

Existence And Uniqueness Of The Adjoint Function For Degree-One Approximation: Dasgupta's Approach

Rishabh Tiwari, P. L. Powar

Abstract— Dasgupta, proposed a method to construct the wedge functions over an element of polygonal discretization of the domain using an analytic approach to determine the denominator of rational wedge function, whereas Wachpress had applied the geometric approach for the same task. We have extended the idea described by Dasgupta and established the conditions, mandatory for the existence and uniqueness of the denominator and consequently the wedge functions in case of the quadrilateral discretization of the domain. A more general form of the linear functions, representing sides of the quadrilateral has been considered, which eradicates restriction on the sides for not passing through the origin. This paper has been furnished with a Mathematica program which computes all the required parameters and finally approximation over the element under consideration.

Index Terms— Finite Element, Quadrilateral Discretization, Rational Form, Adjoint, Mathematica



1 INTRODUCTION

Finite Element Methods(FEM), have a wide range of applications, as in the process, a body with complicated geometry, could be reconstructed by subdividing it with respect to the elements of the discretized domain. In the FEM, we directly pick desired part of the body and analyze it with respect to the corresponding element, hence the entire body can be analyzed by its analysis in the different fragments[12]. It is also used in numerical analysis, where a complicated function is replaced (approximated) by a function having nice mathematical properties such as it can be differentiated, integrated etc. The use of FEM in structural engineering was first noted in 1940s in the work of Hrenikoff[11] and McHenry[17], they separately manufactured different parts of an aircraft, like wings, main body etc., and then assembled them to get the desired structure. At that time this method was termed as framework method. This framework method is known as the forerunner of the FEM. The current practices in FEM are inspired by the work of Courant[5], who used triangles for the discretization and obtained shape functions to solve stress functions. Levy in[14] and [13] suggested flexibility method and stiffness method as promising tools in aircraft construction. Derivation of stiffness matrices by using three node triangular elements was a significant contribution of Turner[20] and Argyris[2] in the area of two dimensional FEM. The name Finite Element Method, for this procedure was first used by Clough[4], when he solved stress problems using three and four node elements.

Möbius[18], calculated the barycentric coordinates by considering a triangle of reference and noted that an arbitrary point can be represented as a linear combination of these barycentric coordinates. Later this concept was used to define

Bézier and Bernstein polynomials. It was noted that these polynomials along with FEM algorithm play a crucial role in solving problems of numerical analysis, mathematical modeling, surface generation and designing, more effectively than existing methods. Applications of this concept evolved rapidly, as these barycentric coordinates form basis of the space of linear functions, and it was found that the execution of the process of interpolation became quite simple. Wachpress[22] used the concepts of algebraic geometry and proposed the idea of rational wedge functions(ratio of two polynomials) known as Wachpress' coordinates. The concept of barycentric coordinates with respect to triangle of reference, has been extended over polygons and polycons[9] and it was noted that the Wachpress' coordinates satisfy the pre-defined properties of barycentric coordinates. It has been clearly mentioned by Clough[3], that the discretization in FEM is not at all restricted to triangles, while on increasing the number of sides of the polygon a better approximation is obtained. It may be noted that, one of the results of algebraic geometry plays a key role in defining the rational form of wedge function, which insists the denominator to be a polynomial passing through the exterior intersection points of polygonal element of the discretization. This form of the rational function assures global continuity of the approximate surface. The process involved in approximation by Rational FEM is mainly comprised of the three important steps [10] viz.

Discretization of the domain: The domain has to be subdivided into non overlapping polygons in such a way that,

- Entire domain is covered.
- The intersection of interiors of any two elements is an empty set.
- The boundaries of any two elements, intersects only at a common edge or at a common node.
- The domain is connected.

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Rational wedge functions: Corresponding to each node of the element (polygon) rational wedge functions are obtained in such a way that the wedge function corresponding to the i^{th} node gives the value 1 on that node and it is zero on the sides other than the adjacent sides, and the denominator is the curve passing through the exterior intersection points of the polygonal element.

Interpolation using wedge basis functions: Finally, a real valued data of two variables, can be interpolated over the considered element with the help of these wedge functions.

