# Sensitive Non-Parametric Control Charts For Monitoring Process Variation

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**Abstract:** Non-parametric or distribution-free control charts are useful in statistical process control when there is limited or a lack of knowledge about essential process distribution. In this article, nonparametric control charts were considered based on Mood and Sukhatme statistics. Two non-parametric statistics were applied on Exponentially-Weighted Moving Average (EWMA) and Cumulative Sum (CUSUM) charts for monitoring process variation. Simulations showed that the EWMA control chart based on Mood statistics was more sensitive for detection of small shifts in process variation, but moderate and large shifts in CUSUM based on Mood statistics were more sensitive than other charts.

**Index Terms:** EWMA, CUSUM, Non-parametric, Mood, Sukhatme, Control Chart, Average Run Length

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## 1 Introduction

Statistical process control (SPC) is widely utilized in industrial processes for detecting, measuring, controlling, and improving the processes. Control charts are important tools for SPC, which were first introduced by Shewhart. Control charts are applied for monitoring the mean, variance and fraction of nonconformity in industrial processes. Shewhart control charts are usually used to detect large changes in processes, while cumulative sum (CUSUM) and exponentially-weighted moving average (EWMA) control charts are more sensitive for small shifts [1]. The common characteristic of a control chart is the average run length (ARL), which is the expectation of alarm time taken to trigger a signal about possible change in process distribution. Ideally, an acceptable ARL for in-control processes should be large enough to detect a small change in the process, which uses a notation in the control process as $ARL_0$. The out-of-control process is denoted by $ARL_1$, where $ARL_1$ should be small. However, process distribution usually assumes normal distribution in parametric control charts. In practice, however, the process observations are not assumed as normality [2]. As such, non-parametric charting statistics should be a good way to proceed. Bakir [3] classified the distribution-free control chart according to driving non-parametric ideas. Petros et al. [4] showed that EWMA control charts are robust for detecting the mean and dispersion in non-normal distribution. Das [5] compared three non-parametric control charts, which were Hodges-Lehmann (HL), sign test and Mann-Whitney (MW), for detecting shifts in location parameters. They found that the three methods were effective for symmetry distributions. Hidetoshi and Takashi [6] used the Mood method for Shewhart control charts to detect changes in dispersion with small sample size. Zombade and Ghute [7] proposed a non-parametric CUSUM control chart for monitoring process changes based on Mood and Sukhatme statistics. They worked with symmetric distributions and found that non-parametric CUSUM was better for detecting small shifts in process variation. In this paper, we propose a non-parametric EWMA control chart using Mood and Sukhatme statistics for monitoring process change. We will compare the performance of the proposed control chart with non-parametric CUSUM based on Mood and Sukhatme statistics.

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## 2 Non-parametric Statistics

Let $X = X_1, X_2, ..., X_m$ be a random sample of size *m* and $Y = Y_1, Y_2, ..., Y_n$ be a random sample of size *n*, which is drawn from continuous distributions. Suppose that $X$ is from an in-control process, then $\sigma_x$ and $\sigma_y$ are standard deviations of *X* and *Y*, respectively. For testing dispersion, $H_0 : \sigma_x = \sigma_y$ versus $H_1 : \sigma_x \neq \sigma_y$. Let $R_1 < R_2 < ... < R_n$ be combine sample ranks of *X* values in increasing order of magnitude.

### 2.1 Sukhatme Statistic

The Sukhatme statistic can be defined as follows:

$$T = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} D(X_i, Y_j) \qquad (1)$$

where $D(X,Y) = \begin{cases} 1; & 0 < X < Y \ or \ Y < X < 0 \\ 0; & otherwise, \end{cases}$ .

mean and variance of *T* are

$$E(T) = \frac{1}{4} \text{ and } V(T) = \frac{(m+n+7)}{48mn} \qquad (2)$$

For a large sample, *T* is a standard normal distribution defined as follows:

$$Z = \frac{T - E(T)}{\sqrt{Var(T)}} \qquad (3)$$

### 2.2 Mood statistic

The Mood statistic can be defined as:

$$M = \sum_{i=1}^{m} (R_i - \frac{N+1}{2})^2 \ ; \ N = m+n \qquad (4)$$

The mean and variance of *M* are:

$$E(T)=\frac{m(N^2-1)}{12}\ ,V(T)=\frac{mn(N+1)(N^2-4)}{180} \qquad (5)$$

For *N* large, the *M* performs on the standard normal distribution, defined as follows:

$$W=\frac{M-E(\mathrm{M})}{\sqrt{Var(\mathrm{M})}} \qquad (6)$$

## 3 Control Charts

This section introduces the design structure for the proposed non-parametric EWMA and non-parametric CUSUM control charts based on Mood and Sukhatme statistics for monitoring process variation.

