Analytical Expression And Numerical Solutions Of Average Run Length For SARMA(P,Q)$_L$ Process On CUSUM Procedure

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Abstract: Statistical process control (SPC) is used to develop and improve the quality of the processes Cumulative sum (CUSUM) chart is an effective tool in SPC for detecting change in a process mean. The main purpose of this paper is to present the analytical expression and the numerical integration of average run length (ARL) of CUSUM chart when observations are seasonal autoregressive and moving average; SARMA(P,Q)$_L$ with exponential white noise. In addition, we compare the accuracy of the average run length obtained from the analytical formula with the results obtained from numerical integration by considering the absolute percentage difference and the computational time to process the data.

Index Terms: Seasonal Autoregressive and Moving Average Process (SARMA), Cumulative Sum (CUSUM), Average Run Length (ARL).

1 INTRODUCTION

STATISTICAL process control (SPC) is an approach to evaluate processes and improve quality in industrial process. Control charts are one of efficient tools of SPC for detecting changes in mean or variations of the processes. It used in monitoring, controlling and improving quality in area. Control charts are prosperously applied in engineering, public health, economics, finance, medicine and in other areas of applications. Traditional control charts are also known Shewhart chart. Walter A. Shewhart [1] proposed Shewhart chart and it is sensitive to detecting relatively large shifts in the process ($\geq 1.5\sigma$). Alternative control charts, such as the Cumulative Sum (CUSUM) chart and the Exponentially Weighted Moving Average (EWMA) chart have been developed to compensate for the inefficiency of Shewhart chart. The CUSUM chart is the good alternative when we are interested in detecting small shifts ($< 1.5\sigma$). Page [2] first introduced the CUSUM chart and used to monitor product quality and detect the occurrence of special causes that may be indicated to out of control state. There are many researchers such as Brodsky and Darkhovsky [3], Basseville and Nikiforov [4] for an introduction to CUSUM chart and its applications. Roberts S.W. [5] originally developed EWMA chart and this approach was based on weighting current samples more heavily than past data, with each sample being assigned a weight and with the weights exponentially decreasing from the present to the past. According to the assumption of SPC techniques, the process should be an independent. In real applications, there are several situations in which the processes are autocorrelated such as in chemical process. Therefore, it is important to be able to calculate the average run length (ARL) when observations are correlations. The average run length (ARL) is a traditional measurement of control chart's performance. Generally, the expected number of observations taken from an in-control process until the control charts falsely signal out-of-control is the ARL_0. And the expected number of observations taken from an out-of-control process until the control chart signals that the process out-of-control is the ARL_1. There are several methods that can be used to find the ARL such as Markov Chain approach (MCA), Integral Equation approach (IE) and Monte Carlo simulations (MC). Sukparungsee [6] have used the Martingale approach to derive the analytical formulas of the Average Run Length (ARL) and the Average Delay (AD) in the case of Gaussian and some Non-Gaussian distributions. Areepong [7] derived an analytical expression for the ARL of EWMA control chart when observations are exponential distribution. Mititelu [8] presented an analytical solution for the ARL by using the Fredholm integral equation approach for CUSUM chart when observations are hyperexponential distribution. Busaba [9] was analyzed the explicit formulas of ARL for CUSUM chart, its corresponding in the case of a stationary first order autoregressive; AR(1) process with exponential white noise. Phanyaem [10] used the integral equation technique to derive the explicit formula for the ARL of CUSUM chart for an autoregressive and moving average; ARMA(1,1) process with exponential white noise. Petcharat [11] derived an analytical expression for the ARL of CUSUM chart when the observations are moving average of order q (MA(q)) process. Paichit [12] proposed the numerical integration of ARL for EWMA chart when observations are SARMA(P,Q)$_L$. Consequently, the objective of this paper is to propose the analytical expression and numerical integration of ARL for CUSUM chart for the seasonal autoregressive and moving average; SARMA(P,Q)$_L$ process with exponential white noise. The organization of this paper is as follows: In Section 2, the characteristic of CUSUM chart for SARMA(P,Q)$_L$ process is presented. The proposed explicit formula for ARL of CUSUM chart is presented in Section 3. The numerical integration of ARL for CUSUM chart is proposed in Section 4. In Section 5, we compared the results from the analytical expression with the results from the numerical solution of an integral equation. Finally, Section 6 provides a conclusion.

