) = 2\), let \(A\) and \(B\) be two dominating \(\chi\) -color sets of \(P_n\). Let \(u\) and \(v\) be the pendant vertices of \(P_n\). If \(n\) is odd, then \(u, v \in A\). In that case, all the vertices outside \(A\) can be strongly dominated by \(A-\{u,v\}\). So, \(A\) strongly dominates \(B\). All the vertices outside \(B\) can be strongly dominated by \(B-\{u,v\}\). So, \(B\) strongly dominates \(A\). Consequently both \(A\) and \(B\) strong dominating \(\chi\) -color sets of \(P_n\). Thus, \(sd_\chi(P_n) = 2\).

For \(n = 2\), the path \(P_2 = K_2\), \(wd_\chi(P_2) = 2\). The next theorem gives the weak dominating \(\chi\) -color sets of \(P_n\) for \(n \neq 2\).

Theorem 9: Let \(G\) be a path \(P_n\), \(n \neq 2\), then \(wd_\chi(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}\)

Proof:
Since \(P_n\) is a bipartite graph, \(d_\chi(P_n) = \chi(P_n) = 2\), let \(A\) and \(B\) be two dominating \(\chi\) -color sets of \(P_n\). Let \(u\) and \(v\) be the pendant vertices of \(P_n\). If \(n\) is odd, then \(u, v \in A\). In that case, all the vertices outside \(A\) can be weakly dominated by \(A-\{u,v\}\). So, \(A\) weakly dominates \(B\). But the pendant vertices \(u, v\) are outside \(B\), they cannot be weakly dominated by any vertex of \(B\). Even though \(B\) has \(\chi\) -color set, it is not a weak dominating \(\chi\) -color set. Consequently, \(wd_\chi(P_n) = 1\). If \(n\) is even, then \(u \in A\) and \(v \in B\). In that case, both \(A\) and \(B\) are not weakly dominating \(\chi\) -color sets. Because, the pendant vertices \(u\) and \(v\) is only dominated by its support vertices, which has degree two. Thus \(wd_\chi(P_n) = 0\).

Theorem 10: For any graphs \(G\) and \(H\),
\(sd_\chi(G \cup H) = \min\{sd_\chi(G), sd_\chi(H)\}\).

Proof:
Let \(D_1, D_2, \ldots, D_{sd_h(H)}\) be the strong dominating \(\chi\) -color sets of \(H\) and let \(C_1, C_2, \ldots, C_{sd_\chi(G)}\) be the strong dominating \(\chi\) -color sets of \(G\). Without loss of generality, let us assume that \(sd_\chi(G) > sd_\chi(H)\).
Now combine the vertices of \(D_i\) and \(C_i\), for all \(1 \leq i \leq sd_\chi(H)\). Then \(D_1 \cup C_1, D_2 \cup C_2, \ldots, D_{sd_h(H)} \cup C_{sd_\chi(H)}, C_{sd_\chi(H)+1}, \ldots, C_{sd_\chi(G)}\) are the
color classes of \( G \cup H \). By definition of \( G \cup H \), no vertices of \( G \) dominates the vertices of \( H \) and vice versa. Therefore, only \( D_1 \cup C_1, D_2 \cup C_2, \ldots, D_{sd_H(G)} \cup C_{sd_H(G)} \) are the strong dominating \( \chi \) –color sets of \( G \cup H \). Hence \( sd_X(G \cup H) = \min \{sd_X(G), sd_X(H) \} \).

5 Existence of graphs with \( sd_{\chi}(G) = 1 \) and \( wc_{\chi}(G) = 1 \)

Arunagam et al. [8] showed that “For every integer \( k \geq 0 \), there exists a connected graph \( G \) with \( \delta(G) = k \) and \( d_1(G) = 1 \).” In the proof, they begin with the statement “For \( k = 0 \), take \( G = K_n \).” But \( K_n \) is not connected graph for \( n > 1 \), which contradicts the statement “there exists a connected graph”. So, we need to start the proof as: For \( k = 0 \), take \( G = K_n \). Similarly, we next show that even if the minimum degree of \( G \) is arbitrarily large, the strong dominating \( \chi \) –color number may be 1.

