Discrete Wavelet Transforms Of Haar's Wavelet

Bahram Dastourian, Elias Dastourian, Shahram Dastourian, Omid Mahnaie

Abstract: Wavelet play an important role not only in the theoretic but also in many kinds of applications, and have been widely applied in signal processing, sampling, coding and communications, filter bank theory, system modeling, and so on. This paper focus on the Haar's wavelet. We discuss on some command of Haar's wavelet with its signal by MATLAB programming. The base of this study followed from multiresolution analysis.

Keyword: approximation; detail; filter; Haar's wavelet; MATLAB programming, multiresolution analysis.

1 INTRODUCTION

Wavelets are a relatively recent development in applied mathematics. Their name itself was coined approximately a decade ago (Morlet, Arens, Fourgeau, and Giard [11], Morlet [10], Grossmann, Morlet [3] and Mallat[9]); in recently years interest in them has grown at an explosive rate. There are several reasons for their present success. On the one hand, the concept of wavelets can be viewed as a synthesis of ideas which originated during the last twenty or thirty years in engineering (subband coding), physics (coherent states, renormalization group), and pure mathematics (study of Calderon-Zygmund operators). As a consequence of these interdisciplinary origins, wavelets appeal to scientists and engineers of many different backgrounds. On the other hand, wavelets are a fairly simple mathematical tool with a great variety of possible applications. Already they have led to exciting applications in signal analysis (sound, images) (some early references are Kronland-Martinet, Morlet and Grossmann [4], Mallat [6], [8]; more recent references are given later) and numerical analysis (fast algorithms for integral transforms in Beylkin, Coifman, and Rokhlin [1]); many other applications are being studied. This wide applicability also contributes to the interest they generate. P.J. Wood in [15] developed the wavelet in the Hilbert C*-module case. Also J.A. Packer and M.A. Rieffel [12] and Z. Liu, X. Mu and G. Wu [5] studies this concept in $L^2(R^d)$.

2 PRELIMINARIES

Detailed In this section we give some Definition and Theorem that we need for the results. For more details about following concepts and proof of Theorems one can see [2, 7, 14]. In the sequel, we denoted the integer, real and complex numbers by Z, R and C, respectively.

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Definition 4. Let H be a Hilbert space

i) A family $(\boldsymbol{x_j})_{j\in J} \subset \boldsymbol{H}$ is an orthonormal system if

$$\forall i,j \in J \ : \ \langle x_i, x_j\rangle = \begin{cases} 1 & i=j, \\ 0 & i\neq j. \end{cases}$$

ii) A family $(\boldsymbol{x_j})_{j\in J} \subset \boldsymbol{H}$ is total if for all $\boldsymbol{x} \in \boldsymbol{H}$ the following implication holds

$$\left(\forall j \in J : \langle x, x_j\rangle = 0\right) \Rightarrow x = 0.$$

iii) A total orthonormal system is called orthonormal basis.

Definition 5. Let H and H_0 be Hilbert spaces and $T: H \to H_0$ be a bounded linear operator. Then T is an (orthogonal) projection if $T = T^* = T^2$.

Definition 6. Let (X, β, μ) be a measure space.

i) $f: X \to R^n$ is called measurable if $f^{-1}(B) \in \beta$ for all Borel sets B. The space of measurable functions $X \to C$ is denoted by $M(X)$.

ii) For $1 \le p < \infty$, the space $L^p(X,\mu) = L^p(X)$ is defined by

$$L^p(X) = \{f \in M(X) : \int_X |f(x)|^p \, d\mu(x) < \infty$$

endowed with the L^p-norm

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

For $\boldsymbol{p} = \infty$ we let

$$L^\infty(X) = \{f \in M(X) : \ \|f\|_\infty < \infty\}$$

Where

$$\|f\|_\infty = \inf\{\alpha : |f(x)| < \alpha \ \ a.e.\}.$$

$L^p(X)$ is complete with respect to the metric $d(f,g) = \|f-g\|_p$; this makes $L^p(X)$ a Banach space. $L^2(X)$ is a Hilbert space, with scalar product

$$\langle f, g\rangle = \int_X f(x)\,\overline{g(x)}\,d\mu(x).$$

For more details one can saw [13].

