Data Interpolation Using C^2 Rational Quartic Spline

Adila Harim, Samsul Ariffin Abdul Karim, Mahmod Othman, Azizan Saaban

Abstract: This study discusses the construction of new C^2 rational quartic spline scheme in from the quartic numerator and quadratic denominator with three parameters. Unlike the derivation of C^2 continuity for some existing splines interpolation, the proposed rational quartic spline can achieve C^2 continuity without the need to solve any system linear of equations. The main advantages of the proposed scheme is that we will obtain various C^2 rational just by manipulating the free parameters. We test the proposed scheme to interpolate data and we compare the performance against established schemes in terms of root mean square error (RMSE) and coefficient of determination (R^2). From numerical results, the proposed scheme is best than some existing schemes in terms of smaller RMSE and higher R^2 value.

Index Terms: Quartic polynomial, Rational Quartic Spline, Error Analysis, Absolute Error, RMSE, R^2, C^2 continuity

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1 INTRODUCTION

In the area of computer graphics (CG), computer-aided design (CAD) computer-aided geometric design (CAGD), the uses of data interpolation are important for interpolating curve that enables modify the final shape of the interpolated data to refine by manipulating the values of parameters. But the current demand in geometric modeling is higher degree of continuity, such as C^2 or more. In this study, we are particularly concerned with the construction of curve interpolation scheme with C^2 continuity. Some studies in this area are given in the following paragraphs:

Lamberti and Manni [11] described the global C^2 continuity shape preserving interpolation function using the cubic curves of the Hermite spline interpolation. The tension parameters were used to control the final shape of interpolating curves. The necessary and sufficient condition for convexity were given in detail. Motivated by the shortage of cubic spline interpolation, several researchers have proposed several types of rational spline interpolants in order to preserve the positivity of the positive data sets, monotonicity of the monotone data and the convexity of the convex data sets. Duan et al. [4] have constructed the weighted rational cubic spline though combination between rational cubic spline [1] and [2]. By choosing suitable values of the parameter then the weighted rational cubic spline will reduce to the scheme of [1] and [2] respectively. In their scheme, the degree smoothness attained is C^1. But the C^2 continuity is more possible by substituting the most suitable values of the parameters and solve the tri-diagonal system of the equations to calculate the first derivative values. Their schemes were also only applicable for equally spaced knots.

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Duan et al. [5] discussed the approximation properties of three's type of rational cubic spline i.e. rational cubic spline with linear denominator by [2], rational cubic spline with quadratic denominator by Gregory et al [3] and the rational cubic spline with cubic denominator by Sarfraz [13]. Their scheme assumed that the function to interpolate is C^2 continuity in the given interval. They concluded that the rational cubic spline with the linear denominator gives the best approximation to the function that is interpolated. Wang and tan [14] have constructed the C^2 continuity monotonicity preserving interpolation curves without the need to solve any nonlinear or linear system of equation. From several numerical testing, it may be not suitable to construct the monotonic interpolating curves with C^2 continuity. Since the derivative obtained from C^2 condition may not satisfy the sufficient condition for the monotonicity but the shape preserving interpolation for monotone data with C^2 continuity is always positive. Piah and Unsworth [12] extended the idea of Karim [10] to improve the sufficient condition for monotonicity of Wang and Tan [14]. Even though the sufficient conditions were improved as what the authors claimed, but it is still not suitable to construct the C^2 rational quartic spline that preserves the monotonicity of the data. Hence, Wang and Tan spline have their own limitation and not optional for C^2 monotonicity preserving interpolation. Han [6] described a new family of piecewise rational quartic interpolation (quartic/quadratic) with one tension parameter. The C^2 continuity can be achieved without requiring to solve the linear or nonlinear systems of equations for the unknown first derivative. The numerical result presented including the convergence analysis of the scheme, but the rational quartic spline was proposed neither have the positivity nor monotonicity properties. Therefore, it not suitable to preserve the shape of positive and monotone data. To overcome his scheme Zhu and Han [16] extended the rational quartic spline of by [10] introduced two tension parameters. The data dependent sufficient condition for C^2 rational quartic interpolant to be positive, convex and monotone are respectively derived on one parameter meanwhile the other is free to be utilized. Karim [9] has constructed C^2rational cubic spline (cubic/quadratic) with three parameter for the monotonic interpolating curve. Zhu [15] constructed new rational quartic spline (quartic/cubic) with four parameters. He claims that his proposed scheme achieves C^2 continuous without the need to modify the first derivative. But based on the mathematical derivation, his scheme is very difficult to use as well as require more computation compare with our proposed scheme is this study.