Theorem 11: For every integer \( k \geq 0 \), there exists a graph \( G \) with \( \delta(G) = k \) and \( sd_{\chi}(G) = 1 \) and \( wc_{\chi}(G) = 1 \).

Proof: For \( k = 0 \), take \( G = K_n \). Hence we may assume that \( k > 0 \). Let \( G \) be obtained from a complete bipartite graph with partite set \( V_1 \) and \( V_2 \) each of cardinality \( k \) by adding a new vertex \( v \) and adding all the edges between vertex \( v \) and vertices of \( V_1 \). By construction, \( \delta(G) = k \). Since, \( G \) is a bipartite graph and \( |V_1| \neq |V_2| \), \( sd_{\chi}(G) = 1 \) and \( wc_{\chi}(G) = 1 \).

Theorem 12: For all integers \( a \geq b \geq 0 \), there exists a graph \( G \) with \( \chi(G) = a \) and \( sd_{\chi}(G) = b \).

Proof: For \( a = b \), take \( G = K_n \). Hence we may assume that \( a > b \). Suppose that \( a > 1 \) and \( b = 1 \). Let \( G \) be obtained from a complete graph \( K_n \) by adding a new vertex \( v \) and adding an edge between vertex \( v \) and any vertices of \( K_n \). By construction, \( \chi(G) = a \) and \( sd_{\chi}(G) = b \).

Suppose that \( a > 1 \) and \( b > 0 \). Let \( G \) be obtained from complete graphs \( K_n \) with vertex set \( \{v_1, v_2, \ldots, v_n\} \) and \( K_b \) with vertex set \( \{u_1, u_2, \ldots, u_b\} \) by adding \( b \) non-adjacent edges \( \{v_1u_1, v_2u_2, \ldots, v_bu_b\} \) between them.

If \( G \) is colored by the coloring function \( c(v_i) = i \) and \( c(u_i) = j + 1 \), then \( \chi(G) = a \). Since the coloring of \( G \) has \( b \) color classes which are strong dominating sets in \( G \), say the color classes associated with the colors \( 2, 3, \ldots, b + 1 \), we have \( sd_{\chi}(G) \geq b \).

The color classes \( \{u_1, v_2, u_2, v_3, \ldots, u_b, v_{b+1}\} \) are strong dominating sets of \( G \). The remaining \( a - b \) color classes \( \{v_1, v_{b+2}, v_{b+3}, \ldots, v_n\} \) are not dominating sets of \( G \). Consequently, \( \chi(G) = a \) and \( sd_{\chi}(G) = b \). Correspondingly, we can show the following theorem for weak dominating \( \chi \) –color number.

Theorem 13: For all integers \( a \geq b \geq 0 \), there exists a graph \( G \) with \( \chi(G) = a \) and \( wc_{\chi}(G) = b \).

6 Nordhaus-Gaddum Inequalities

For any graph \( G \) with parameter \( \psi \), sharp upper and lower bounds for \( \psi(G) + \psi(\overline{G}) \) and \( \psi(G) \cdot \psi(\overline{G}) \) are referred as Nordhaus-Gaddum inequalities. In this section, some bounds and these inequalities are derived for \( sd_{\chi}(G) \) and \( wc_{\chi}(G) \).