Definition 7. The Fourier transform of a function $f \in L^1(R)$ is $\hat{f}(\omega) = \int f(x) e^{-2\pi i w x}\, d\mu(x)$, for $\omega \in R$. If $\hat{f} \in L^1(R)$ then f is continuous with $f(x) = \int \hat{f}(w)e^{2\pi iwx}\, d\mu(w)$. The Fourier transform is the linear operator $F: f \to \hat{f}$.

Definition 8. Let $\omega, x, a \in R$, with $a \neq 0$. We define

$$(T_x f)(y) = f(y - x), \qquad Translation\ operator$$
$$(D_a f)(y) = |a|^{-1/2} f(a^{-1}y), \qquad Dilation\ operator$$
$$(M_\omega f)(y) = e^{-2\pi i\omega y} f(y), \quad Modulation\ operator$$

All these operators are easily checked to be unitary on $L^2(R)$. Dilation and translation have a simple geometrical meaning. T_x amounts to shifting a graph of a function by x, whereas D_a is a dilation of the graph by a along the x-axis, with a dilation by$|a|^{-1/2}$ along the y-axis: If $supp f = [0, 1]$, then $supp(D_a f) = [0, a]$ and $supp(T_x f) = [x, x + 1]$, where $supp(g) = \overline{\{x : g(x) \neq 0\}}$.

Definition 9. If f, g are complex-valued functions defined on R, their convolution $f * g$ is the function $(f * g)(x) = \int f(x - y)g(y)d\mu(y)$, provided that the integral exists.

Definition ???. Let $m: R \to C$ be a bounded measurable function. Then the associated multiplication operator $S_m: L^2(R) \to L^2(R)$, $S_m(f)(x) = m(x)f(x)$ is a bounded linear operator. An operator $Q: L^2(R) \to L^2(R)$ is called (linear) filter if there exists $m \in L^\infty(R)$ such that

For all $f \in L^2(R)$, $(Qf)\hat{}(\omega) = \hat{f}(\omega)m(\omega)$.

Definition 1. A multiresolution analysis (MRA) is a sequence of closed subspaces $(V_j)_{j\in Z}$ of $L^2(R)$ with the following list of properties:

1. $\forall \mathbf{j} \in \mathbf{Z} : \mathbf{V_j} \subset \mathbf{V_{j+1}}$,
2. $\overline{\bigcup_{\mathbf{j}\in\mathbf{Z}} \mathbf{V_j}} = \mathbf{L^2(R)}$,
3. $\bigcap_{\mathbf{j}\in\mathbf{Z}} \mathbf{V_j} = \{\mathbf{0}\}$,
4. $\forall \mathbf{j} \in \mathbf{J}, \forall \mathbf{f} \in \mathbf{L^2(R)} : \mathbf{f} \in \mathbf{V_j} \Leftrightarrow \mathbf{D_{2^j}(f)} \in \mathbf{V_0}$,
5. $\mathbf{f} \in \mathbf{V_0} \Rightarrow \forall \mathbf{m} \in \mathbf{Z} : \mathbf{T_m f} \in \mathbf{V_0}$,
6. **There exists $\boldsymbol{\varphi} \in \mathbf{V_0}$ such that $(\mathbf{T_m}\boldsymbol{\varphi})_{\mathbf{m}\in\mathbf{Z}}$ is an orthonormal basis of $\mathbf{V_0}$.**

In “6” φ is called scaling function of $(V_j)_{j\in Z}$. The properties “1”-“6” are somewhat redundant. They are just listed for a better understanding of the construction. The following discussion will successively strip down “1”-“6” to the essential.

Remarks 2. (a) Properties “4” and “6” imply that the scaling function φ uniquely determines the multiresolution analysis: We have

$$V_0 = \overline{span(T_k\varphi : k \in Z)}$$

by “6” (which incidentally takes care of “5”) and

$$V_j = D_{2^{-j}} V_0$$

is prescribed by “4”. What is missing are criteria for “1”-“3” and “6”. However, note that in the following the focus of attention shifts from the spaces V_j to the scaling function φ.