Computer programs like poly2014[21] with several input options, many programs using MATLAB, Mathematica, C++, for computing rational basis made work of Wachspress more significant.

Every method has its own significance, eg. several iterative methods in numerical analysis lead to a same or similar results, some make the approximations very close to the exact value some produce a greater error but each one is important in one or the other point of application.

Dasgupta[8] emphasizes the approximations beyond 'isoparametric shapes', which has no geometric foundation and is so ad-hoc that 'isoparametric formulation' is restricted to a subset of convex quadrilaterals as compared to the projective geometry's vista that encompasses concavity, and goes to n-gons in a systematic way that are beyond what the isoparametrics could achieve.

The Wachspress Coordinates[15], mostly employed on convex regions, can be extended to concave quadrilaterals and (degenerated) triangles with a side node. Another important aspect of the derivation is that the square root singularity, which occurs in all isoparametric maps for nontrapezoidal regions, can be traced to the degenerated triangles.

This paper is concerned with the rational shape functions, and it is a well known fact that the Padé's approximants adhere different mathematical ideas than Taylor's series.

Computational issues rather than calculational ones, are algebraic in nature and modern technology in software/hardware address the entire class of problems. In order to determine adjoint of the wedge functions over a domain with polygonal discretization, Dasgupta in [8] proposed a technique different from the well established Wachspress' method (see [21], [22]), which was later appreciated and elaborated by Wachspress[7]. Unlike Wachspress' approach, the method used in [8] has a very little dependency on the geometry, rather it deals with the analysis. This new method allowed an alternative technique of solving problems involving FEM.

Using the property, that sum of wedge functions is a partition of unity, Dasgupta[8] claimed that the denominator is nothing but the sum of numerators of wedge functions, with appropriate choice of the normalizing constants K_i 's, which are obtained by solving a system of linear equations.

In this paper, a more general method of defining the linear forms, required to define wedge functions has been introduced, which allows the quadrilateral to have sides passing through the origin. Moreover, analytic approach is

applied to establish the existence and uniqueness of the denominator function (adjoint) which was an open problem in [8]. This paper also comprises of a Mathematica program, to execute the entire process accurately, which supports assertions made in this paper.

2 Prerequisites

Some basic definitions, which are required for construction and analysis are given in this section.

Let $Q_m = (1, 2, \dots, m)$, $m \geq 4$ be closed and convex polygon with m sides, in \mathbb{R}^2 and $i \in \mathbb{Z}_m$ be the vertices of Q_m which are called nodes, where $i-1$ and i are the consecutive nodes and \mathbb{Z}_m denotes the set of integers under modulo m .

Let the Cartesian coordinates of the nodes $i-1$ and i be (x_{i-1}, y_{i-1}) and (x_i, y_i) respectively. Then, equation of the straight line passing through $i-1$ and i is

$$l_i(x, y) = (i-1, i) \cong (x-x_i)(y_{i-1}-y_i) - (y-y_i)(x_{i-1}-x_i) = 0$$

Definition 1[21](see also[22]) Sides of the polygon containing the node i , are called **adjacent** to i and remaining sides are called **opposite** to the node i .

Definition 2[19] Let $s_i, i \in \mathbb{Z}_m$ be straight line passing through the nodes i and $i-1$, referring relation (1), l_i denotes the **linear form** of s_i .

Definition 3[21](see also[22]) Wedge functions are the functions $N_i: Q_m \rightarrow \mathbb{R}$ of the form

$$N_i(x, y) = K_i \frac{\text{Num}_i(x, y)}{D(x, y)} \quad (i \in \mathbb{Z}_m)$$

where K_i 's are the normalizing constants.