The purpose of this paper is to present a rational quartic spline interpolation with three parameters which can be used to construct C^2 interpolation curve for data interpolation without solving the linear system equation of the consistency equation for the values of the derivatives at the knots. The control parameters have been manipulated the final shape of the interpolation. The simple sufficient condition concerning the value of the second derivative for the C^2 interpolation curve and also for error analysis are developed.

This paper is organized as follows. In Section 1, we give some basic introduction as well as the related literature review. Section 2 is devoted to the rational quartic spline interpolant. In section 3, we discussed the sufficient condition for C^2 continuity and derivative estimation to find out the derivative for its function. Section 4, we discuss on data interpolation using the proposed scheme and dedicated for Results and Discussion including the comparison with existing schemes and default quartic polynomial using error analysis such as RMSE and R^2. The summary will be given in the final section.

2 Construction Rational Quartic Spline

This section discusses the construction rational quartic spline with three parameters including the definition from Harim et al. [7, 8]. This section will introduce the C^2 rational quartic spline interpolant with three parameters. Given the scalar data $\{(x_i, f_i), i = 0,1,..., n\}$ where $x_0 < x_1 < ... < x_n$ and the second derivative d_i, at the respective point $x_i, i = 0,1,..n$. Then the rational quartic spline with three parameters $\alpha_i, \beta_i > 0$ and $\gamma_i \geq 0$ on the interval $[x_i, x_{i+1}], i = 0,1,..., n-1$ is given by:

$$S_i(x) = \frac{P_i(\theta)}{Q_i(\theta)} \tag{1}$$

where

$$p_i(\theta) = (1-\theta)^4 \alpha_i f_i + (1-\theta)^3 \theta A_i + (1-\theta)^2 \theta^2 B_i + (1-\theta)\theta^3 C_i + \theta^4 \beta_i f_{i+1}$$

$$Q_i(\theta) = \alpha_i (1-\theta)^2 + \gamma_i (1-\theta)\theta + \beta_i \theta^2$$

$$s_i(x) = \frac{(1-\theta)^4 \alpha_i f_i + (1-\theta)^3 \theta A_i + (1-\theta)^2 \theta^2 B_i + (1-\theta)\theta^3 C_i + \theta^4 \beta_i f_{i+1}}{\alpha_i (1-\theta)^2 + \gamma_i (1-\theta)\theta + \beta_i \theta^2} \tag{2}$$

With $h_i = x_{i+1} - x_i$, $\Delta_i = \dfrac{f_{i+1} - f_i}{h_i}$ and a local variable, $\theta = \dfrac{x - x_i}{h_i}$ where the, $\theta \in [0,1]$

The following conditions, it will assure that the rational quartic spline interpolant in Equation (1) has C^2 continuity.

$$\begin{aligned} s(x_i) &= f_i \\ s(x_{i+1}) &= f_{i+1} \\ s^{(1)}(x_i) &= d_i \\ s^{(1)}(x_{i+1}) &= d_{i+1} \\ s^{(2)}(x_{i+}) &= s^{(2)}(x_{i-}) \end{aligned} \tag{3}$$

Where the $s^{(1)}(x_i)$ and $s^{(2)}(x_i)$ denote the first and second derivative respectively while for $s^{(2)}(x_{i+})$ and $s^{(2)}(x_{i-})$ shows the right and left hand side of the second derivative. By using condition in Equation (3), the C^2 rational quartic spline interpolant with three parameters defined in Equation (1) has the unknowns:

$$\begin{aligned} A_i &= (2\alpha_i + \gamma_i) f_i + \alpha_i h_i d_i \\ B_i &= (\alpha_i + \gamma_i) f_{i+1} + (\beta_i + \gamma_i) f_i \\ C_i &= (2\beta_i + \gamma_i) f_{i+1} - \beta_i h_i d_{i+1} \end{aligned} \tag{4}$$

