Bayesian Control Charts Using Gamma Prior

Amin Shaka Aunali; Venkatesan D. and Michele Gallo

Abstract: The SPC (statistical process control) method is the most common method for the most efficient evaluation of the production process based on the sampling inspection and the chart performance. This problem is currently being studied in the economic planning of control charts and more recently in adaptive control charts. The traditional approach to the control charts' design uses the traditional structure of the control charts to determine the values of the parameters of the chart, that is, the sample size, sampling intervals, and control limits for meeting economic or statistical needs. Under the Bayesian approach, one can focus on defining the best control policy based on the posterior, thereby reducing the total expected costs in the finite time horizon or the average expected long-term costs. In this paper, the Bayesian control chart is developed using Bayesian approach by employing Gamma prior distribution, which is considered as generalization of exponential prior distribution.

Index Terms: Statistical Process Control, Bayesian control chart, control limits, Prior and Posterior distribution, Gamma distribution.

1. INTRODUCTION

SPC (statistical process control) is a statistical methodology used to monitor a process control and is a proactive approach. Unlike acceptance sampling plans, it is usually used as a result of the process where the nonconformities products have already occurred, SPC is used to signal the process when it is out of control, and then one has taken necessary preventive steps to regulate the production process. Specifically, it is an application of statistical methods for the collection of the data observations, charting the data, and then monitoring the variability of a particular process of interest over a time range relative to the lower and upper control limits normally set at three standard deviations below and above from the process target line. One of the interesting new approaches in industrial SQC (statistical quality control), is the Bayesian approach, based on subjective probability of prior information about the process in which many forms of uncertainty are included in the model and expressed. Bayesian context includes an overview of the belief in the process prior and the product after observing the data, also the ability to dynamically update the control chart components and estimated parameters as they occur for the collection of new data. Bayesian models are designed to define interesting system parameters as variables that behave in a distribution of unknown probability. The Bayesian model presents the structure of observable and unobservable variables, parameters and their dependencies. This structure keeps in view more flexibility and tries to solve the parameters of the production system. As opposed to the traditional approach, the Bayes approach combines sample information with prior information to address the uncertainty about the relevant parameter and is widely used in the process monitoring, evaluation of the process control. Ulrich (2002, 2007) used the Bayes framework to verify the target and the variability of control charts, respectively. Constructed control chart, proposed by Ulrich (2010), when the target of the process and its variance are unknown of normal distribution. In 2015, Aamir proposed the $\overline{X}$ chart using posterior control limits where he used the non-informative and informative priors' context to update the process mean and display the control limits. Recently, Aunali and Venkatesan (2017) discussed the comparison between Bayesian technique and classical control charts in the detection of shifts which is small. Later Aunali and Venkatesan (2019), they discussed on Bayesian approach in control charts techniques. Many researchers have also extended the MLE approach to a situation in which the prior is not known. The only hypothesis they formulated was that the shape of the changes belongs to a set of monotonous effects. This paper is presented as follows. The Bayesian approach for Gamma Prior Distribution is provided in section 2. Section 3 contains the control limits for Gamma prior distribution. Section 4 illustrates a numerical example, while section 5 describes the conclusion.

- *Amin Shaka Aunali, Department of Statistics, Annamalai University, India. E-mail: aunali473@gmail.com*
- *Venkatesan D., Department of Statistics, Annamalai University, India. E-mail: sit_san@hotmail.com*
- *Michele Gallo, Department of Social Science, University of Naples – L'Orientale, Naples, Italy. E-mail: mgallo@unior.it*

2 BAYESIAN APPROACH FOR GAMMA PRIOR DISTRIBUTION

The Bayesian approach to modeling and statistical analysis forms the basis for estimating a summary of prior distributions and current sample data. From the viewpoint of statistical theory, the statistical process is obtained by looking for the average efficiency of all possible data. This is the contradiction to the Bayesian process because more concentration is paid to the behaviour of the process in a given situation. In addition, unlike frequentist procedures, Bayesian approaches officially use the information available from sources other than statistical surveys. This information, past experiences, describes all potential values' probability distribution of the unknown parameter in a statistical model. Bayesian approaches are an absolute model for questionable statistical decisions and statistical conclusions. It is the frequently statistical methods, which are used to solve many of the problems faced by standard statistical methods and to increase the usefulness of statistical methods. Statistical inferences for the exact interest in the Bayesian structure are called adjustments of the unpredictability caused by evident values. Bayes' theorem clearly states how this adjustment should be implemented:

