Flow Of Heat Through A Plane Wall, And Through A Finite Fin Insulated At The Tip

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Abstract: Generally, the general heat flow equations describing the flow of heat through a Plane wall, and through a finite fin insulated at the tip are analyzed by calculus method. Laplace transformation has been applied to analyze boundary value problems arising in the areas of engineering, science and technology. The flow of heat through a Plane wall with equally scattered heat sources between its faces, and through a finite fin with one end attached to a thermal source maintained at fixed temperature and other end (i.e. tip) insulated can be obtained easily by analyzing the general heat flow equations via the Laplace transform method and it will come out to be very effective mathematical tool applied to analyze the general heat flow equations. This paper presents the application of Laplace transformation to the general heat flow equations for obtaining the flow of heat through a Plane wall with equally scattered heat sources between its faces, and through a finite fin with one end attached to a thermal source maintained at fixed temperature and other end (i.e. tip) insulated.

Index Terms: Heat Flow equations, Plane Wall, Finite Fin and Laplace Transformation.

Introduction:

Heat is one of the forms of energy which flows in the direction of decreasing temperature by virtue of temperature gradient irrespective of the amounts of heat possessed by the bodies in contact. In many practical situations, heat is generated at a uniform rate itself within the conducting medium and is dissipated from its surface to the surroundings. The rise in temperature due to heat generated within the conducting medium leads to the failure of the system and so it has to be controlled. The flow of heat through a Plane wall with equally scattered heat sources between its parallel faces, and through a finite fin with one end attached to thermal source and other end (i.e. tip) insulated assume significant importance in the construction of thermal systems. The rate of heat flow across the material of a solid block between its parallel faces is given by the equation $H = -KA\frac{dt}{dy}$, known as Fourier's law, where K is the thermal conductivity, Ais the area of the cross-section of the material of solid block and $\frac{dt}{dy}$ is the temperature gradient. The negative sign shows that the temperature decreases as the distance increases in the direction of heat flow [1-4].

Basic Definitions:

The Laplace transform of g(y) is defined as $L[g(y)] = \int_0^\infty e^{-py} g(y)dy = G(p)$, provided that the integral is convergent for some value of p, a parameter which may be a real or complex. If we write [g(y)] = G (p), then L^{-1}[G (p)] = g(y) is the inverse Laplace transform of G (p) [5-7].

Laplace Transformation of derivatives:

The Laplace transformation of derivatives of g(y)are given by [5-7]

$$L\{g'(y)\} = pG(p) - g(0),$$

$$L\{g''(y)\} = p^2G(p) - pg(0) - g'(0),$$

$$L\{g'''(y)\} = p^3G(p) - p^2g(0) - pg'(0) - g''(0),$$

and so on.

Material and Method:

A. Heat Flow through a Plane wall

The general heat flow equation through a plane wall with uniformly distributed heat sources between its faces is given by [1]

$$\ddot{t}(y) + \frac{Q}{K} = 0 \;\ldots\ldots\ldots\ldots (1) \quad \dot{} \equiv \frac{d}{dy}$$

Where Q is the uniform volumetric heat generation within the plane wall, t is the temperature distribution and K is constant thermal conductivity.

On taking the Laplace Transform [4, 6] of equation (1), we get

$$L[\ddot{t}(y)] + \frac{Q}{K} L[1] = 0$$

Or

$$p^2\bar{t}(p) - pt(0) - \dot{t}(0) + \frac{Q}{K}\frac{1}{p} = 0 \;\ldots\ldots\ldots\ldots (2)$$

Where $\bar{t}(p)$ is the Laplace transform of t(y).

Case I: Heat flow through a Plane wall when both the surfaces of the wall are maintained at same temperature

In this case, the boundary conditions [1] are written as $t(0) = t(d) = T$.

Since t(0) = T, therefore, equation (2) becomes,

$$p^2\bar{t}(p) - pT - \dot{t}(0) + \frac{Q}{K}\frac{1}{p} = 0$$

Or

$$p^2\bar{t}(p) - pT + \frac{Q}{K}\frac{1}{p} = \dot{t}(0) \;\ldots\ldots. (3)$$

As $\dot{t}(0)$ is a constant, therefore, substituting $\dot{t}(0) = C$, equation (3) can be rewritten as

$$p^2\bar{t}(p) - pT + \frac{Q}{K}\frac{1}{p} = C$$

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Or $\bar{t}(p) = \frac{C}{p^2} + \frac{T}{p} - \frac{Q}{K}\frac{1}{p^3}$ .... (4)