The C^2 continuity, $s^{(2)}(x_{i+}) = s^{(2)}(x_{i-})$, $i = 1,2,3,..., n-1$ will give that

$$\frac{2[(\beta_{i-1} + \gamma_{i-1})(d_i - \Delta_{i-1})]}{h_{i-1}\beta_{i-1}} = \frac{2[(\alpha_i + \gamma_i)(\Delta_i - d_i)]}{h_i \alpha_i} \tag{5}$$

3 Derivative Estimation

When the data interpolation is not from the true function, the second derivative need to be determined. There is a common method to estimate the second derivative which is the arithmetic mean method (AMM) by Delbourgo and Gregory [3]. Suppose a data $\{(x_i, f_i), i = 0,1,2,..., n\}$ is considered that $x_0 < x_1 < ... < x_n$. The derivation of d_i at the point $x_i, i = 1,2,...,n$ are estimated as follow: Let assume $h_i = x_{i+1} - x_i$ and $\Delta_i = \dfrac{f_{i+1} - f_i}{h_i}$.

For endpoint derivatives d_1 and d_n described as:

$$s^{(1)}(x_1) = d_1, \quad s^{(1)}(x_n) = d_n \tag{6}$$

Which can be estimated by using the derivative estimation of Arithmetic Mean Method (AMM).

$$d_1 = \Delta_1 + (\Delta_1 - \Delta_2)\left(\frac{h_1}{h_1 + h_2}\right) \tag{7}$$

$$d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})\left(\frac{h_{n-1}}{h_{n-2} + h_{n-1}}\right) \tag{8}$$

At the other point, $x_i, i = 2,3,...,n-1$ the value d_i is calculated by using Equation (5)

$$d_i = \left[\frac{\alpha_i\beta_{i-1}\Delta_{i-1}h_i + \alpha_i\beta_{i-1}\Delta_i h_{i-1} + \alpha_i\gamma_{i-1}\Delta_{i-1}h_i + \beta_{i-1}\gamma_i\Delta_{i1}h_{i-1}}{\alpha_i\beta_{i-1}h_{i-1} + \alpha_i\beta_{i-1}h_2 + \alpha_i\gamma_{i-1}h_2 + \beta_{i-1}\gamma_i h_{i-1}}\right] \tag{9}$$

After some simplification we obtain

$$d_i = \left[\frac{h_i\alpha_i(\beta_{i-1}\Delta_{i-1} + \gamma_{i-1}\Delta_{i-1}) + h_{i-1}\beta_{i-1}(\alpha_i\Delta_i + \gamma_i\Delta_i)}{h_i\alpha_i(\beta_{i-1} + \gamma_{i-1}) + h_{i-1}\beta_{i-1}(\alpha_i + \gamma_i)}\right], \quad i = 2,3,...,n-1 \tag{10}$$

Meanwhile or Wang and Tan [14] scheme,

$$d_i = \frac{h_i\Delta_{i-1} + h_{i-1}\Delta_i}{\frac{2}{3}(h_{i-1} + h_i) + \frac{1}{3}\left(\frac{\delta_i}{\lambda_i}h_{i-1} + \frac{\lambda_{i-1}}{\delta_{i-1}}h_i\right)} \quad i = 2,3,... n-1 \tag{11}$$

But for zhu [15] scheme, using the derivative estimation of Arithmetic Mean Method (AMM).

$$d_i = \left(\frac{h_{i-1}\Delta_i + h_i \Delta_{i-1}}{h_{i-1} + h_i} \right) \quad i = 2,3,\ldots, n-1 \tag{12}$$

4 RESULT AND DISCUSSION

This section discusses the numerical example of the data interpolation using the proposed scheme including comparison with some existing schemes for the comparison in C^2 continuity. As a validation, we plotting the interpolating curve for each scheme and calculate error analysis such as absolute error, RMSE and R^2 for each example. We also calculate absolute error for different value of h_i. This section discusses the example of the data interpolation using the proposed scheme including comparison with existing schemes. As validation, we calculate error analysis for each data sets. Throughout this section, let *f(x)* represent true function for each example, *p(x)* considered as quartic polynomial, w(x) is Wang and Tan [14] scheme, *z(x)* considered as Zhu [15] scheme and *s(x)* represents the proposed scheme with suitable parameters value. Formula for absolute error, RMSE and R^2 are described as in Harim et al. [7] as:

1. Absolute error

$$A_e = |v_a - v_e| \tag{13}$$

Where

v_a = Approximation value.

v_e = Exact value from true function.