### 3.1 EWMA Control Chart

In this section, we describe the design of the proposed control chart. For the structure of the EWMA control chart, let $X_1, X_2, ..., X_n$ be a random sample of size *n*. The EWMA statistic is defined recursively by:

$$X_t=(1-\lambda)X_{t-1}+\lambda Z_t\ ,\quad t=1,2,... \qquad (7)$$

where $\lambda$ is a constant, $0\le\lambda\le1$ , $X_t$ represents the $i^{th}$ statistic and $Z_t$ is $i^{th}$ observation. Let $X_0=\mu_0$ represent the starting value. Control limits are given by:

$$CL=\mu_0 \text{ and } LCL\,/\,\mathrm{UCL}=\mu_0\pm L\sigma\left(\frac{\lambda}{2-\lambda}\right).$$

The values of *L* depends on choices *n* and *ARL* for the in-control process. To develop a non-parametric EWMA control chart to detect process variation, non-parametric statistic *T* according to (1) is obtained from two independent random samples $X=X_1, X_2, ..., X_m$ and $Y=Y_1, Y_2, ..., Y_n$. Now, the non-parametric EWMA statistic is written as:

$X_t=(1-\lambda)X_{t-1}+\lambda T_t$ , $t=1,2,...,$ and $T_t$ is $i^{th}$ observation. The EWMA control chart based on Mood W statistic is given by:

$$X_t=(1-\lambda)X_{t-1}+\lambda W_t\ ,\quad t=1,2,...,$$

and $W_t$ is $i^{th}$ observation base on the Mood statistic. Then EWMA control chart based on the Mood statistic is denoted by EWMA-M. The EWMA control chart based on Sukhatme Z statistic is given by:

$$X_t=(1-\lambda)X_{t-1}+\lambda Z_t\ ,\quad t=1,2,...,$$

and $Z_t$ is $i^{th}$ observation based on the Sukhatme statistic. Then EWMA control chart based on the Sukhatme statistic is denoted by EWMA-S.

### 3.2 CUSUM Control Chart

Similar to [7], the one-sided CUSUM control chart based on non-parametric statistic *T* is defined as:

$$S_t=Max(0,S_{t-1}+T_t-k)\ ,$$

where $S_0=0$, $h>0$ and $k>0$. Parameter k is a reference value and k is a control limit. Hence, $W_t$ is $i^{th}$ observation based on the Mood statistic. The CUSUM chart based on the Mood statistic is given by:

$$S_t=Max(0,S_{t-1}+W_t-k)\ ,$$

and the CUSUM control chart based on the Mood statistic is denoted by CUSUM-M. Hence, $Z_t$ is $i^{th}$ observation based on the Sukhatme statistic. The CUSUM chart based on non-parametric statistic $Z_t$ is defined as:

$$S_t=Max(0,S_{t-1}+Z_t-k)\ ,$$

and the CUSUM control chart based on the Sukhatme statistic is denoted by CUSUM-S.

## 4 Results and Comparison

The performance of the control chart is measured by average run length (*ARL*). The notation of *ARL* for the in-control process is denoted by $ARL_0$, while the out-of-control process is denoted by $ARL_1$. In this study, we set $ARL_0$=370. The process observations are logistic and uniform distributions. Consider a random sample *X* as distributed with mean $\mu$ and standard deviation (SD) $\sigma$. Let $\mu=\mu_0$ and $\sigma=\sigma_0$ be the mean and SD for the in-control process. When process shift $\delta$ in variation occurs, let $\sigma_1=\delta\sigma_0$, $(0<\delta\neq1)$ be standard deviation for the out-of-control process. The optimal parameters *L, k* and *h* of each control chart are obtained by in-control $ARL_0$=370. The results of ARL values are computed by a Monte Carlo simulation technique with 50,000 simulations using sample size n=15 and 20 and shift size $\delta=1, 1.2, 1.3, 1.4, 1.6, 1.8, 2, 3$ .