Observation 1: For all graphs \( G \),
\[ sd_{\chi}(G) \leq 2 \] and \( wc_{\chi}(G) \leq 2.5. \]

Theorem 14: If any graph \( G \) is a non-regular graph with \( n \) vertices and it has a connected complement, denoted by \( \overline{G} \), then
\[ sd_{\chi}(G) \leq n(G) \text{ and } wc_{\chi}(G) \leq n(G). \]

Proof: It is necessary to have a maximum degree and minimum degree vertices of \( G \) in strong dominating \( \chi \) –color set of \( G \) respectively. And number of maximum degree \( n_{\chi}(G) \) and minimum degree \( n_{\chi}(G) \) vertices of \( G \) equal to the number of minimum degree and maximum degree vertices of \( G \) respectively. Due to this fact, (a) and (b) can be easily proved. From (a) and (b), \( sd_{\chi}(G) + sd_{\chi}(\overline{G}) \leq n_{\chi}(G) + n_{\chi}(G) \). Since \( G \) is not a regular graph, \( sd_{\chi}(G) < n \) and \( sd_{\chi}(G) + n_{\chi}(G) = n \). The following are immediate from (a) and (b).
\[ sd_{\chi}(G) + sd_{\chi}(\overline{G}) \leq n_{\chi}(G) + n_{\chi}(G) \leq n. \]

Further, we can sharpen the bounds of \( sd_{\chi}(G) \) and \( wc_{\chi}(G) \) in the succeeding theorems. Let \( V_a \) and \( V_b \) be the set of all vertices which has maximum and minimum degree of a graph \( G \) respectively.

Theorem 15: Let \( G \) be any graph, then \( sd_{\chi}(G) \leq \chi(V_a) \) \( \text{ and } wc_{\chi}(G) \leq \chi(V_b) \).

Proof: Let \( G \) be any graph. The maximum degree vertices of \( G \) occur minimum \( \chi(V_a) \) number of color classes of \( G \). If all the maximum degree vertices occur in \( \chi(V_a) \) number of color classes, then \( sd_{\chi}(G) \leq \chi(V_a) \). If not, there are maximum \( n_{\chi}(G) - \chi(V_a) \) vertices occur in \( \chi(G) - \chi(V_a) \) number of color classes which are not strong dominating sets of \( G \). Because such maximum degree vertices are non adjacent and has different color with \( \chi(V_a) \) number of maximum degree vertices. Also, those vertices cannot be strongly dominated by any other vertices of \( G \). In that case, \( sd_{\chi}(G) = 0 \). Consequently, \( sd_{\chi}(G) \leq \chi(V_a) \). Similarly, we can prove that \( wc_{\chi}(G) \leq \chi(V_b) \).

Theorem 16: Let \( G \) be any graph, then \( sd_{\chi}(G) \leq \nu(G) \) \( \text{ and } wc_{\chi}(G) \leq \mu(G) \).

Proof: Let \( G \) be any graph. The maximum degree vertices of \( G \) occur minimum \( \chi(V_a) \) number of color classes of \( G \). It is clear that, out of this \( \chi(V_a) \) number of color classes of \( G \), \( \nu(G) \)
number of color classes of $G$ can be a strong dominating set of $G$. Even though the remaining color classes of $G$ has maximum degree vertices, they not dominate atleast one maximum degree vertex of $G$. So, they are not strong dominating sets of $G$. Therefore, $sd_x(G) \leq \omega((V_\Delta))$. Similarly we can prove that $wd_x(G) \leq \omega((V_\delta))$.

**Corollary 3:** If $(V_\Delta)$ is a null graph, then $sd_x(G) \leq 1$

**Corollary 4:** If $(V_\delta)$ is a null graph, then $wd_x(G) \leq 1$.

**Theorem 17:** Let $G$ be any graph with $n$ vertices. Then

(a) $0 \leq sd_x(G) + sd_x(\overline{G}) \leq n_\Delta + n_\delta$ and $0 \leq sd_x(G) \cdot sd_x(\overline{G}) \leq n_\Delta \cdot n_\delta$

(b) $0 \leq wd_x(G) + wd_x(\overline{G}) \leq n_\Delta + n_\delta$ and $0 \leq wd_x(G) \cdot wd_x(\overline{G}) \leq n_\Delta \cdot n_\delta$

**Proof:** Since $0 \leq sd_x(G) \leq n_\Delta$ and $0 \leq sd_x(\overline{G}) \leq n_\delta$, we can easily derive (a) and (b).

**REFERENCES**


