

A Review On Groebner Basis And Its Applications

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Abstract: This paper is a survey on Groebner basis and its applications on the some areas of Science and Technology. Here we have presented some of the applications of concepts and techniques from Groebner basis to broader area of science and technology such as applications in steady state detection of chemical reaction network(CRN), determining kinematic equations in the investigation and design of robots. Groebner basis applications could be found in vast area in circuits and systems. Also in pure mathematics Groebner basis is used to determine that a polynomial is radical membership, zero divisors or invertible about an ideal. A short note is being presented on Groebner basis and its applications.

Index Terms: Groebner basis, polynomials, polynomials rings, ideals, division algorithm, applications of Groebner basis.

1 INTRODUCTION

Polynomial systems are available in many areas of mathematics, natural sciences and engineering, and the theory of Groebner basis is one of the most advance tools for solving polynomial systems. Groebner basis are the basis for the finitely generated ideals of a polynomial ring with several variables. The three steps monomial ordering, division algorithm and Hilbert basis theorem are the shortest way which leads to the definition of Groebner basis [15]. The notion of Groebner basis was introduced by Buchberger in 1965 to describe ideals of commutative algebras. This notion generalizes both the Gaussian elimination method for solving systems of polynomial equations and the division of polynomials in multiple indeterminates. In the case of systems of linear polynomial equations, Groebner basis give the Gaussian elimination algorithm. In the case of the division, of a polynomial by a set of polynomials, the remainder is not always unique. Hence if the remainder of the division of f by f_1, f_2, \dots, f_s is equal to zero then f is in the ideal generated by f_1, f_2, \dots, f_s , but if the remainder is not equal to zero we cannot determine whether f is in the ideal generated by f_1, f_2, \dots, f_s . Also if we choose any divisor and the remainder is unique regardless of the order of divisors. These divisors are the Groebner basis. Groebner basis can also be used to decide whether or not an element of a commutative algebra is in some ideal. Given I an ideal of the ring $K[x_1, \dots, x_n]$ of polynomials in n indeterminates, a Groebner basis of I gives us an algorithm deciding whether or not an element of $K[x_1, \dots, x_n]$ is in I . A Groebner basis is a set of multivariate nonlinear polynomials that reduces many fundamental problems in mathematics into simple algorithmic solutions such as finding intersection of ideal, finding Hilbert dimension and envelopes in commutative algebra; geometric theorem proving, graph coloring, summation and integration, linear integer optimization in the field of mathematics; also it has many applications in coding theory, robotics, software engineering and in petroleum industry for natural sciences and engineering [3].

Groebner basis theory in [1] and [2], is of great importance in various scientific fields with the progress in Computer Algebra System (CAS) such as mathematical modeling and simulations for problems. To name a few problems that the Groebner basis technique solved recently are determining genetic connection between the species, reverse engineering problems, inverse kinematics in robotics and to control oil platforms using artificial intelligence.

2 PRILIMINARIES

Let K indicate a field generally $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ or Z_p where p is

prime. Let $R = K[x_1, x_2, \dots, x_n]$ represent the polynomial

ring in the indeterminates x_1, x_2, \dots, x_n over the field K . A

monomial is nothing but a product of the variables, that is $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ where $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N}^n$.

Furthermore, the total degree of the monomial x^α could be

define as $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

A polynomial of R can be define as a linear combination of

monomials such as $f = c_1 x_1^{\alpha_1} + c_2 x_2^{\alpha_2} + \dots + c_n x_n^{\alpha_n}$.

That is all the polynomials of R is of the form

$f = c_t x^t + c_{t-1} x^{t-1} + \dots + c_1 x + c_0$ where t is the

degree of f . And so, each polynomial will have a largest

monomial of degree t . We call $c_t x^t$ the leading term of f and

x^t the leading monomial.

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2.1 Example

x^7y^2z, x^6y^3, y^5 and x^3z are monomials in x, y, z . They are of total degree 10, 9, 5 and 3 respectively.