2 THE CUSUM PROCEDURE FOR SARMA(P,Q)$_L$

This section we describe the characteristics of CUSUM chart which was originally introduced by Page [2] in quality control in order to detect a small shift in the mean of a process as soon

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as it occurs. Let X_1, X_2, … be a sequential observation of some process are modeled as seasonal autoregressive and moving average; $SARMA(P,Q)_L$ with exponential white noise.

Let C_t be the CUSUM statistics, the recursive CUSUM based on $SARMA(P,Q)_L$ process is defined as:

$$C_t = \max(C_{t-1} + X_t - a,\ 0)\ ;\ t = 1,2,.... \quad (1)$$

where X_t is a sequence of $SARMA(P,Q)_L$ process, $C_0 = u$ is an initial value, a is a reference value of CUSUM chart.

The recursive equation of $SARMA(P,Q)_L$ process with exponential white noise is defined as:

$$X_t = \mu + \varphi_1 X_{t-L} + ... + \varphi_P X_{t-PL} + \xi_t - \theta_1\xi_{t-L} - ... - \theta_Q\xi_{t-QL} \quad (2)$$

where ξ_t is an exponential white noise

φ_i is an autoregressive coefficient, $i = 1, 2,…, P$

θ_i is a moving average coefficient, $i = 1, 2,…, Q$.

Let ξ_{t-L}, ξ_{t-2L}, …, ξ_{t-QL} and X_{t-L}, X_{t-2L}, …, X_{t-PL} be an initial value of $SARMA(P,Q)_L$ process.

The stopping time of CUSUM chart is given by

$$\tau_h = \inf\{t > 0;\ C_t > h\},\ h > u \quad (3)$$

where τ_h is the stopping time

h is the constant parameter as upper control limit.

Suppose H(u) denote the ARL for $SARMA(P,Q)_L$ process with an initial value $C_0 = u$. To define H(u) as follows

$$\text{ARL} = H(u) = \mathbf{E}_\infty(\tau_h) < \infty. \quad (4)$$

where $\mathbf{E}_\infty$(.) is the expectation under density function f(x,α).

3 EXPLICIT FORMULAS OF AVERAGE RUN LENGTH

We derive exact solution of Fredholm Integral Equation of the second kind, which is called explicit formulas of ARL for $SARMA(P,Q)_L$ process based on CUSUM chart. Let the notations $\mathbf{P}_c$ denote the probability measure and $\mathbf{E}_c$ denote expectation corresponding to initial value $C_0 = u$.

The solution of integral equation can be written as:

$$H(u) = 1 + \mathbf{E}_c\left[I\{0 < C_1 < h\}H(C_1)\right] + \mathbf{P}_c\{C_1 = 0\}H(0). \quad (5)$$

Therefore, the integral equation of CUSUM control chart is

$$H(u) = 1 + \alpha e^{\alpha(u-a+\mu+\varphi_1 X_{t-L}+...+\varphi_P X_{t-PL}-\theta_1\xi_{t-L}-...-\theta_Q\xi_{t-QL}} \int_0^h H(y)e^{-\alpha y}dy$$
$$+\left(1 - e^{-\alpha(a-u-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})}\right)H(0). \quad (6)$$

Theorem 3.1 The explicit formulas of ARL for $SARMA(P,Q)_L$ process is

$$H(u) = e^{\alpha h}(1 + e^{\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} - \alpha h) - e^{\alpha u}$$

Proof.

To consider the integral equation

$$H(u) = 1 + \alpha e^{\alpha(u-a+\mu+\varphi_1 X_{t-L}+...+\varphi_P X_{t-PL}-\theta_1\xi_{t-L}-...-\theta_Q\xi_{t-QL}} \int_0^h H(y)e^{-\alpha y}dy$$

$$+\left(1 - e^{-\alpha(a-u-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})}\right)H(0);\ u \in [0,a]$$

Let d be a constant as $d = \int_0^h H(y)e^{-\alpha y}dy.$ Thus, the function H(u) can be written as

$$H(u) = 1 + \alpha e^{\alpha(u-a+\mu+\varphi_1 X_{t-L}+...+\varphi_P X_{t-PL}-\theta_1\xi_{t-L}-...-\theta_Q\xi_{t-QL})}d$$
$$+\left(1 - e^{-\alpha(a-u-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})}\right)H(0)$$

(7)