2. Root mean square error (RMSE)

$$\sqrt{\frac{\sum_{n=1}^{n} (v_e - v_a)^2}{n}} \tag{14}$$

3. coefficient of determination (R^2)

$$R^2 = \frac{n(\sum xy) - \sum x \sum y}{\sqrt{(n\sum x^2 - (\sum x)^2)(n\sum y^2 - (\sum y)^2)}} \tag{15}$$

Where

n =Number of samples

x =The value of data x

y =The value of data y

4.1 Example 1

Table 1 shows function of $f(x) = e^x$ for $x = \{0,1,2,3,4\}$ and C^2 derivative for each scheme.

TABLE 1: DATA POINT

x_i	0	1	2	3	4
y_i	1	2.71823	7.3891	20.0855	54.5982
d_i	0.2420	3.1945	8.6836	23.6045	45.4207
d_i (Wang and Tan [14])	0.2420	2.7382	7.4431	20.2325	45.4207
d_i (Zhu [15])	0.2420	3.1945	8.6836	23.6045	45.4207
d_i (proposed Scheme)	0.2420	2.9836	8.11038	22.0462	45.4207

TABLE 2: MAXIMUM ERROR FOR EXAMPLE 1

Scheme		Value of parameter				Maximum Error
		α_i	β_i	γ_i	e_i	
Quartic Polynomial	*p(x)*	1	1	2	-	2.773692
Wang and Tan[13]	*w(x)*	1	0.3	-	-	5.161171
Zhu[14]	*z(x)*	1	1	0.3	1	1.777924
Quartic Spline	*s1(x)*	1	1	0.03	-	1.499161
	s2(x)	1	1	0.3	-	1.794232
	s3(x)	1	1	3	-	3.048759

TABLE 3: ERROR ANALYSIS

Scheme		Error analysis	
		RMSE	R^2
Quartic Polynomial	*p(x)*	0.95455	0.800131
Wang and Tan[13]	*w(x)*	1.73901	0.805687
Zhu[14]	*z(x)*	0.625889	0.797214
Quartic Spline	*s1(x)*	0.51369	0.793111
	s2(x)	0.59970	0.794729
	s3(x)	1.06817	0.801698

TABLE 4: MAXIMUM VALUE OF ABSOLUTE ERROR FOR DIFFERENT VALUE FOR h_i

Scheme		Values of h_i		
		$h_i = 0.05$	$h_i = 0.1$	$h_i = 0.2$
Quartic Polynomial	*p(x)*	2.812673	2.773692	2.773692
Wang and Tan [14]	*w(x)*	5.161171	5.161171	5.161171
Zhu [15]	*z(x)*	1.777924	1.777924	1.777924
Quartic Spline	*s1(x)*	1.499161	1.499161	1.499161
	s2(x)	1.794232	1.794232	1.794232
	s3(x)	3.098658	3.0576628	3.048759

Table 2 shows the maximum value for absolute error that has been calculated in Harim et al. [7]. When $\gamma_i = 0.3$, Wang and Tan [14] has the highest value for maximum absolute error followed by the proposed scheme. Zhu [15] scheme have the smallest value but when $\gamma_i = 0.03$, the proposed scheme has the smallest value of maximum absolute error. For the proposed scheme, we can conclude the smallest value, the smallest maximum value for the absolute error. Table 3 shows error analysis for RMSE and R^2. Scheme of Wang and Tan still have the highest values for error analysis and followed by s3(x) for the proposed scheme. Table 4 shows the maximum value for absolute error for a different value h_i. The maximum value for absolute error are the same for both $h_i = 0.1$ and

$h_i = 0.2$ but s3(x) has a slightly different value for $h_i = 0.05$.