2.2 Definition

A multivariate polynomial f in x_1, x_2, \dots, x_n with coefficients

in a field K is a finite linear combination $\sum_{\alpha} a_{\alpha} x^{\alpha}$, where x^{α}

is the monomials and $a_{\alpha} \in K$ are the coefficients. The total

degree of the polynomial f is the maximum $|\alpha|$ such that

$a_{\alpha} \neq 0$ (Mvngu, 2009).

2.3 Definition

A monomial ordering is a total ordering on the variables x_1, x_2, \dots, x_n (that is there are no elements which are not

comparable) $>$ on the set of monomials of

$R = K[x_1, x_2, \dots, x_n]$ such that

$$i. \quad x^{\alpha} > x^{\beta} \Rightarrow x^{\alpha} x^{\gamma} > x^{\beta} x^{\gamma} \forall x^{\gamma}$$

$$ii. \quad x^{\alpha} > 1 \quad \forall x^{\alpha} \neq 1.$$

We know that a monomial ordering is well-ordering, for every set of monomials there is a minimal element on the set of monomials.

i. A Lexicographic order is an ordering $\alpha >_{lex} \beta$ iff,

the left-most nonzero entry in $\alpha - \beta$ is positive.

ii. A Graded Lex order is an ordering $\alpha >_{grlex} \beta$ iff,

$|\alpha| > |\beta|$ or $(|\alpha| = |\beta|)$ and $\alpha >_{lex} \beta$.

iii. A Graded Reverse Lex order is an ordering $\alpha >_{grevlex} \beta$ iff, $|\alpha| > |\beta|$ or $(|\alpha| = |\beta|)$ and the

right most nonzero entry in $\alpha - \beta$ is negative.

2.4 Definition

Let us suppose $>$ be a monomial ordering and f be a

polynomial of $K[x]$. Let us also assume that $f = cx^{\alpha} +$

monomial terms of the form x^{β} with $\beta \neq \alpha$, where $c \neq 0$

and for every term of the form x^{β} , $x^{\alpha} > x^{\beta}$ with respect to

the ordering $>$. Then

i. $LT(f) = cx^{\alpha}$ is the leading term.

ii. $LM(f) = x^{\alpha}$ is the leading monomial.

2.5 Definition

Let us suppose $>$ be a monomial ordering and f be a

polynomial of $K[x]$, we can write

$$f = c_1 x^{\alpha_1} + c_2 x^{\alpha_2} + \dots + c_r x^{\alpha_r} \quad \text{where}$$

$a_i \in K \setminus \{0\}$, x^{α_i} are power products, and . Then the leading

power product of f , $LP(f) = x^{\alpha_1}$.

$$x^{\alpha_1} > x^{\alpha_2} > \dots > x^{\alpha_r}$$

2.6 Definition

The multivariate division algorithm consists of a sequence of reduction steps as follows:

Let $f, g, h \in K[x_1, x_2, \dots, x_n]$ with $g \neq 0$. We say that f

reduces to h modulo g in one step, denoted by $f: g \rightarrow h$, if

and only if $LP(g)$ divides a nonzero term cx^{α} that appears in

$$f \text{ and } h = f - \left(\frac{cx^{\alpha}}{LT(g)} \right) g.$$

2.7 Division Algorithm [8]

Input: $f, f_1, f_2, \dots, f_s \in K[x_1, x_2, \dots, x_n]$ with

$$f_i \neq 0, (1 \leq i \leq s)$$

Output: a_1, a_2, \dots, a_s, r such that i.e $LT(f)$ and $LT(g)$ of the polynomials f and g have no $f = a_1f_1 + a_2f_2 + \dots + a_sf_s + r$ and r is reduced with common elements then $S\text{-poly}(f, g)$ can be treated as zero respect to $\{f_1, f_2, \dots, f_s\}$ and with respect to $\{f, g\}$. Therefore in Buchberger Algorithm $\max(LP(a_1)\{LP(f_1), \dots, LP(a_s) LP(f_s), LP(r)\}) = LP(f)$ such kind of computation could be avoided.