Taking inverse Laplace transform [4, 6] of equation (4), we get

$t(y) = Cy + T - \frac{Q}{K}\frac{y^2}{2}$ .... (5)

To find the value of C, applying the boundary condition, t(d) = T, we get

$$T = Cd + T - \frac{Q}{K}\frac{d^2}{2}$$

Or

$$C = \frac{Q}{K}\frac{d}{2}$$

Put the value of C in equation (5), we get

$$t(y) = \frac{Q}{K}\frac{d}{2}y + T - \frac{Q}{K}\frac{y^2}{2}$$

Or

$$t(y) = \frac{Q}{2K}y(d-y) + T \;\dots\dots\; (6)$$

To find the maximum value of temperature and its location within the wall, differentiating equation (6) w.r.t. y, we get

$$\dot{t}(y) = \frac{Q}{K}\frac{d}{2} - \frac{Q}{K}y$$

Or

$$\dot{t}(y) = \frac{Q}{2K}(d - 2y)$$

Putting $\dot{t}(y) = 0$, we get

$y = \frac{d}{2}$ .... (7)

Thus the temperature is maximum at the mid of the wall. This leads to the transmission of heat towards both the surfaces of the wall. The maximum temperature at the mid of the wall is obtained by substituting equation (7) in equation (6) and is given by

$t_{maximum} = \frac{Q}{8K}d^2 + T$ .... (8)

The flow of heat towards each surface of the wall is given by

$$H_{y=0\ or\ y=d} = -KA\{\dot{t}(y)\}_{y=0\ or\ y=d}$$

Or

$$H_{y=0\ or\ y=d} = \frac{1}{2}AdQ \;\dots\dots\dots.(9)$$

The flow of heat towards both the surfaces of the wall is given by

$H_{total} = H_{y=0} + H_{y=d} = AdQ$ ..... (10)

This shows that the flow of heat towards both the surfaces of the wall is dependent of internal generation of heat Q. If there is no internal heat generation i.e. Q = 0, then $H_{total} = 0$.

The heat conducted to the wall surface is finally dissipated to the surrounding atmosphere at temperature T_s by convection. For each surface, we have [1, 2]

$$\frac{1}{2}AdQ = \sigma A(T - T_s),$$

Or

$T = T_s + \frac{dQ}{2\sigma}$.......... (11)

Substituting equation (11) in equation (6), we get the temperature distribution in terms of temperature T_s of surrounding atmosphere as

$t(y) = \frac{Q}{2K}y(d-y) + T_s + \frac{dQ}{2\sigma}$ ...... (12)

Case II: Heat flow through a Plane wall when both the surfaces are maintained at different temperatures

In this case, the boundary conditions [1] are written as $t(0) = \tau_1$, $t(d) = \tau_2$

Since $t(0) = \tau_1$, therefore, equation (2) becomes,

$$p^2\bar{t}(p) - p\tau_1 - \dot{t}(0) + \frac{Q}{K}\frac{1}{p} = 0$$

Or

$$p^2\bar{t}(p) - p\tau_1 + \frac{Q}{K}\frac{1}{p} = \dot{t}(0)$$

As $\dot{t}(0)$ is a constant, therefore, substituting $\dot{t}(0) = \mu$, a constant, we can write

$$p^2\bar{t}(p) - p\tau_1 + \frac{Q}{K}\frac{1}{p} = \mu$$

Or

$$\bar{t}(p) = \frac{\mu}{p^2} + \frac{T}{p} - \frac{Q}{K}\frac{1}{p^3} \dots\dots(13)$$

Taking inverse Laplace transform [4, 6] of equation (13), we get

$t(y) = \mu y + \tau_1 - \frac{Q}{K}\frac{y^2}{2}$ ...... (14)

To find the value of μ, applying the boundary condition, $t(d) = \tau_2$, we get

$$\tau_2 = \mu d + \tau_1 - \frac{Q}{K}\frac{d^2}{2}$$

Or

$$\mu = \frac{Q}{K}\frac{d}{2} + \frac{\tau_2 - \tau_1}{d}$$

Put the value of μ in equation (14), we get

$$t(y) = \left(\frac{Q}{K}\frac{d}{2} + \frac{\tau_2 - \tau_1}{d}\right)y + \tau_1 - \frac{Q}{K}\frac{y^2}{2}$$

$t(y) = y\left[\frac{Q}{2K}(d-y) + \frac{\tau_2 - \tau_1}{d}\right] + \tau_1$....... (15)

To find the maximum value of temperature and its location within the wall, differentiating equation (15) w.r.t. y, we get

$\dot{t}(y)=\frac{Q}{K}\frac{d}{2}+\frac{\tau_2-\tau_1}{d}-\frac{Q}{K}y$

Putting $\dot{t}(y)=0$, we get

$y=\frac{d}{2}+\frac{(\tau_2-\tau_1)K}{dQ}$ …….. (16)

This shows that the maximum temperature occurs within the wall.