Within the context of uniqueness, it is important to note that $D(x, y)$ has a unique geometrical meaning for convex polygons. It is the equation of the straight line (see [21], [22]) connecting the exterior intersection points (cf. Fig. 1)

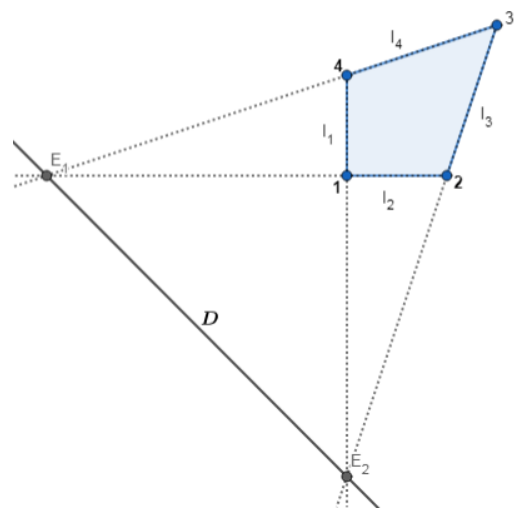


Figure 1: Figure representing the exterior intersection points

For the analysis of this paper, following properties of wedge functions described in [21](see also[22]) for linear approximation over a polygonal element, have been considered:

Properties of wedge functions[21](see also[22]) The wedge functions are functions which are smooth within their associated element and posses the following properties:

1. There is a node at each vertex of the polygon. For each node there is an associated wedge within each polygon containing the node.
2. Wedge $N_i(x, y)$ associated with node i is normalized to unity at node i ,
($i \in \mathbb{Z}_m$).
3. Wedge $N_i(x, y)$ is linear on sides adjacent to i , ($i \in \mathbb{Z}_m$).
4. Wedge $N_i(x, y)$ vanishes on sides opposite to node i and at all nodes j for which $j \neq i$.
5. The wedges associated with Q_m form a basis for linear functions over it. For the polygon Q_m , there must be at least m nodes. For these to suffice, one should have(cf. G.15 of [6]):

$$\begin{aligned} \sum_{i=1}^m N_i(x, y) &= 1 \\ \sum_{i=1}^m x_i N_i(x, y) &= x \\ \sum_{i=1}^m y_i N_i(x, y) &= y \end{aligned}$$

6. Each wedge function and all its derivatives are continuous within the polygon for which the wedge is a basis function.

The following concept of interpolation is also required for the analysis:

Definition 4 [16] Let $W: W_1, W_2, \dots, W_n$ be a system of real valued functions on a set A , let x_1, x_2, \dots, x_n be given distinct points of A and let c_1, c_2, \dots, c_n be given real numbers. The polynomial $N(x) = \sum_{i=1}^n c_i W_i$ is said to **interpolate** the values c_k if $N(x_k) = c_k, k = 1, \dots, n$.

3 CONSTRUCTION OF WEDGE FUNCTIONS

For a convex m -gon, a rational wedge function for degree k approximation, corresponding to the i^{th} node has been constructed as follows[21]:

$$N_i^k(x, y) = \frac{P^{m+k-3}(x, y)}{P^{m-3}(x, y)}$$

where $P^{(m)}(x, y)$ is a polynomial of degree m , in x and y such that it satisfies the wedge properties. For this, the numerator is the product of all the edges other than the ones which are adjacent to the given node, with the multiple of a constant K_i to normalize it on the i^{th} node(see [21], [22]). The denominator of the wedge function is desired to be such that the wedge function remains a bi-variate polynomial of degree k on the adjacent sides.

3.1 Computation of denominator of the rational form of wedges

Using properties of wedge functions, now the denominator can be determined.

Referring relation (6), it is quite clear that in case of the quadrilateral element, numerator and denominator of N_i, s

reduce to degree two and degree one respectively. In view of relation (3), the denominator is identically equal to the sum of the terms of the numerator. For the validity of this identity, it is mandatory to equate the coefficients of the terms of degree higher than one to zero. In this process, a system of three linear equations in four unknowns, $K_i, (i=1, \dots, 4)$ is obtained. Without loss of generality $K_1 (\neq 0)$ may be normalized to one.

Finally, a system of three equations in three unknowns K_2, K_3 and K_4 , is left which can be expressed as

$$AK = M$$

where A is a 3×3 matrix and K is the column vector $[K_2 \ K_3 \ K_4]^T$. As a first step, invertibility of the matrix A is established, to assure the existence and uniqueness of the solution of the system.

The value of K_i s thus obtained have been substituted in the left hand side of (3) to determine linear form of the denominator function.