$$Posterior \propto \left(Likelihood \times Prior\right)$$

where, "Posterior" is later information related to the size of the sample data. "Prior" means knowing the interest's quantity of the probability distribution; and "Likelihood" is a prototype observation. Therefore, Bayes' theorem is attributed to three random quantities of X and scale and shape parameters α and β respectively as;

$$p(\alpha,\beta|X) \propto L(X) \times p(X|\alpha,\beta)$$

The prior distribution is combined with the likelihood function to give another distribution called posterior distribution $p(\alpha,\beta|X)$.

Fixing of prior distribution is a very difficult task in Bayesian approach and depends upon the problem and expertize, viz., non-informative and informative. This article takes into account informative prior, viz., the Gamma prior distribution is considered.

This can be well demonstrated by common relations.

$$E[\alpha,\beta] = E[E(\alpha,\beta|X)] \quad (1)$$

$$Var[\alpha,\beta] = E[Var(\alpha,\beta|X)] + Var[E(\alpha,\beta|X)] \quad (2)$$

The first equation shows the mean's prior is the expected of all possible posterior means (based on all possible process control datasets). The second shows that, on average, the variance's posterior is smaller than the prior one. The size of the differences will depend on the posterior means' variation.

The prior information can be quantified in the form of Gamma distribution, which is used to develop the control chart and is given by

$$P(X|\alpha,\beta) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1}; \qquad x \geq 0;\ \alpha,\beta > 0$$

If $X_1, X_2, \ldots, X_n$ be random sample then, one can write the likelihood function as

$$L \propto \left[\alpha^\beta / \Gamma(\beta)\right]^n e^{-\alpha \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left(x_i^{\beta-1}\right)$$

Using Bayes' theorem the joint posterior density function, after simplifications, can be expressed as:

$$P(\alpha,\beta|X) \propto \left[\left\{\frac{\alpha^\beta}{\Gamma(\beta)}\right\}^{n+1} e^{-\alpha \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\beta-1}\right] \times \left[\frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1}\right]$$

$$= \left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^{n+1} e^{-\alpha \sum_{i=1}^{n+1} x_i} \prod_{i=1}^{n+1} x_i^{\beta-1} \quad (3)$$

The posterior marginal r^th moments E(α,β|X^r) can be obtained, after simplifications, as follows

$$E(\alpha,\beta|X^r) = \int_0^\infty x^r \left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^{n+1} e^{-\alpha \sum_{i=1}^{n+1} x_i} \prod_{i=1}^{n+1} x_i^{\beta-1}\, dx$$

$$= \left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^{n+1} \frac{\Gamma(\beta+r+1)}{\alpha^{\beta+r-1}}$$

Further, after simplifications, one gets the first posterior moment and second posterior moment by substituting r = 1 and r = 2 respectively and are given by

$$E(\alpha,\beta|X) = \left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^{n+1} \frac{\Gamma(\beta+2)}{\alpha^\beta} = \beta(\beta+1)\left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^n \quad (4)$$

$$E(\alpha,\beta|X^2) = \left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^{n+1} \frac{\Gamma(\beta+3)}{\alpha^{\beta+1}} = \beta(\beta+1)(\beta+2)\frac{\alpha^{n\beta-1}}{[\Gamma(\beta)]^n} \quad (5)$$

and after simplifications, one can obtain the posterior variance by using equations (4) and (5) and is given by

$$V(a,b|X) = \beta(\beta+1)(\beta+2)\frac{\alpha^{n\beta-1}}{[\Gamma(\beta)]^n} - \beta^2(\beta+1)^2\left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^{2n} \quad (6)$$

$$= \frac{\beta(\beta+1)}{[\Gamma(\beta)]^{2n}}\left\{(\beta+2)\alpha^{n\beta-1}[\Gamma(\beta)]^n - \beta(\beta+1)\alpha^{2n\beta}\right\}$$

3 CONTROL LIMITS FOR GAMMA PRIOR DISTRIBUTION

Bayesian control chart illustrates the probability of two out of control levels and is used to make decisions about the process control's state. This requires more information about the structure of the process than the most popular control charts, but having this knowledge can bring real benefits. Another distribution according to the distribution's prior and data, so if the size of the sample is not large enough, the posterior variance is smaller than the prior variance.