(b) The parameter j can be interpreted as resolution or (inverse) scale or (with some freedom) frequency parameter. Thus the inclusion property “1” has a quite natural interpretation: Increasing resolution amounts to adding information. If we denote by P_j the projection onto V_j, we obtain the characterizations

$$\text{"2"} \quad \Leftrightarrow \forall f \in L^2(R): \|f - P_j f\| \to 0, as\ j \to \infty$$
$$\text{"3"} \quad \Leftrightarrow \forall f \in L^2(R): \|P_j f\| \to 0 \quad, as\ j \to -\infty$$

$P_j f$ can be interpreted as an approximation to f with resolution 2^j. Thus “2” implies that this approximation converges to f, as resolution increases.

Definition 3. Let $(V_j)_{j\in Z}$ denote an MRA. A space V_j is called approximation space of scale j. Denote by W_j the orthogonal complement of V_j in V_{j+1}, so that we have the orthogonal decomposition $W_j \oplus V_j = V_{j+1}$. W_j is called the detail space of scale j.

3 Main Results

As Throughout this section we fix a multiresolution analysis $(V_j)_{j\in Z}$ with scaling function φ, wavelet ψ and scaling coefficients $(a_k)_{k\in Z}$. Given $f \in V_0$, we can expand it with respect to two different orthonormal basis:

$$f = \sum_{k\in Z} c_{0,k} T_k\varphi = \sum_{k\in Z} d_{-1,k}\, D_2 T_k\psi + \sum_{k\in Z} c_{-1,k}\, D_2 T_k\varphi$$

Here we use the notations

$$c_{j,k} = \langle f, D_{2^{-j}} T_k\varphi\rangle, \qquad \text{"}approximation\ coefficient\text{"}$$

$$d_{j,k} = \langle f, D_{2^{-j}} T_k\psi\rangle, \qquad \text{"}detail\ coefficient\text{"}$$

In the following we will use $d^j = (d_{-j,k})_{k\in Z}$ and $c^j = (c_{-j,k})_{k\in Z}$. At the heart of the fast wavelet transform are explicit formulae for the correspondence

$$c^0 \leftrightarrow (c^1, d^1)$$

For this purpose we require one more piece of notation. Definition 10. The up- and downsampling operators $\uparrow 2$ and $\downarrow 2$ are given by

$$(\downarrow 2f)(n) = f(2n), \qquad \text{"}down\text{"}$$

$$(\uparrow 2f)(n) = \begin{cases} 0 & n\ odd, \\ f(k) & n = 2k. \end{cases} \quad \text{"}up\text{"}$$

We have $(\uparrow 2)\hat{} * = (\downarrow 2)$, as well as $\downarrow 2\ o \uparrow 2 = I$, where I is the identity operator. In particular, $\uparrow 2$ is an isometry. Theorem 11. The bijection $c^0 \leftrightarrow (c^1, d^1)$ is given by

$$Analysis: \qquad c^{j+1} = \downarrow 2(c^j * \ell), d^{j+1} = \downarrow 2(c^j * h)$$

And

$$Synthesis: \qquad c^j = (\uparrow 2c^{j+1}) * h^* + (\uparrow 2d^{j+1}) * \ell^*$$

Here the filters h, ℓ are given by $h(k) = (-1)^{1+k} a_{1+k}/\sqrt{2}$ and $\ell(k) = \overline{a_{-k}}/\sqrt{2}$.

Remark 12. The “Cascade Algorithm”: Iteration of the analysis step yields c^j , d^j for $j \geq 1$.

$$\begin{array}{ccccccccccc} c^0 & \rightarrow & c^1 & \rightarrow & c^2 & \rightarrow & \dots & \rightarrow & c^{j-1} & \rightarrow & c^j \\ & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\ & & d^1 & & d^2 & & & & d^{j-1} & & d^j \end{array}$$

Whereas the synthesis step is given by

$$\begin{array}{ccccccccccc} c^j & \rightarrow & c^{j-1} & \rightarrow & c^{j-2} & \rightarrow & \dots & \rightarrow & c^1 & \rightarrow & c^0 \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ d^j & & d^{j-1} & & & & d^2 & & d^1 & & \end{array}$$

The decomposition step corresponds to the orthogonal decompositions

$$\begin{array}{ccccccccccc} V_0 & \rightarrow & V_{-1} & \rightarrow & V_{-2} & \rightarrow & \dots & \rightarrow & V_{-j+1} & \rightarrow & V_{-j} \\ & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\ & & W_{-1} & & W_{-2} & & & & W_{-j+1} & & W_{-j} \end{array}$$

For example we give the Haar wavelet as follow.