3.2 Existence of the denominator function

The main result, which establishes the existence and uniqueness of solution of the system of linear equations(7) under certain geometric constraints on the element under consideration, is proved in this part of the paper. Let Ω be the quadrilateral discretization of the domain \mathbb{R}^2 (see Figure 2) and Q_4^1 be an arbitrary quadrilateral element of Ω .

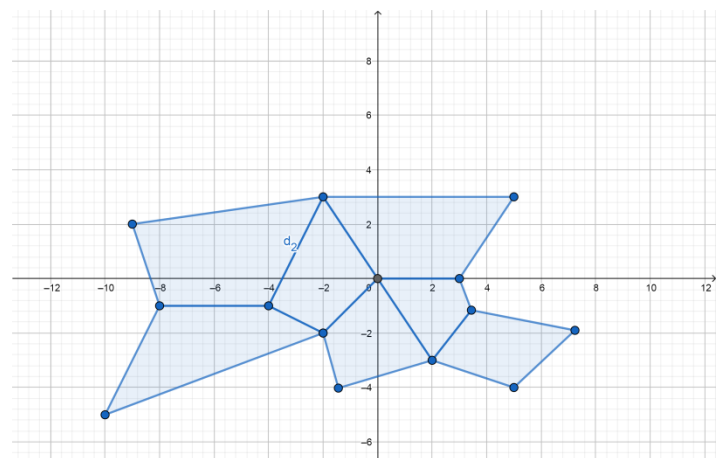


Figure 2: Quadrilateral discretization, Ω

Theorem 1 The denominator of the wedge functions corresponding to the element $Q_4^1 \in \Omega$ exists and is unique if the following hold:

- (a) No two opposite sides of Q_4^1 are parallel.
- (b) No three vertices of Q_4^1 are collinear.

Proof of Theorem 1

Consider the quadrilateral $Q_4^1 = (a, b, c, d) \in \Omega$ (cf. Figure 3) having cartesian coordinates $a = (s_1, t_1), b = (s_2, t_2), c = (s_3, t_3)$ and $d = (s_4, t_4)$ respectively.

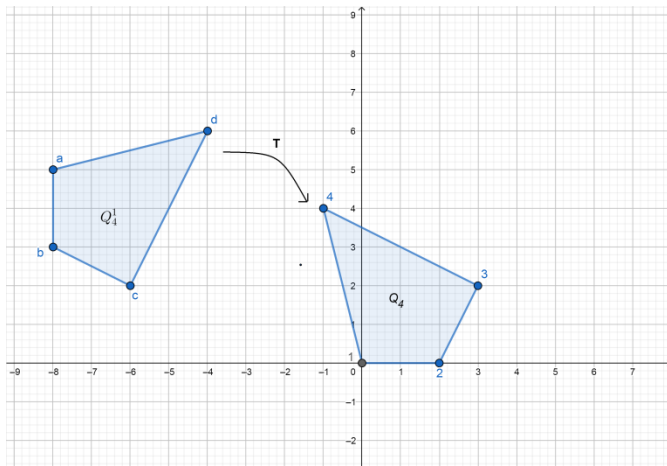


Figure 3: Transformation of Quadrilateral Element

By applying the transformations, viz. rotation and translation, the quadrilateral Q_1 has been transformed into Q_4 with vertices (1,2,3,4) having Cartesian coordinates (0,0), $(x_2, 0)$, (x_3, y_3) and (x_4, y_4) respectively (cf. Figure 3). This transformation does not cause any loss to the generality.

Using notation of linear form described in (1), $l_{i,s}$ ($i = 1, \dots, 4$) have been computed

$$\begin{aligned}
 l_1 &= (4,1) \cong xy_4 - yx_4, \quad l_2 = (1,2) \cong y, \quad l_3 = \\
 &= (2,3) \cong y_3x - (x_3 - x_2)y - x_2y_3, \\
 l_4 &= (3,4) \cong (y_3 - y_4)x - (x_3 - x_4)y - x_4y_3 + \\
 &+ x_3y_4 \quad (8)
 \end{aligned}$$