For the case $u = 0$, thus we have the function $H(0)$ as following form:

$$H(0) = 1 + \alpha e^{\alpha(-a+\mu+\varphi_1 X_{t-L}+...+\varphi_P X_{t-PL}-\theta_1\xi_{t-L}-...-\theta_Q\xi_{t-QL})}d$$
$$+\left(1 - e^{-\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})}\right)H(0)$$
$$= e^{\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} + \alpha d$$

Hence, substituting $H(0)$ into Equation (7) as following form:

$$H(u) = 1 + \alpha e^{\alpha(u-a+\mu+\varphi_1 X_{t-L}+...+\varphi_P X_{t-PL}-\theta_1\xi_{t-L}-...-\theta_Q\xi_{t-QL})}d$$
$$+\left(1 - e^{-\alpha(a-u-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})}\right)$$
$$* e^{\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} + \alpha d$$
$$= 1 + e^{\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} + \alpha d - e^{\alpha u}. \quad (8)$$

To find the constant d as following form:

$$d = \int_0^h H(y)e^{-\alpha y}dy.$$
$$= \int_0^h (1 + \alpha d + e^{\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} - e^{\alpha y})e^{-\alpha y}dy.$$
$$= \int_0^h (1 + \alpha d + e^{\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} - e^{\alpha y})dy - \int_0^h e^{\alpha y - \alpha y}dy.$$
$$= \frac{e^{\alpha h}}{\alpha}(1 - e^{-\alpha h})(1 + e^{\lambda(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})}) - he^{\alpha h}.$$

Consequently, the explicit formulas obtained by substituting the constant d into Equation (8) as following form:

$$H(u) = e^{\alpha h}(1 + e^{\alpha(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} - \alpha h) - e^{\alpha u}.$$

As mentioned above, the value of the parameter α is equal to α_0 when the process is in-control. Therefore, we obtain the explicit formula for ARL_0 as follows:

$$ARL_0 = e^{\alpha_0 h}(1 + e^{\alpha_0(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} - \alpha_0 h) - e^{\alpha_0 u}.$$

On the other hand, the process is out-of-control, the value of parameter α is equal to α_1; where $\alpha_1 = \alpha_0(1+\delta)$. The explicit formula for ARL_1 can be written as follows:

$$ARL_1 = e^{\alpha_1 h}(1 + e^{\alpha_1(a-\mu-\varphi_1 X_{t-L}-...-\varphi_P X_{t-PL}+\theta_1\xi_{t-L}+...+\theta_Q\xi_{t-QL})} - \alpha_1 h) - e^{\alpha_1 u}.$$

where $0 \le \varphi_i \le 1$ is autoregressive coefficient and $0 \le \theta_i \le 1$ is the moving average coefficient and α is a parameter of the exponential distribution, and h is the constant parameter as upper control limit and X_0, ζ_0 are the initial values.

4 NUMERICAL INTEGRATION OF AVERAGE RUN LENGTH

In this section, we propose the scheme to evaluate numerically the solution of the integral equation by using Gauss-Legendre quadrature rule [13]. Since $y \sim Exp(\alpha)$, then

$$F(y) = 1 - e^{-\alpha y} \text{ and } f(y) = \frac{dF(y)}{du} = \alpha e^{-\alpha y}.$$

By quadrature rule approach we can approximate the integral by finite sum of areas of rectangles with base h/m with heights chosen as the values of f at midpoints of intervals of length h/m beginning at zero. Particularly, once the choice of a quadrature rule is made, the interval $[0,h]$ is divided into a partition $0 \le a_1 \le a_2 \le ... \le a_m \le h$ and a set of constant weights $w_j = h/m \ge 0.$

The approximation for an integral equation as follows

$$\int_0^h W(y)F(y)dy \approx \sum_{j=1}^{m} w_j F(a_j), \tag{9}$$

where W(y) and F(y) are given functions, a_j is a set of point and w_j is a weight define different quadrature rules.