Put the value of y in equation (15), the maximum temperature is given by

$$t_{maximum}=\frac{Q}{8K}d^2+\frac{\tau_2-\tau_1}{2}\left(1+\frac{K}{d^2Q}\right)+\tau_1 \text{ …… (17)}$$

The rate of flow of heat towards any surface of the wall is given by

H = $-KA\,\dot{t}(y)$

Or

$H=-KA\left[\frac{Q}{K}\frac{d}{2}+\frac{\tau_2-\tau_1}{d}-\frac{Q}{K}y\right]$….. (18)

The flow of heat towards the face located at y = 0 is given by

$$H_{y=0}=-KA\,\dot{t}(0)$$

Or

$$H_{y=0}=-KA\left[\frac{Q}{K}\frac{d}{2}+\frac{\tau_2-\tau_1}{d}\right]$$

Or

$H_{y=0}=KA\left[\frac{\tau_1-\tau_2}{d}-\frac{Q}{K}\frac{d}{2}\right]$ ……. (19)

The flow of heat towards the face located at y = d is given by

$$H_{y=d}=-KA\,\dot{\emptyset}(d)$$

Or

$$H_{y=d}=-KA\left[\frac{Q}{K}\frac{d}{2}+\frac{\tau_2-\tau_1}{d}-\frac{Q}{K}d\right]$$

Or

$$H_{y=d}=KA\left[\frac{Q}{K}\frac{d}{2}-\frac{\tau_2-\tau_1}{d}\right]$$

Or

$H_{y=d}=KA\left[\frac{\tau_1-\tau_2}{d}+\frac{Q}{K}\frac{d}{2}\right]$ ……… (20)

The flow of heat towards both the surfaces of the wall is given by

$$H_{total}=H_{y=0}+H_{y=d}$$

Or

$H_{total}=2\,KA\,\frac{\tau_1-\tau_2}{d}$ …….. (21)

This shows that the flow of heat towards both the surfaces of the wall is independent of internal generation of heat Q.

B. Flow of Heat through a finite fin insulated at the tip

The general heat flow equation through the finite fin is given by [1, 4]

$\ddot{t}(y)-\frac{\sigma P}{KA}[t(y)-T_s]=0$ ……… (22)

Where T_s is the temperature of the surrounding of the fin and is kept constant, A and P are uniform area of cross-section and perimeter of the finite fin, K is thermal conductivity of material of finite fin, σ represents the coefficient of heat transfer by convectionand t is temperature distribution.
To simplify the equation (22), let us substitute
$\left(\frac{\sigma P}{KA}\right)^{\frac{1}{2}}=\beta$……. (23)
$and\ define\ \ t(y)-T_s=\tau(y)$ …. (24)
Where $\tau(y)$ is called excess temperature at length y of the rod, then $\frac{d}{dy}[t(y)-T_s]=\dot{\tau}(y)$.
As T_s is constant, therefore, we can write $\ddot{t}(y)=\ddot{\tau}(y)$,
Therefore, equation (22) can be rewritten as
$\ddot{\tau}(y)-\beta^2\,\tau(y)=0$ ……….. (25)
Where β is constant provided that σ is constant over the entire surface of finite fin and K is constant within the range of temperature considered.

Solution of the differential equation:
To solve equation (25) for a finite fin with one end at y = 0 attached to a heat source maintained at fixed temperature T and the other end at y = L i.e. tip insulated, the relevant boundary conditions as follows [1]:as the end at y = 0 is maintained at fixed temperature T, therefore, t(y = 0) = T, and as no heat is transferred from the end at y = L (i.e. tip), therefore, $\dot{t}(y=L)=0$.
In terms of excess temperature $\tau(y)$, we can write, t(y = 0) $-T_s$ = T- T_s or $\tau(0)=\tau_1$ and $\dot{\tau}(y=L)=0$
Taking Laplace Transform [4, 6] of equation (25), we get
$L[\ddot{\tau}(y)]-\beta^2L[\tau(y)]=0$
This equation gives
$p^2\bar{\tau}(p)-p\tau(0)-\dot{\tau}(0)-\beta^2\bar{\tau}(p)=0$... (26)
Applying boundary condition: $\tau(0)=\tau_1$, equation (26) becomes,
$p^2\bar{\tau}(p)-p\tau_1-\dot{\tau}(0)-\beta^2\bar{\tau}(p)=0$
Or
$p^2\bar{\tau}(p)-\beta^2\bar{\tau}(p)=\dot{\tau}(0)+p\tau_1$….. (27)
In this equation, $\dot{\tau}(0)$is some constant so let us substitute $\dot{\tau}(0)=\varepsilon, a\ constant$, then equation (27) becomes
$p^2\bar{\tau}(p)-\beta^2\bar{\tau}(p)=\varepsilon+p\tau_1$
Or
$\bar{\tau}(p)=\frac{\varepsilon}{(p^2-\beta^2)}+\frac{p\tau_1}{(p^2-\beta^2)}$……….. (28)
Taking inverse Laplace transform [4, 6] of equation (28), we get
$\tau(y)=\frac{\varepsilon}{\beta}\sin h\beta y+\tau_1\cos h\beta y$… (29)