The value of numerator of the wedge function N_i corresponding to the i^{th} node can be computed. In view of the wedge properties, N_i should vanish on the opposite sides of the node i and turns out to be linear on the adjacent sides. Denoting the numerator of N_i by Num_i , we have

$$\begin{aligned}
 Num_1 &= K_1 l_3 l_4, \quad Num_2 = K_2 l_4 l_1 \\
 Num_3 &= K_3 l_1 l_2, \quad Num_4 = K_4 l_2 l_3
 \end{aligned}$$

Applying the technique, described in [8], consider $\sum_{i=1}^4 Num_i$ and equate the coefficients of terms $x^i y^j$ ($i + j = 2, i, j = 0, 1, 2$) to zero. Thus, the following system of three linear equations in three unknowns K_2, K_3 and K_4 is obtained for $K_1 = 1$.

$$\begin{aligned}
 K_2 y_3 y_4 - K_2 y_4^2 &= -(-y_3^2 + y_3 y_4) \\
 -K_4 x_2^2 + K_4 x_2 x_3 - K_3 x_2 x_4 + K_2 x_3 x_4 - \\
 K_2 x_4^2 &+ = -(x_2 x_3 - x_3^2 - x_2 x_4 + x_3 x_4) \\
 -K_4 x_2 y_3 + K_3 x_2 y_4 - K_2 x_4 y_3 - K_2 x_3 y_4 + \\
 2K_2 x_4 y_4 &= -(2x_3 y_3 - x_4 y_3 + x_2 y_4 - \\
 &+ x_3 y_4 - x_2 y_3)
 \end{aligned}$$

It may be verified easily that the coefficient matrix of the system (10) is invertible if

- $y_4 \neq 0$
- $y_3 \neq y_4$
- $x_4 \neq \frac{y_4(x_3 - x_2)}{y_3}$

These conditions in turn imply assertions (a) and (b). Hence, the proof of Theorem 1 is completed. Substituting the value of K_i s, $i=1,2,3,4$, obtained on solving the system (10), the denominator of the wedge functions yields directly.

4 EXISTENCE OF DENOMINATOR AND COMPUTATION OF APPROXIMATION

Algorithm of a program in Mathematica which supports the claim has been written here. In case of existence of the denominator function, it determines the wedge function. Further, it executes the task of computing approximant to the data provided and also generates the approximate surface. Algorithm illustrating this program is as follows:

Algorithm Start

Step1:

Step2:

Declare variables $n, x[4], y[4], nodes[4], k[4], R[3][3], K.$

Step3:

Repeat steps until $(i \leq 4); i = 1.$

Let $[ls(x,y)[i]] = call\ function\ nodesToLines(nodes(x[i],y[i]),x,y).$

Step4:

Repeat steps until $(i \leq 4); i = 1.$

$m := Mod[i+2,4];$

If $[m == 0, num[i] := k[[i]]*ls[[i+2]]*ls[[1]], num[i] := k[[i]]*ls[[m]]*ls[[m+1]]$

Step5:

Let $deno(x,y) = deno(x,y) + call$

$Expand([num(x,y),[i]]); i = 1; i \leq 4.$

Step6:

Let $L = \{call\ Coefficient(deno(x,y), pow(x,2)), call\ Coefficient(deno(x,y), pow(y,2)), call\ Coefficient(deno(x,y), pow(x*y,1))\}.$

Step7:

Repeat $i=1, i \leq 3.$

Repeat $j=2, j \leq 4.$

$R[i][j] = call\ Coefficient(k[j]\ in\ L[[i]]).$

Step 8:

(9)

If $call\ Det(R) = 0$ then

exit.

Else

Call function Solve for $L[[i]], i=1$ to 3 equated to 0

with respect to $k_2, k_3, k_4.$

Repeat until $i \leq 4; i \neq 1$

Let $[N(x,y), [i]] = [num(x,y), [i]] / deno(x,y).$

Input function $f(x,y)$, to be approximated.

Repeat until $i \leq 4; i = 1.$

$a[i] = f(nodes(x[i],y[i])).$

The approximant $Ap(x,y) = Ap(x,y) + call$

$Expand(a[i]*N[i](x,y)); i = 1; i \leq 4.$

Plot $Ap(x,y)$ and $f(x,y)$ over the considered quadrilateral.