Example. The Haar wavelet is the following simple step function:

$$\psi(x) = \chi_{[0,1/2[} - \chi_{[1/2,1[} = \begin{cases} 1 & 0 \leq x < 1/2, \\ -1 & 1/2 \leq x < 1 \\ 0 & \text{o. w.} \end{cases}$$

Define

$$V_0 = \{f \in L^2(R) : \forall n \in Z : f|_{[n,n+1]} is\ constant\}$$

If we set $\varphi = \chi_{[0,1[}$, it is not hard to see that $(T_k\varphi)_{k\in Z}$ is an orthonarmal basis of V_0. φ induces an MRA. Note that in this case

$$V_j = \{f \in L^2(R) : \forall n \in Z : f|_{[n,n+1]} is\ constant\}$$

Thus the closed subspace $(V_j)_{j\in Z}$ is indeed a multiresolution analysis, so called Haar multiresolution analysis. The Haar wavelet and it’s Fourier is as follows:

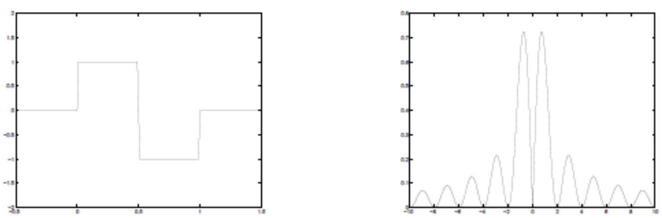


Fig 1: The Haar wavelet and its Fourier transform (only the absolute value)

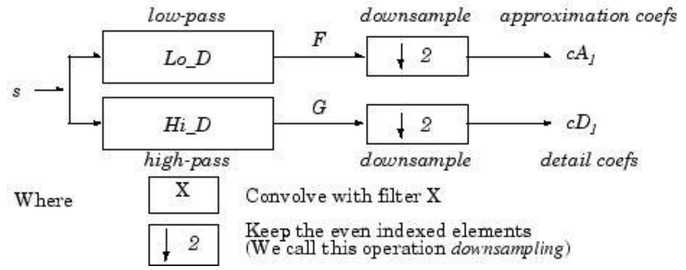
A. DWT

dwt command performs a single-level one-dimensional wavelet decomposition with respect to either a particular wavelet (*'wname'*) or particular wavelet decomposition filters (Lo_D and Hi_D) that you specify. [cA,cD]=dwt(X,*'wname'*) computes the approximation coefficients vector cA and detail coefficients vector cD, obtained by a wavelet decomposition of the vector X. The string *'wname'* contains the wavelet name. [cA,cD]=dwt(X,Lo_D,Hi_D) computes the wavelet decomposition as above, given these filters as input:

- Lo_D is the decomposition low-pass filter.
- Hi_D is the decomposition high-pass filter.

Lo_D and Hi_D must be the same length.

Starting from a signal *s*, two sets of coefficients are computed: approximation coefficients CA_1, and detail coefficients CD_1. These vectors are obtained by convolving *s* with the low-pass filter Lo_D for approximation and with the high-pass filter Hi_D for detail, followed by dyadic decimation. More precisely, the first step is



Note that in MATLAB we have,

$\ell = Li\_D,\ h = Hi\_D.$

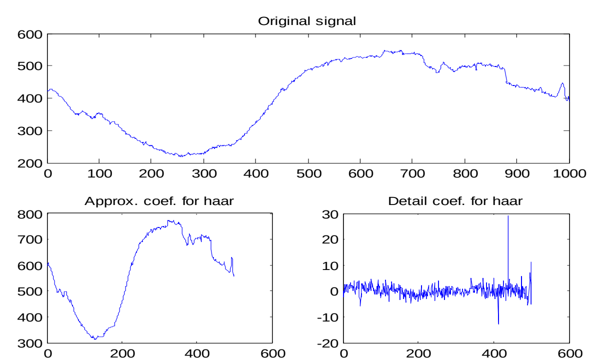
We can compute the high pass and low pass filters h, ℓ as follows:

$\ell = [1/\sqrt{2}\ \ 1/\sqrt{2}],\quad h = [-1/\sqrt{2}\ \ 1/\sqrt{2}]$.