Firstly, the integral equation (6) of CUSUM chart can be rewritten as:

$$H(u) = 1 + H(0)F(a-u-\mu-\varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL} + \theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL})$$
$$+ \int_0^h H(y) f(y+a-u-\mu-\varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL} + \theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL})dy,$$

where $F(u) = 1 - e^{-\alpha u}$ and $f(u) = \frac{dF(u)}{du} = \alpha e^{-\alpha u}.$

Let $\tilde{H}(u)$ denote the approximated solution of $H(u)$ by using the quadrature rule, then the integral equation can be approximated by

$$\tilde{H}(a_i) = 1 + \tilde{H}(a_1)F(a-a_i-\mu-\varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL}$$
$$+\theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL}) + \sum_{j=1}^{m} w_j \tilde{H}(a_j) f(a_j + a - a_i$$
$$-\mu - \varphi_1 X_{t-L} - .. - \varphi_P X_{t-PL} + \theta_1\xi_{t-L} + .. + \theta_Q\xi_{t-QL}) \tag{10}$$

Equation (10) is a system of m linear equations in the unknowns $\tilde{H}(a_1), \tilde{H}(a_2), ..., \tilde{H}(a_m)$ can be re-arranged as

$$\tilde{H}(a_1) = 1 + \tilde{H}(a_1)\big[F(a-a_1-\mu-\varphi_1 X_{t-L} - .. - \varphi_P X_{t-PL} + \theta_1\xi_{t-L}$$
$$+ ... + \theta_Q\xi_{t-QL}) + w_1 f(a-\mu-\varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL}$$
$$+\theta_1\xi_{t-1} + ... + \theta_Q\xi_{t-QL}) + \sum_{j=2}^{m} w_j \tilde{H}(a_j) f(a_j + a - a_1$$
$$-\mu - \varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL} + \theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL})$$

$$\tilde{H}(a_2) = 1 + \tilde{H}(a_1)\big[F(a-a_2-\mu-\varphi_1 X_{t-L} - .. - \varphi_P X_{t-PL} + \theta_1\xi_{t-L}$$
$$+ .. + \theta_Q\xi_{t-QL}) + w_1 f(a_1 + a - a_2 - \mu - \varphi_1 X_{t-L} - .. - \varphi_P X_{t-PL}$$
$$+\theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL}) + \sum_{j=2}^{m} w_j \tilde{H}(a_j) f(a_j + a - a_2$$
$$-\mu - \varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL} + \theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL})$$

$$\vdots$$

$$\tilde{H}(a_m) = 1 + \tilde{H}(a_1)\big[F(a-a_m-\mu-\varphi_1 X_{t-L} - .. - \varphi_P X_{t-PQ} + \theta_1\xi_{t-L}$$
$$+ .. + \theta_Q\xi_{t-QL}) + w_1 f(a_1 + a - a_m - \mu - \varphi_1 X_{t-L} - .. - \varphi_P X_{t-PL}$$
$$+\theta_1\xi_{t-1} + ... + \theta_Q\xi_{t-QL}) + \sum_{j=2}^{m} w_j \tilde{H}(a_j) f(a_j + a - a_m$$
$$-\mu - \varphi_1 X_{t-1} - ... - \varphi_P X_{t-PL} + \theta_1\xi_{t-1} + ... + \theta_Q\xi_{t-QL})$$

or in matrix form as follows:

$$\mathbf{H}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{H}_{m\times 1} \tag{8}$$

where $\mathbf{H}_{m\times 1} = \begin{pmatrix} \tilde{H}(a_1) \\ \tilde{H}(a_2) \\ \vdots \\ \tilde{H}(a_m) \end{pmatrix}$, $\mathbf{1}_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

and I_m = diag(1,1,…,1) is the unit matrix of order m. If there exists $(I_m - R_{mxm})^{-1}$, then the solution of matrix equation as follows

$$\mathbf{H}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}.$$

Solving set of equations for the approximate values of $\tilde{H}(a_1), \tilde{H}(a_2), .., \tilde{H}(a_m)$. Therefore, the numerical integration of ARL for CUSUM chart based on $SARMA(P,Q)_L$ as follows:

$$\tilde{H}(u) = 1 + \tilde{H}(a_1)F(a-u-\mu-\varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL}$$
$$+\theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL}) + \sum_{j=1}^{m} w_j \tilde{H}(a_j) f(a_j + a - u$$
$$-\mu - \varphi_1 X_{t-L} - ... - \varphi_P X_{t-PL} + \theta_1\xi_{t-L} + ... + \theta_Q\xi_{t-QL}) \tag{11}$$

with $w_j = \frac{h}{m}$ and $a_j = \frac{h}{m}\left(j - \frac{1}{2}\right); j = 1, 2, ..., m.$

TABLE 2

COMPARISON OF ARL COMPUTED USING EXPLICIT FORMULAS AGAINST NUMERICAL INTEGRAL EQUATION FOR $SARMA(2,1)_{12}$ PROCESS WITH THE PARAMETER a = 2.5 AND h = 3.5289