Step9:

Stop.

User defined functions

Intercept	$Intercept((x_1, y_1), (x_2, y_2)) = (y_2 - y_1, x_1 - x_2, -x_1 * y_2 + y_1 * x_2)$
nodesToIntercepts	$nodesToIntercepts(p) = Intercepts(p, RotateRight(p))$
nodesToLines	$nodesToLines(p, (x, y)) = -Dot(nodesToIntercept(p), (x, y, 1))$
ls	$ls = nodesToLines(nodes, (x, y))$

5 Numerical Examples

In support of the assertions of this paper, following numerical examples have been illustrated:

Example 1(When assertions (a) and (b) are true) Consider the quadrilateral $Q_4 = (1,2,3,4)$ with Cartesian coordinates of the vertices as (cf. Figure4) 1=(0,0), 2=(2,0), 3=(1.5,1) and 4=(0.5,0.75). Then, referring relation(7) the matrix **A** will be

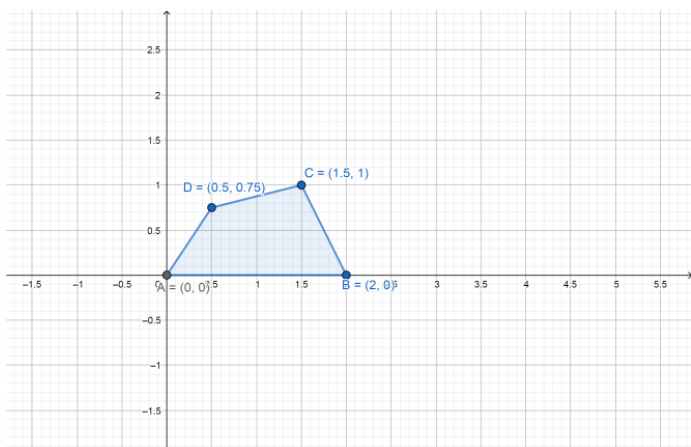


Figure 4: Quadrilateral Element

$$A = \begin{bmatrix} -7 & 3 & -2 \\ 8 & 2 & -2 \\ 1 & -1 & -1 \\ 2 & 3 & 0 \\ 3 & 0 & 0 \\ 16 & 0 & 0 \end{bmatrix}$$

Clearly, determinant of **A** is nonzero. Hence, the denominator function $D(x,y)$ exists uniquely and

$$D(x, y) = \frac{1}{16}(8x - 17y + 20)$$

the wedge function N_i , corresponding to the i^{th} node, ($i = 1,2,3,4$) will be:

$$N_1 = -\frac{4x^2 - 14xy - 8y^2 + 2x + 37y - 20}{8x - 17y + 20}$$

$$N_2 = \frac{12x^2 - 56xy + 32y^2 + 30x - 20y}{3(8x - 17y + 20)}$$

$$N_3 = \frac{18xy - 12y^2}{8x - 17y + 20}$$

$$N_4 = -\frac{4(10xy + 5y^2 - 20y)}{3(8x - 17y + 20)}$$

Case 1 Let $f(x, y) = 2x + y$ is to be approximated, it may be verified easily that the resulting approximant is $f(x,y)$ itself.

Case 2 If any non linear function is to be approximated, say $f(x, y) = \sin(xy)$ then the approximant is $\phi(x, y)$ (say) = $\frac{[54\sin^3\frac{3}{2} - 40\sin^3\frac{3}{8}]xy - [36\sin^3\frac{3}{2} + 20\sin^3\frac{3}{8}]y^2 + 80\sin^3\frac{3}{8}y}{3(8x - 17y + 20)}$. Figure 5, represents $f(x,y)$ over the element Q_4 and its approximation is displayed in Figure 6.