Therefore, the limits of the process control are obtained from equations (4) and (6) as follows:

$$UCL = \beta(\beta+1)\left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^n + 3\sqrt{\frac{\beta(\beta+1)}{[\Gamma(\beta)]^{2n}}\left\{(\beta+2)\alpha^{n\beta-1}[\Gamma(\beta)]^n - \beta(\beta+1)\alpha^{2n\beta}\right\}}$$

$$CL = \beta(\beta+1)\left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^n$$

$$LCL = \beta(\beta+1)\left[\frac{\alpha^\beta}{\Gamma(\beta)}\right]^n - 3\sqrt{\frac{\beta(\beta+1)}{[\Gamma(\beta)]^{2n}}\left\{(\beta+2)\alpha^{n\beta-1}[\Gamma(\beta)]^n - \beta(\beta+1)\alpha^{2n\beta}\right\}}$$

4 NUMERICAL ILLUSTRATIONS

If an example regarding the illustration of control limits is considered for illustrating the applications of the method proposed in the above. The Bayesian control limits using prior Gamma distribution are obtained by using simulated data set for parameters α and β with different sample sizes n, the Tables 1, 2, 3 and 4 are constructed. Using the generated samples UCL and LCL are reported in below tables.

Control limits for Gamma Prior Distribution for different sample sizes

TABLE 1: *Sample size n=3*

α	β	CL	UCL	LCL
0.3	1	1.03E-52	2.20	0
	1.5	1.39E-78	1.32	0
	2	1.6E-104	0.72	0
	2.5	1.7E-130	0.38	0
	3	1.3E-186	0.07	0
0.5	1	1.58E-30	3.60	0
	1.5	2.63E-45	3.19	0
	2	3.73E-60	2.58	0
	2.5	4.84E-75	1.97	0
	3	4.6E-120	0.51	0
0.7	1	6.47E-16	4.71	0
	1.5	2.18E-23	5.37	0
	2	6.28E-31	5.64	0
	2.5	1.65E-38	5.62	0
	3	3.2E-76	1.96	0
1	1	2	6.24	0

TABLE 2: *Sample size n=5*

α	β	CL	UCL	LCL
0.3	1	1.03E-52	0.66	0
	1.5	1.39E-78	0.22	0
	2	1.6E-104	0.07	0
	2.5	1.7E-130	0.02	0
	3	1.3E-186	0.001	0
0.5	1	1.58E-30	1.83	0
	1.5	2.63E-45	1.14	0
	2	3.73E-60	0.65	0
	2.5	4.84E-75	0.35	0
	3	4.6E-120	0.03	0
0.7	1	6.47E-16	3.46	0
	1.5	2.18E-23	3.32	0
	2	6.28E-31	2.91	0
	2.5	1.65E-38	2.40	0
	3	3.2E-76	0.34	0
1	1	2	6.24	0

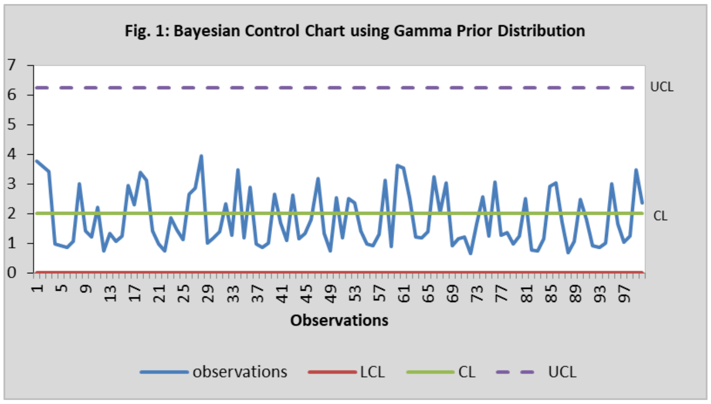
TABLE 3: *Sample size n=7*

α	β	CL	UCL	LCL
0.3	1	1.03E-52	0.20	0
	1.5	1.39E-78	0.04	0
	2	1.6E-104	0.01	0
	2.5	1.7E-130	0.001	0
	3	1.3E-186	0.0001	0
0.5	1	1.58E-30	0.92	0
	1.5	2.63E-45	0.40	0
	2	3.73E-60	0.16	0
	2.5	4.84E-75	0.06	0
	3	4.6E-120	0.02	0
0.7	1	6.47E-16	2.47	0
	1.5	2.18E-23	1.98	0
	2	6.28E-31	1.44	0
	2.5	1.65E-38	0.99	0
	3	3.2E-76	0.66	0
1	1	2	6.24	0