Initialization: $a_1 := 0, a_2 := 0, \dots, a_s := 0, h = f$

While $h \neq 0$ Do

IF $LP(f_i)$ divides $LP(h)$ **THEN** take i least such that $LP(f_i)$ divides $LP(h)$

$$a_i := a_i + \frac{LT(h)}{LT(f_i)}$$

$$h := h - \left(\frac{LT(h)}{LT(f_i)}\right) f_i$$

ELSE

$$r := r + LT(h)$$

$$h := h - LT(h)$$

End

2.8 Definition

The notion of S -polynomial is very important in determining the Groebner basis. Let f, g be non-zero polynomials in $K[x_1, x_2, \dots, x_n]$ and let $L = \text{lcm}(LP(f), LP(g))$. Then the polynomial $S(f, g) = \left(\frac{L}{LT(f)}\right) f - \left(\frac{L}{LT(g)}\right) g$ is called the S -polynomial of f and g .

Calculation of greatest common divisor (gcd) in most computer algebra system for calculating S -polynomial and to convert to

a normal form involves only two polynomials at one time. Therefore it is a time taking and space consuming process to compute more than two polynomials. So, if the leading terms

2.8 Hilbert Basis Theorem

Let K be a field. Then every ideal of $K[x_1, x_2, \dots, x_n]$ is finitely generated [8].

2.9 Theorem(Buchberger Criterion)

A subset $G \subseteq K[x]$ is a Groebner basis of the Ideal I it generates, iff for every $f, g \in G$ we have $S(f, g) \xrightarrow{G} 0$ i.e. the remainder of division of $S(f, g)$ by G is 0.

2.10 Buchberger Algorithm [2]

Input: A polynomial set $f = (f_1, f_2, \dots, f_s)$ that generates an ideal I .

Output: A Groebner basis $G = (g_1, g_2, \dots, g_r)$ that generates the same ideal I with $f \subset G$.

$$G := f$$

$$M := \{\{f_i, f_j\} | f_i, f_j \in G \text{ and } f_i \neq f_j\}$$

Repeat

$$\{p, q\} := \text{a pair in } M$$

$$M := M - \{\{p, q\}\}$$

$$S := S\text{-poly}(p, q)$$

$$h := \text{NormalForm}(S, G)$$

IF $h \neq 0$ **THEN**

$$M := M \cup \{\{g, h\}\} \forall g \in G$$

$$G := G \cup \{h\}$$

Until $M := \emptyset$.

In many areas of mathematics and engineering problems we encounter the systems of equations of the type

$$f_1(x_1, x_2, \dots, x_n) = 0, f_2(x_1, x_2, \dots, x_n) = 0, \dots, \\ f_m(x_1, x_2, \dots, x_n) = 0.$$

Such type of system of equation can be computed and is solvable if and only if the Groebner basis of the equations f_1, f_2, \dots, f_m is not one.

2.11 Example

Let us assume that $\{x + xy^2 = 1, x^2y + y = 1, x^2 + y^2 = 1\}$ be the

system of equations. To see whether it is solvable or not we just use MAPLE to compute the Groebner basis:

$$> L := [x + xy^2 - 1, x^2y + y - 1, x^2 + y^2 - 1]$$

$$> G1 := \text{Groebner}[Basis](L, \text{plex}(x, y, z));$$

$$G1 := [1]$$

So the given system of equations is not solvable.

2.12 Example

Consider the following systems of equations:

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + y^2 + z^2 = 2x$$

$$2x - 3y - z = 0$$

Let

$$J = \langle x^2 + y^2 + z^2 - 1, x^2 + y^2 + z^2 - 2x, 2x - 3y - z \rangle$$

, we can compute Groebner basis for J with respect to the lex

ordering using maple:

$$> \text{Ideal}(J) := [x^2 + y^2 + z^2 - 1, x^2 + y^2 + z^2 - 2x, 2x - 3y - z]$$

$$> G2 := \text{Groebner}[Basis](\text{Ideal}(J), \text{plex}(x, y, z));$$

$$G2 := [40z^2 - 8z - 23, -1 + 3y + z, -1 + 2x]$$

So the Groebner basis $G2 = (g_1, g_2, g_3)$ is given by:

$$g_1 = 40z^2 - 8z - 23$$

$$g_2 = 3y + z - 1$$

$$g_3 = 2x - 1$$

By examining the generators in $G2$, we have one generator,

g_1 , in the variable $x_n = z$ alone. The other variables have

been eliminated during the course of finding the Groebner basis. This equation has a finite number of roots that could be solved by single variable method. So, there is exactly one generator in the variables $x_{n-1} = y$ and $x_n = z$. Because

we have all possible roots of z we can solve for y . Generator

g_3 is in x alone so roots of x could be easily determine.