The programming in the MATLAB is as follows:

```
clc;
clear;
close all;

load leleccum;
x = leleccum(1:1000);

[a,d] = dwt(x,'haar');

figure(1);
subplot(2,1,1); plot(x); title('Original signal');
subplot(2,2,3); plot(a); title('Approx. coef. for haar');
subplot(2,2,4); plot(d); title('Detail coef. for haar');
```

With the following figure.

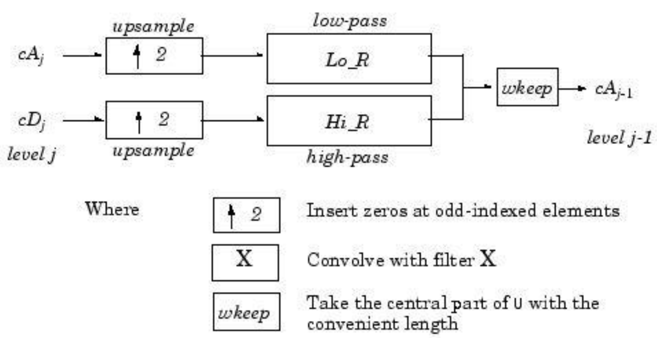


B. IDWT

The idwt command performs a single-level one-dimensional wavelet reconstruction with respect to either a particular wavelet (*'wname'*) or particular wavelet reconstruction filters (Lo_R and Hi_R) that you specify. X=idwt(cA,cD,*'wname'*) returns the single-level reconstructed approximation coefficients vector X based on approximation and detail coefficients vectors cA and cD, and using the wavelet *'wname'*. X = idwt(cA,cD,Lo_R,Hi_R) reconstructs as above using filters that you specify.

- Lo_R is the reconstruction low-pass filter.
- Hi_R is the reconstruction high-pass filter.

Lo_R and Hi_R must be the same length. idwt is the inverse function of dwt in the sense that the abstract statement idwt(dwt(X,*'wname'*),*'wname'*) would give back X. Starting from the approximation and detail coefficients at level *j*, *cAj* and cD_j, the inverse discrete wavelet transform reconstructs cA_{j-1}, inverting the decomposition step by inserting zeros and convolving the results with the reconstruction filters.



Note that in MATLAB we have,

$$\ell^* = Lo\_R, \quad h^* = Hi\_R.$$

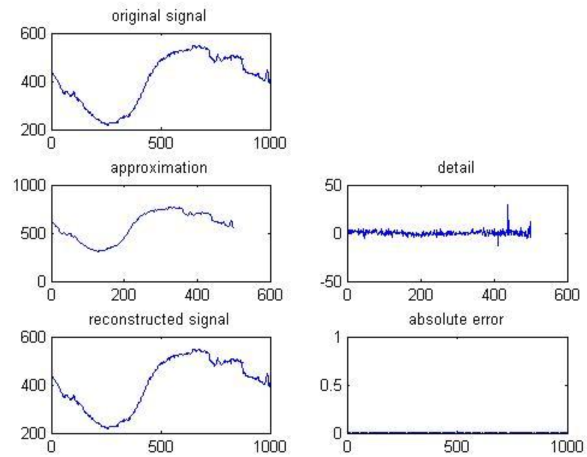
We can compute the high and low pass filters h, ℓ as follows:

$$\ell = [1/\sqrt{2} \;\; 1/\sqrt{2}], \quad h = [-1/\sqrt{2} \;\; 1/\sqrt{2}]$$

The programming in the MATLAB is as follows:

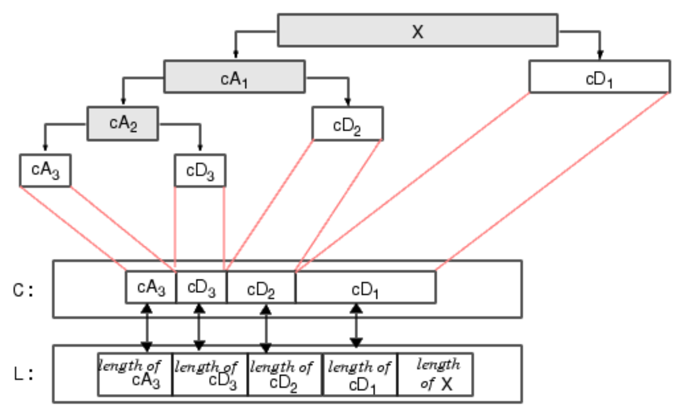
```
clc;
clear;
close all;

load leleccum; x=leleccum(1:1000);

[a,d] = dwt(x,'haar');

x_hat = idwt(a,d,'haar');

figure(1);
subplot(3,2,1); plot(x); title('original signal');
subplot(3,2,3); plot(a); title('approximation');
subplot(3,2,4); plot(d); title('detail');
subplot(3,2,5); plot(x_hat); title('reconstructed signal');

subplot(3,2,6); plot(abs(x-x_hat)); ylim([0,1]); title('absolute error');
```

With the following figure.



C. WAVDEC

wavedec performs a multilevel one-dimensional wavelet analysis using either a specific wavelet (*'wname'*) or a specific wavelet decomposition filters (Lo_D and Hi_D). [C,L] = wavedec(X,N,*'wname'*) returns the wavelet decomposition of the signal X at level N, using *'wname'*. N must be a strictly positive integer. The output decomposition structure contains the wavelet decomposition vector C and the bookkeeping vector L. The structure is organized as in this level-3 decomposition example.



[C,L]=wavedec(X,N,Lo_D,Hi_D) returns the decomposition structure as above, given the low- and high-pass decomposition filters you specify.

D. DETCOEF

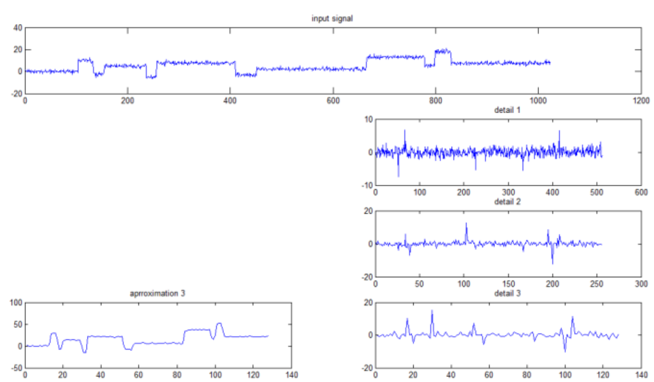
detcoef is a one-dimensional wavelet analysis function. D = detcoef(C,L,N) extracts the detail coefficients at level *N* from the wavelet decomposition structure [C,L]. Level N must be an integer such that 1 ≤ N ≤ NMAX where NMAX = length(L)-2. D = detcoef(C,L) extracts the detail coefficients at last level NMAX.

E. APPCOEF

appcoef is a one-dimensional wavelet analysis function.appcoef computes the approximation coefficients of a one-dimensional signal. A = appcoef(C,L,*'wname'*,N) computes the approximation coefficients at level *N* using the wavelet decomposition structure [C,L]. *'wname'* is a string containing the wavelet name. Level *N* must be an integer such that 0 ≤ N ≤ length(L)-2. Instead of giving the wavelet name, you can give the filters. For A = appcoef(C,L,Lo_R,Hi_R) or A = appcoef(C,L,Lo_R,Hi_R,N), *Lo_R* is the reconstruction low-pass filter and *Hi_R* is the reconstruction high-pass filter. The following programming is in level-3.

```
clc;
clear;
close all;

load noisbloc; x = noisbloc;
wname = 'db1';
[c,l] = wavedec(x,3,wname);

d1 = detcoef(c,l,1);
d2 = detcoef(c,l,2);
d3 = detcoef(c,l,3);

a3 = appcoef(c,l,wname,3);

figure(1);
subplot(4,1,1); plot(x); title('input signal');
subplot(4,2,7); plot(a3); title('aproximation 3');
subplot(4,2,4); plot(d1); title('detail 1');
subplot(4,2,6); plot(d2); title('detail 2');
subplot(4,2,8); plot(d3); title('detail 3');
```

Also we have the following figure for this.



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