TABLE 4: *Sample size n=100*

α	β	CL	UCL	LCL
0.3	1	1.03E-52	9.63E-26	0
	1.5	1.39E-78	1.21E-38	0
	2	1.6E-104	1.38E-51	0
	2.5	1.7E-130	1.5E-64	0
	3	1.3E-186	1.39E-92	0
0.5	1	1.58E-30	9.23E-15	0
	1.5	2.63E-45	4.07E-22	0
	2	3.73E-60	1.64E-29	0
	2.5	4.84E-75	6.26E-37	0
	3	4.6E-120	2.05E-59	0
0.7	1	6.47E-16	1.58E-07	0
	1.5	2.18E-23	3.13E-11	0
	2	6.28E-31	5.68E-15	0
	2.5	1.65E-38	9.76E-19	0
	3	3.2E-76	1.44E-37	0
1	1	2	6.24	0

It is observed that from table 1 to table 4, for the fixed value of the parameter α, the control limits decrease whenever the parameter β increases. For the fixed value of the parameter β, the control limits increase whenever the parameter α increases. For the increasing of the sample size n, the deviation of the control limits decreases and is much closed to zero [Montgomery (2012), Duncan (1986)]. The UCL and LCL should be symmetric around the center line. However, in the case the LCL becomes negative; it can be rounded to zero. The control chart for the proposed Bayesian model is shown in the Fig. 1 for special case α = 9 and β = 10 with n = 100 as sample size.



It is also observed that the observations of the process control must be α, β > 0; otherwise the process may not be in control. That is, depends on the manufacturing products, the manufacturing engineers should fix the parameter's value and sample size based on what type of data they are working with. As one can see from Table 1 to Table 4, when the sample size increases, the width of the control limits is reducing.

5 CONCLUSION

Bayes' approach preceding knowledge about unknown parameters is included into the process by assigning Gamma distribution as a prior distribution to the parameters. In order to find the posterior distribution, the combination of information of the prior with the likelihood function has been made to construct the control charts using the Bayesian's approach. A future distribution of standardized mean is obtained from the posterior distribution. It is advisable to keep the sample size at the minimum level otherwise the deviations of control limits will be much closer to zero. In case where the quality engineer has the high measure observations and high sample size, it is advisable to use the parameters α = 1 and β = 1 with α, β > 0.

ACKNOWLEDGMENT

The authors wish to thank reviewers and Editor for their comments which helped to improve the work.

REFERENCES

[1] Aamir Saghir, "Phase-I design scheme for x-chart based on posterior distribution", Communications in Statistics: Theory and Methods, Vol. 44, No. 3, pp. 644–655, 2015.
[2] Amin S. Aunali and Venkatesan D. , "Comparison of Bayesian Method and Classical Charts in Detection of Small Shifts in the Control Charts", International Journal of Operations Research and Optimization, Vol. 8, No. 1, pp. 23-35, 2017.
[3] Amin S. Aunali and Venkatesan D., "Bayesian Approach in Control Charts Techniques", International Journal of Scientific Research in Mathematical and Statistical Sciences, Vol. 6, No. 2, pp. 217-221, 2019.
[4] Duncan, A. J., "Quality Control and Industrial Statistics", 5th ed., Irwin, Homewood, IL, 1986.
[5] Montgomery, D. C., "Introduction to Statistical Quality Control", 7th ed., Wiley, New York, NY, 2012.
[6] Ulrich Menzefricke, "On the evaluation of control chart limits based on predictive distributions", Communications in Statistics-Theory and Methods, Vol. 31, No. 8, pp. 1423–1440, 2002.
[7] Ulrich Menzefricke, "Control charts for the generalized variance based on its predictive distribution", Communications in Statistics-Theory and Methods, Vol. 36, No. 5, pp. 1031–1038, 2007.
[8] Ulrich Menzefricke , "Control charts for the variance and coefficient of variation based on their predictive distribution", Communications in Statistics-Theory and Methods, Vol. 39, No. 16, pp. 2930–2941, 2010.