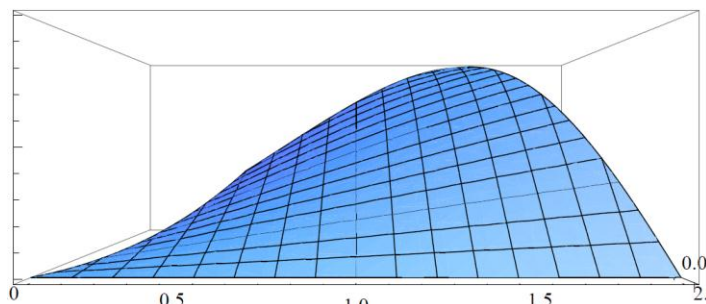


Figure 5: Sin(xy)

Figure 6: Approximant to Sin(xy)

Example 2 (When assertions (a) and (b) of Theorem 1 are violated) Consider the quadrilateral $Q_4 = (1,2,3,4)$ such that 1=(0,0), 2=(2,0), 3=(1,1) and 4=(0.5,1) (cf Figure 7).

Figure 7: Element with parallel sides

Then, the matrix **A**(cf. relation (7)) will be

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & -1 & -2 \\ 1 & -1 & -2 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $|A| = 0$, obviously the matrix is not invertible and thus, the denominator function $D(x,y)$ does not exist uniquely.

6 CONCLUSION

Dasgupta's contribution in computing the wedge functions makes the process more viable to be solved using computers, as it sets the method free from geometry. From application point of view, it may be noted that this approach is more preferable than previous ones. Moreover, it may be easily implemented by using software based on Mathematics, in particular Mathematica. Identifying the conditions of existence and uniqueness of the wedge functions makes the process more plausible.

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APPENDIX

Example of a Quadrilateral:Mathematica program implementing the algorithm in section 4

```

n = 4;
(*Coordinates of the vertices*)
xy = {{0, 2, 3/2, 1/2}, {0, 0, 1, 3/4}}
nodes = Transpose[xy]

(*Linear forms*)
Intercept[{x1_, y1_}, {x2_, y2_}] := {y2 - y1, x1 - x2, -x1*y2 + y1*x2}

nodesToIntercepts[p_?MatrixQ]:= Thread[Intercept[p, RotateRight[p]]]

nodesToLines[p_{{x_,y_}}]:=-Dot[nodesToIntercepts[p], {x, y, 1}]

ls = nodesToLines[nodes, {x, y}]

(*Figure representing the quadrilateral*)
Graphics[Polygon[nodes]]

(*Weights*)
k = {k1, k2, k3, k4};

(*Construction of wedge functions*)
For[i = 1, i < 5, i++, m := Mod[i + 2, 4];
    If[m == 0,
        num[i] := k[[i]]*ls[[i + 2]]*ls[[1]],
        num[i] := k[[i]]*ls[[m]]*ls[[m + 1]]
    ]
]

deno = Expand[Sum[num[i], {i, 1, 4}]]

L={Coefficient[deno,x,2],Coefficient[deno,y,2],Coefficient[deno, x*y, 1]}

ClearAll[R]

R:={{Coefficient[L[1], k2], Coefficient[L[1], k3], Coefficient[L[1], k4]},
{Coefficient[L[2],k2],Coefficient[L[2],k3], Coefficient[L[2], k4]},
{Coefficient[L[3], k2], Coefficient[L[3], k3], Coefficient[L[3], k4]}}}

K := Det[R]

(*verification of the condition of existence*)
If[K == 0, NotebookClose[], Print[Entries are valid]]

k1 := 1;

Table[L[[i]] == 0, {i, 3}];

Solve[%, {k2, k3, k4}]
For[i = 1, i < 5, i++,

```

```

(Subscript[N, i^1])[{x, y}] = Thread[Expand[num[i]/deno] /. %]
ClearAll[f, x, y];

```

(*Function to be approximated*)

```

f[{x_, y_}] = Sin[x * y]
For[i = 1, i < 5, i++, a[i] = f[nodes[[i]]]]
ClearAll[A1];

```

(*Approximant*)

```

A1=
Thread[ApartThread[Apart[Sum[a[i]*(Subscript[N, i^1])[{x, y}], {i, 1, 4}]]]

```

(*Plot of the approximant*)

```

Plot3D[A1, {x, 0, 2}, {y, 0, 1}, ViewPoint -> Front,
RegionFunction -> Function[{x, y}, ls[[1]] ≤ 0 && ls[[2]] ≤ 0 && ls[[3]] ≤ 0 && ls[[4]] ≤ 0]

```

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