4.2 Example 2

Data for Table 5 show function of $f(x) = \cos\left(\frac{\pi x}{2}\right)$ for x= {0, 0.25, 0.5, 0.75, 1}. Table 5 shows the data points and derivative values using Arithmetic Mean Method (AMM) and also the C^2 derivative for each scheme.

***TABLE 5**: SAMPLING DATA FOR EXAMPLE 2*

x_i	0	0.2500	0.5000	0.7500	1.000
y_i	1	0.9239	0.7071	0.3827	0
d_i (AMM)	-0.0232	-0.6390	-1.1808	-1.5428	-1.6473
d_i (Wang and Tan [14])	-0.0232	-0.6390	-1.1808	-1.5428	-1.6473
d_i (Zhu [15])	-0.0232	-0.5858	-1.0824	-1.4142	-1.6473
d_i (proposed Scheme)	-0.0232	-0.6675	-1.1449	-1.4481	-1.6473

***TABLE 6:** MAXIMUM VALUE FOR EXAMPLE 2*

Scheme		Value of parameter				Maximum Error
		α_i	β_i	γ_i	e_i	
Quartic Polynomial	*p(x)*	1	1	2	-	0.011520
Wang and Tan[14]	*w(x)*	1	0.2	-	-	0.035346
Zhu[15]	*z(x)*	1	1	0.2	1	0.010427
Quartic Spline	*s1(x)*	1	1	0.2	-	0.005527
	s2(x)	0.2	1	1	-	0.011920
	s3(x)	1	0.2	1	-	0.009789

***TABLE 7:** ERROR ANALYSIS*

Scheme		Error analysis	
		RMSE	R^2
Quartic Polynomial	*p(x)*	0.00487	0.953963
Wang and Tan[14]	*w(x)*	0.01901	0.956996
Zhu[15]	*z(x)*	0.00459	0.953607
Quartic Spline	*s1(x)*	0.00300	0.953374
	s2(x)	0.00555	0.954376
	s3(x)	0.00414	0.953457

***TABLE 8:** MAXIMUM VALUE OF ABSOLUTE ERROR FOR DIFFERENT VALUE FOR h_i*

Scheme		Values of h_i		
		$h_i = 0.05$	$h_i = 0.1$	$h_i = 0.2$
Quartic Polynomial	*p(x)*	0.011597	0.011520	0.011520
Wang and Tan [14]	*w(x)*	0.036527	0.036527	0.035346
Zhu [15]	*z(x)*	0.011179	0.011179	0.010427
Quartic Spline	*s1(x)*	0.006158	0.006158	0.005527
	s2(x)	0.011202	0.011920	0.011920
	s3(x)	0.010604	0.010604	0.009789

Table 6 shows the maximum value for absolute error. Wang and Tan [14] has the highest value for maximum absolute error followed by the proposed scheme when $\alpha_i = 0.2$. The proposed scheme with $\gamma_i = 0.2$ has the smallest value of the maximum value for the absolute error. From that table, for the proposed scheme, we can conclude the smallest value, the smallest maximum value for the absolute error. Table 7 shows error analysis for RMSE and R^2 for Example 2. For RMSE, Scheme of Wang and Tan [14] has maximum value followed by the quartic polynomial and zhu [15] meanwhile the values for R^2 have slightly different for all schemes. Table 8 shows the maximum value for absolute error for a different value h_i . For quartic polynomial, $h_i = 0.1$ and $h_i = 0.2$ have same value for maximum value but different value for $h_i = 0.05$ meanwhile for Wang and Tan [14] and Zhu [15] scheme, for $h_i = 0.1$ and $h_i = 0.05$ have same value for maximum value but different value for $h_i = 0.2$. For the proposed scheme, s1(x) has the smallest value for maximum value absolute error compared with s2(x) and s3(x).

5 CONCLUSION

In this study, a new C2 continuity of rational quartic spline (quartic/quadratic) with three parameters are constructed for curve and data interpolation. From the error analysis, we observed that the proposed scheme is more suitable for all data sets compared with some existing schemes. It is also obvious from the numerical examples that the absolute error, RMSE and R2 respectively, obtained from the proposed scheme are more accurate as compared to existing scheme. Error estimation shows that the proposed scheme gives high accuracy with smaller error as compared to quartic polynomial, Wang and Tan [14], and Zhu [15] schemes.

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