3 APPLICATIONS ON GROEBNER BASIS

Groebner basis is being used in qualitative analysis of systems of ordinary differential equations in order to determine the center-focus problem, cyclicity problems, critical period perturbations, finding linearizability and isochronicity conditions for a given group of polynomials [4]. It is also applied in steady state detection of chemical reaction network (CRN), using CRN as a model of system of ODE, which is nonlinear equations containing several variables. The qualitative conduct of the network can be found after solving the ODE. Finding the solutions of such type of problem is hard. In [13], the authors used Groebner basis to analyse CRNs on biochemical examples. Groebner basis has been utilized to determine kinematic equations in the investigation and design of robots [31]. Additionally it has been utilized in robotics to decide the unique solutions for the forward kinematic issue uncommonly connected to possessing space and planar parallel manipulator in mechanical technology [12][23][24]. Applications of Groebner basis in robotics goes back to 1990's. Many researchers worked on 6R robot inverse solution algorithm based on quaternion matrix and Groebner basis. To solve the problem in [20], authors applied double quaternion and Groebner basis which encounters 8 irrelevant

roots. So they extracted four $1 + i^2$ common factors in Dixon resultant to eliminate the irrelevant roots. And hence a 16 degree equation with single variable is obtained. X.G. Huang and Q.Z. Liao in [11], worked on the same algorithm and they transformed the sine and cosine of angle rotation into plural form instead of extracting common factors resulting in single variable equation involving numerical calculation without rigorous proof. For the first time in 6R robot inverse kinematics analysis, Z.S. Ni, and R.K. Wu in [19], introduced quaternion matrix modeling to solve inverse inverse kinematics in the three dimensional space. They proposed a new algorithm of quaternion and dual quaternion in matrix form. After transforming into matrix form, five variables have been eliminated using Groebner basis. Thus, in this algorithm they could determined the degree of the equation with single variable without any extraneous roots. So far many researchers, Gelernter, H., Wu Wen-Tsun and so on in [29][10][7][22][28], had studied Automated Geometric Theorem proving. Automatic Geometric Theorem proving is the process of proving mathematical theorems with the help of

computer program and generates automatically complementary condition that is true for the theorem in [9]. Related works have been found on many papers in [6][16][21]. In the paper [5], the authors proposed a method by using parametric Groebner basis for proving geometric theorem automatically. Yao Sun, DingKang Wang and Jie Zhou; introduced a new method for proving and discovering geometric theorems automatically in [26]. They extended Rabinovitch's trick to show that a polynomial is radical membership, zero divisors or invertible about an ideal. Applying Rabinovitch's trick and the properties of Comprehensive Groebner System in automatic geometric theorem proving, they have shown whether a geometric statement is true, true on component, generic false or complete false on non-degenerate case in [26]. Uses of Groebner basis could be found in vast area in circuits and systems, for example, the algebraic representation of error bounds for depicting function applying to nonlinear circuit frameworks and the advancement of productive representative methodology for testability measures and ambiguity groups for resolving simple linear circuits in [4]. Additionally application can be found at solving the load-flow problem as the progressions associated with an electrical network shift [17].

4 OPEN PROBLEM

To determine the factorization $F = G_1 F_1$, G_1 is a square matrix (a unimodular matrix) where determinant is a non zero constant in K of an n -dimensional polynomial matrix F is very difficult, either by conventional methods or by Groebner bases. This was highlighted as one of the most important problems in n -dimensional systems theory by the authors [32].

5 CONCLUSION

In this paper, we have highlighted some of the applications of Groebner basis and provided information about this important technique of Groebner basis. We see that using the properties of Groebner basis the solutions of systems of equations could be easily determined. We can use Computer Algebra Systems like Maple, Singular, Mathematica, Reduce, CoCoCA, Macauley2, etc., to find this technique.

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