Determination of the constantε:

To find the value of constantε, applying boundary condition:$\dot{\tau}(L) = 0$, equation (29) provides,

$0 = \frac{\varepsilon}{\beta}\beta \cos h\beta L + \tau_1\beta \sin h\beta L$

Upon rearranging and simplification of above equation, we get

$\varepsilon = \frac{-\beta\tau_1 \sin h\beta L}{\cos h\beta L}$...... (30)

Substitute the value of ε from equation (30) in equation (29), we get

$$\tau(y) = \frac{-\tau_1 \sin h\beta L}{\cos h\beta L}\sin h\beta y + \tau_1 \cos h\beta y$$

Or

$$\tau(y) = \frac{-\tau_1 \sin h\beta L \ \sin h\beta y + \tau_1 \cos h\beta y \ \cos h\beta L}{\cos h\beta L}$$

Or

$\tau(y) = \frac{\tau_1 \cos h\beta(L-y)}{\cos h\beta L}$.... (31)

Using equation (24) in equation (31), we can write

$$t(y) - T_s = \frac{(T - T_s)\cos h\beta(L-y)}{\cos h\beta L}$$

Or

$$t(y) = T_s + \frac{(T - T_s)\cos h\beta(L-y)}{\cos h\beta L}$$

Put the value of β from equation (23), we get

$t(y) = T_s + \frac{(T - T_s)\cos h\sqrt{\frac{\sigma P}{KA}}(L-y)}{\cos h\sqrt{\frac{\sigma P}{KA}}L}$ ...(32)

The equation (32) provides the distribution of temperature along the length of the finite fin.

The rate of heat flow from the finite fin insulated at the tip can be obtained [7] by using the equation

$H = -K A\dot{t}(0)$...... (33)

On differentiating equation (32) with respect to y and then simplifying, we get

$[\dot{t}(y)]_{y=0} = \frac{-\beta\tau_1 \sin h\beta L}{\cos h\beta L}$..... (34)

Using equation (34) in equation (33), we get

$H = K A\frac{\beta\tau_1 \sin h\beta L}{\cos h\beta L}$

Or

$$H = K A\beta(T - T_s)\tan h\beta L$$

Or

$$H = K A\sqrt{\frac{\sigma P}{KA}}(T - T_s)\tan h\sqrt{\frac{\sigma P}{KA}}L$$

Or

$H = \sqrt{KA\sigma P}(T - T_s)\tan h\sqrt{\frac{\sigma P}{KA}}L$ ..... (35)

This equation (35) provides expression for the rate of heat flow from the finite fin insulated at the tip into its surroundings and confirms that the rate of flow of heat can be increased by increasing its surface area.

Result and Discussion

In this paper, we have successfully obtained the rate of flow of heat through a Plane wall equally scattered heat sources between its faces, and through a finite fin insulated at the tip. When both the surfaces of the wall are maintained at same temperature, the temperature is maximum at the mid of the wall which leads to the transmission of heat towards both the surfaces of the wall. But there is no flow of heat towards surfaces of the wall if there is no internal heat generation. When both the surfaces are maintained at different temperatures, the maximum temperature occurs within the wall and the flow of heat towards both the surfaces of the wall is independent of internal generation of heat. The temperature of the finite fin insulated at the tip decreases with increase in distance from the thermal source. The rate of flow of heat from the finite fin insulated at the tip can be increased by increasing its surface area.

Conclusion

We concluded that the Laplace Transformation method presented in this paper for obtaining the rate of flow of heat through a Plane wall with equally scattered heat sources, and through a finite fin insulated at the tip is an effective method and with its ease of application, the other boundary value problems can also be analyzed easily in engineering, science and technology.

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