Shift size δ	Explicit Formulas Method	Numerical Integral Equation Method	Absolute Percentage Difference
0.00	370.012	370.045 (53.04)	0.009
0.01	347.127	347.321 (54.32)	0.056
0.03	306.642	306.313 (53.79)	0.107
0.05	272.149	272.198 (52.43)	0.018
0.10	205.796	205.706 (54.58)	0.044
0.20	126.203	126.556 (50.35)	0.280
0.30	83.5137	83.5547 (53.42)	0.049
0.40	58.7071	58.7094 (52.51)	0.004
0.50	43.3300	43.3364 (54.58)	0.015
1.00	15.3108	15.2266 (55.21)	0.550
1.50	8.42625	8.42243 (56.75)	0.045

5 NUMERICAL RESULT

The performance of a control chart is generally measured by the ARL, when the process is under control, the ARL should be sufficiently large to avoid false alarms. When the process is out of control, the ARL should be small to detect shifts. In this section, we compare the ARL obtained from the analytical expression with the results obtained from the numerical integral equation with m = 800 nodes. In Table 1-3 show the results of comparison for ARL_0 and ARL_1 when observations are $SARMA(1,2)_{12}$ and $SARMA(2,2)_{12}$ processes with exponential white noise. Given in-control parameter value $\alpha_0 = 1$ and out-of-control parameter values $\alpha_1 = \alpha_0(1+\delta)$ where $\delta = 0.00$, 0.01, 0.03, 0.05, 0.10, 0.20, 0.30, 0.40, 0.50, 1.00 and 1.50 respectively. The value of parameter a and h for CUSUM chart was chosen by given desired $ARL_0 = 370$. In Table 1, we set the values of parameter for $SARMA(1,2)_{12}$ process with $\varphi_1 = 0.10$, $\theta_1 = 0.30$ and $\theta_2 = 0.20$, then the parameter of CUSUM chart are a = 2.50 and h = 3.1466. The numerical values for ARL_0 are computed for an in-control parameter value $\alpha_0 = 1$ and numerical values for ARL_1 are computed for range of out-of-control parameter values $\alpha_1 = \alpha_0(1+\delta)$, δ = 0.01, 0.02, 0.03, 0.04, 0.05, 0.10, 0.20, 0.30, 0.40 and 0.50. Similarity, in Table 2, we set the values of parameter of $SARMA(2,1)_{12}$ process with $\varphi_1 = 0.10$, $\varphi_2 = 0.10$ and $\theta_1 = 0.30$, then the parameter of CUSUM chart are a = 2.50 and h = 3.5289. Finally in Table 3, we determine the values of parameter of $SARMA(2,2)_{12}$ process with $\varphi_1 = 0.10$, $\varphi_2 = 0.10$, $\theta_1 = 0.10$, and $\theta_2 = 0.10$ then the parameter of CUSUM chart are a = 2.50 and h = 3.6681.

TABLE 1

COMPARISON OF ARL COMPUTED USING EXPLICIT FORMULAS AGAINST NUMERICAL INTEGRAL EQUATION FOR $SARMA(1,2)_{12}$ PROCESS WITH THE PARAMETER $a = 2.5$ AND $h = 3.1466$

Shift size δ	Explicit Formulas Method	Numerical Integral Equation Method	Absolute Percentage Difference
0.00	370.032	370.013 (54.15)	0.005
0.01	347.731	347.093 (50.89)	0.183
0.03	308.174	307.623 (52.86)	0.179
0.05	274.357	273.878 (54.72)	0.175
0.10	208.949	208.605 (49.76)	0.165
0.20	129.659	129.468 (49.95)	0.147
0.30	86.5740	86.4598 (49.76)	0.132
0.40	61.2628	61.1899 (51.43)	0.119
0.50	45.4292	45.3803 (53.31)	0.108
1.00	16.1455	16.1344 (52.94)	0.069
1.50	8.82817	8.8240 (51.83)	0.047

a. The values in parentheses are CPU times in numerical integration methods (Minutes)

The results from Table 1 to Table 3, show that the absolute percentage difference are less than 1.0% by the numerical integration for the case of division points m = 800, and the CPU times of approximately 50-60 minutes. However, the CPU times from the proposed explicit formulas are less than 1 second.