***TABLE 1:*** *ARL values for EWMA-M and CUSUM-M, when n=15 and 20 for Logistics (5,2)*

| $\delta$ | n=15 | | n=20 | |
|---|---|---|---|---|
| | EWMA-M $\lambda=0.01$ L=0.0901 | CUSUM-M $k=1$ $h=2.1$ | EWMA-M $\lambda=0.01$ L=0.0905 | CUSUM-M $k=1$ $h=2.1285$ |
| 1.0 | 370.968 | 370.386 | 371.508 | 369.372 |
| 1.2 | 20.458 | 45.332 | 17.289 | 32.605 |
| 1.4 | 10.892 | 11.905 | 7.344 | 7.955 |
| 1.6 | 7.984 | 5.407 | 6.849 | 3.611 |
| 1.8 | 6.486 | 3.292 | 5.596 | 2.209 |
| 2.0 | 5.721 | 2.249 | 4.880 | 1.551 |
| 3.0 | 4.053 | 0.894 | 3.478 | 0.524 |

***TABLE 2:*** *ARL VALUES FOR EWMA-S AND CUSUM-S, WHEN N=15 AND 20 FOR LOGISTICS (5,2)*

| $\delta$ | n=15 | | n=20 | |
|---|---|---|---|---|
| | EWMA-S $\lambda=0.01$ L=3.054 | CUSUM-S $k=4$ $h=7.259$ | EWMA-S $\lambda=0.01$ L=3.595 | CUSUM-S $k=1$ $h=7.955$ |
| 1.0 | 370.794 | 370.352 | 369.094 | 370.413 |
| 1.2 | 196.695 | 286.269 | 176.652 | 286.605 |
| 1.4 | 128.797 | 244.493 | 117.897 | 217.615 |
| 1.6 | 102.185 | 193.329 | 92.285 | 174.422 |
| 1.8 | 45.135 | 163.615 | 45.135 | 140.327 |
| 2.0 | 29.548 | 131.027 | 21.468 | 110.504 |
| 3.0 | 18.839 | 67.979 | 15.921 | 52.580 |

***TABLE 3:*** *ARL VALUES FOR EWMA-M AND CUSUM-M, WHEN N=15 AND 20 FOR UNIFORM (1,2)*

| $\delta$ | n=15 | | n=20 | |
|---|---|---|---|---|
| | EWMA-M $\lambda=0.01$ L=0.022 | CUSUM-M $k=2$ $h=0.451$ | EWMA-M $\lambda=0.01$ L=0.027 | CUSUM-M $k=3$ $h=0.4525$ |
| 1.0 | 370.081 | 370.713 | 370.251 | 370.013 |
| 1.2 | 13.741 | 19.878 | 10.682 | 19.878 |
| 1.4 | 7.514 | 6.662 | 5.114 | 4.921 |
| 1.6 | 5.519 | 4.219 | 4.819 | 3.012 |
| 1.8 | 4.554 | 3.247 | 3.034 | 2.044 |
| 2.0 | 3.988 | 1.306 | 2.278 | 1.113 |
| 3.0 | 2.956 | 0.887 | 1.746 | 0.487 |

***TABLE 4:*** *ARL VALUES FOR EWMA-S AND CUSUM-S, WHEN N=15 AND 20 FOR UNIFORM (1,2)*

| $\delta$ | n=15 | | n=20 | |
|---|---|---|---|---|
| | EWMA-S $\lambda=0.01$ L=2.021 | CUSUM-S $k=2$ $h=4.259$ | EWMA-S $\lambda=0.01$ L=2.757 | CUSUM-S $k=1$ $h=7.955$ |
| 1.0 | 370.430 | 369.918 | 369.127 | 370.928 |
| 1.2 | 275.488 | 292.915 | 161.387 | 280.015 |
| 1.4 | 212.705 | 236.452 | 93.115 | 204.272 |
| 1.6 | 141.923 | 192.351 | 58.677 | 152.958 |
| 1.8 | 128.027 | 156.393 | 38.928 | 112.940 |
| 2.0 | 21.854 | 89.485 | 18.980 | 89.054 |
| 3.0 | 15.759 | 25.161 | 12.745 | 38.061 |

The performance evaluation for the proposed control chart and CUSUM control chart is under different situations, advocating the following:

- Non-parametric EWMA and non-parametric CUSUM based on Mood and Sukhatme statistics showed decreasing behavior of ARLs with increasing sample sizes (*n*) and shift sizes ( $\delta$ ).
- The proposed EWMA-S and EWMA-M control charts have the ability to detect changes in dispersion.
- The proposed EWMA-S control charts are more sensitive than CUSUM-S control charts for all magnitudes of shift.
- The proposed EWMA-M control charts are more sensitive than CUSUM-M control charts for monitoring small shifts, while CUSUM-M control charts are more sensitive for monitoring moderate to large shifts.

## 5 CONCLUSIONS

This paper proposed a non-parametric EWMA control chart based on Mood and Sukhatme statistics for efficient monitoring of dispersion in processes. The performance of the proposed control chart was evaluated in terms of average run length (ARL) using the Monte Claro simulation technique. The results indicate that EWMA charts based on Mood statistics are more sensitive than other charts.

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