TABLE 3

COMPARISON OF ARL COMPUTED USING EXPLICIT FORMULAS AGAINST NUMERICAL INTEGRAL EQUATION FOR $SARMA(2,2)_{12}$ PROCESS WITH THE PARAMETER $a = 2.5$ AND $h = 3.6681$

Shift size δ	Explicit Formulas Method	Numerical Integral Equation Method	Absolute Percentage Difference
0.00	370.031	370.015 (55.73)	0.004
0.01	346.883	346.891 (53.89)	0.002
0.03	305.978	305.235 (55.93)	0.243
0.05	271.182	271.135 (56.73)	0.017
0.10	204.414	204.243 (55.25)	0.084
0.20	124.706	124.738 (58.45)	0.026
0.30	82.2039	82.2440 (59.17)	0.049
0.40	57.6263	57.6336 (58.93)	0.013
0.50	42.4521	42.4649 (57.75)	0.030
1.00	14.9775	14.9867 (56.35)	0.061
1.50	8.27150	8.27245 (55.23)	0.011

6 CONCLUSION

In this paper, we have proposed the explicit formulas for the Average Run Length (ARL) of Cumulative Sum (CUSUM) chart for $SARMA(P,Q)_L$ process with exponential distribution white noise. We derived the explicit formulas by using integral equation technique. In addition, we developed numerical integration for evaluating the ARL of CUSUM charts for $SARMA(P,Q)_L$ process with exponential distribution white noise by using Gauss-Legendre quadrature rule. The accuracy of formula is verified by the absolute percentage difference between the explicit formulas solution and the numerical integration solution. The results of research show that the ARL from explicit formulas is close to the numerical integration. Therefore, the proposed analytical formulas are sufficiently high accuracy and easy to calculate in comparison with numerical integration technique. In addition, the CPU times for evaluating the proposed explicit formulas takes less than 1 second while the numerical integration method takes approximately 50-60 minutes in the case of $SARMA(P,Q)_L$ process. Thus, the explicit formulas can reduce in the computational time much better than the numerical integration.

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REFERENCES

[1] W.A. Shewhart, Economic Control of Quality of Manufactured Product. Van Nostrand, New York, 1993.
[2] E.S. Page, "Continuous Inspection Schemes," Biometrika, vol. 41, pp. 100-114, 1954.
[3] B.E. Brodsky and B.S. Darkhovsky, "Nonparametric Methods in Change-Point Problems," Mathematics and its applications. 1993; 243.
[4] M. Basseville and I.V. Nikiforov, Detection of abrupt changes: Theory and application, New Jersey: Prentice Hall, 1993.

[5] S.W. Roberts, "Control Chart Tests Based on Geometric Moving Average," Technometrics, vol. 1, pp. 239-250, 1959.
[6] S. Sukparungsee and A.A. Novikov, "On EWMA Procedure for Detection of a Change in Observations via Martingale Approach," An International Journal of Science and Applied Science, vol. 6, pp. 373-380, 2006.
[7] Y. Areepong and A.A. Novikov, "An Integral Equation Approach for Analysis of Control Charts," Ph.D. dissertation, University of Technology, Australia, 2009.
[8] G. Mititelu, Y. Areepong, S. Sukparungsee and A.A. Novikov, "Explicit Analytical Solutions for the Average Run Length of CUSUM and EWMA Charts," East West Journal of Mathematics, vol. 1, pp.253-265, 2010.
[9] J. Busaba, S. Sukparungsee, Y. Areepong and G. Mititelu "Numerical Approximations of Average Run Length for AR(1) on Exponential CUSUM," Proc. The International Muti Conference of Engineers and Computer Scientists, Hong Kong, pp. 1268-1273, March 7-10, 2012.
[10] S. Phanyaem, Y. Areepong, S. Sukparungsee and G. Mititelu, "Explicit Formulas of Average Run Length for ARMA(1,1)," International Journal of Applied Mathematics and Statistics, vol. 43, pp. 392-405, 2013.
[11] K. Petcharat, S. Sukparungsee and Y. Areepong, "Exact Solution of the Average Run Length for the Cumulative Sum Chart for a Moving Average Process of Order q," ScienceAsia, vol. 41, pp. 141-147, 2015.
[12] P. Paichit, Y. Areepong and S. Sukparungsee, "Average Run Length of EWMA Chart for $SARMA(P,Q)_L$ Processes," Far East Journal of Mathematical Sciences, vol.93(2), pp. 229-241, 2014.
[13] K. Atkinson and W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework. Springer Verlag, New York, 2001.