Common Fixed Point Theorems For Finite Number Of Mappings Without Continuity And Compatibility In Menger Spaces

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Introduction

Sessa [9] generalized the notion of commuting maps given by Jungck [2] and introduced weakly commuting mappings. Further, Jungck [3] introduced more generalized commutativity called compatibility. In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not true. Menger [5] introduced the notion of probabilistic metric space, which is generalization of metric space and study of these spaces was expanded rapidly with pioneering work of Schewizer and Sklar [7], [8]. The existence of fixed points for compatible mappings on probabilistic metric space is shown by Mishra [6]Most of the fixed point theorems in Menger spaces deal with conditions of continuity and compatibility or compatibility of type (α) or compatible of type (β). There are maps which are not continuous but have fixed points. Also weakly compatible maps defined by Jungck and Rhoades [4] are weaker than that of compatibility. To prove existence of common fixed point for finite number of mappings some commutativity conditions are required.

Preliminaries

Let R denote the set of reals and R^+ the non-negative reals. A mapping $F : R \to R^+$ is called a distribution function if it is non- decreasing and left continuous with inf F = 0 and sup F = 1. We will denote by L the set of all distribution functions. A probabilistic metric space is a pair (X, F), where X is non empty set and F is a mapping from $X \times X$ to L. For $(p, q) \in X \times X$, the distribution function F(p, q) is denoted by Fp,q. The function Fp,q are assumed to satisfy the following conditions:

(P_1) Fp,q (x) = 1 for every x > 0 if and only if p = q,

(P_2) Fp,q (0) = 0 for every p, q $\in$ X,

(P_3) Fp,q (x) = Fq,p (x) for every p, q $\in$ X,

(P_4) if Fp,q (x) = 1 and F q,r(y) = 1 then F p,r (x + y) = 1 for every p, q, r $\in$ X and x, y > 0.

In metric space (X, d) the metric d induces a mapping F: X $\times$ X $\to$ L such that F(p, q) (x) = Fp,q (x) = H(x - d(p, q)) for every p, q $\in$ X and x $\in$ R, where H is a distributive function defined by

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Definition 1 : A function t: $[0, 1] \times [0, 1] \to [0, 1]$ is called a T- norm if it satisfies the following conditions:

(t_1) t (a, 1) = a for every a $\in$ [0,1] and t(0, 0) = 0,

(t_2) t(a, b) = t(b, a) for every a, b $\in$ [0,1],

(t_3) If c ≥ a and d ≥ b then t(c, d) ≥ t(a, b), for every a, b, c $\in$ [0, 1],

(t_4) t(t(a, b), c) = t(a, t(b, c)) for every a, b, c $\in$ [0, 1].

Definition 2 : A Menger space is a triple (X, F, t), where (X, F) is a PM-space and t is a T-norm with the following condition: (P_5) Fp,r (x+y) ≥ t (Fp,q (x), Fq,r (y)) for every p, q, r $\in$ X and x, y $\in$ R^+. An important T-norm is the T-norm t(a, b) = min{a, b} for all a, b $\in$ [0,1] and this is the unique T-norm such that t(a, a) ≥ a for every a $\in$ [0,1]. Indeed if it satisfies this condition, we have

min{a, b} ≤ t(min{a, b}, min{a, b}) ≤ t(a, b)

≤ t(min{a, b},1) = min {a, b}

Therefore t = min.

Definition 3 : Let (X, F, t) be a Menger space with continuous T- norm t. A sequence $\{x_n\}$ of points in X is said to be convergent to a point x $\in$ X if for every ε > 0

$$\lim_{n\to\infty} Fx_n,x\,(\varepsilon) = 1.$$

Definition 4 : Let (X, F, t) be a Menger space with continuous T-norm t. A sequence $\{x_n\}$ of points in X is said to be Cauchy sequence if for every ε > 0 and λ > 0, there exists an integer N = N(ε, λ) > 0 such that Fx_n,x_m (ε) > 1 - λ for all m, n $\in$ N.

Definition 5 : A Menger space (X, F, t) with the continuous T-norm t is said to be complete if every Cauchy sequence in X converges to a point in X.

Theorem A : [7] Let t be a T- norm defined by t(a, b) = min{a, b}. Then the induced Menger space (X, F, t) is complete if a metric space (X, d) is complete.

Definition 6 : [6] Self mappings A and S of a Menger space (X, F, t) are called compatible if $F_{ASx_n,SAx_n}(x) \to 1$ for all x > 0, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to u$ for some u in X as $n \to \infty$.

Definition 7 : [4] Two maps A and B are said to be weakly compatible if they commute at coincidence point.

Lemma 1 : Let $\{x_n\}$ be a sequence in a Menger space (X, F, t) with continuous t- norm and t(x, x) ≥ x. Suppose for all x ∈ [0, 1] there exists k ∈ (0, 1) such that for all x > 0 and n ∈ N

$F_{x_n,x_{n+1}}(kx) \geq F_{x_{n-1},x_n}(x)$.

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2 : Let (X, F, t) be a Menger space. If there exists k ∈ (0, 1) such that for p, q ∈ X

$F_{p,q}(kx) \geq F_{p,q}(x)$.

Then p = q.

Sharma and Bamboria [247] defined the (S-B) property in the following way:

Definition 8 : Let S and T be two self mappings of a Menger space (X, M, *). We say that S and T satisfy the property (S-B) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z ∈ X. On the basis of the above definition we give following examples :

Example 1 : Let X = [0, + ∞ [. Define S,T : X → X by

Tx = x/2 and Sx = 3x/2, ∀ x ∈ X.

Consider the sequence $x_n = 1/n$. Clearly $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$.

Then S and T satisfy (S-B).

Example 2 : Let X = [1, + ∞ [. Define S,T : X → X by

Tx = x +1/2 and Sx = 2x +1/2 , ∀ x ∈ X.

Suppose property (S-B) holds; then there exists in X a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z \text{ for some } z \in X.$$

Therefore

$$\lim_{n\to\infty} x_n = z - 1/2 \text{ and } \lim_{n\to\infty} x_n = (2z-1)/4.$$

Then z = 1/2, which is a contradiction since 1/2 ∉ X. Hence S and T do not satisfy (S-B). hat

$F_{Au,Bv}(kx) \geq t(F_{Au,Su}(x), t(F_{Bv,Tv}(x), t(F_{Au,Tv}(\alpha x), F_{Bv,Su}(2x - \alpha x))))$,

for all u,v ∈ X , x > 0 and α ∈ (0,2).

(iii) one of A(X), B(X), S(X) or T(X) is complete subspace of X,

Then

(a) A and S have a coincidence point,

(b) B and T have a coincidence point.

Further if

(iv) the pairs {A,S} and {B,T} are weakly compatible, then

Then A, B, S and T have a unique common fixed point in X. Sharma, Deshpande and Tiwari [10] proved the following.

Theorem D : Let Let A, B, S, T, I, J, L, U, P and Q be self maps on a Menger space (X, F, t) with t(a, a) ≥ a for all a ∈ [0, 1], satisfying

(1) P(X) ⊂ ABIL(X), Q(X) ⊂ STJU(X)

(2) there exists k ∈ (0, 1) such that

$F_{Px,Qy}(ku) \geq \min\{F_{ABILy,STJUx}(u), F_{Px,STJUx}(u), F_{Qy,ABILy}(u), F_{Qy,STJUx}(\alpha u), F_{Px,ABILy}((2-\alpha)u)\}$

for all x, y ∈s X, α ∈ (0, 2) and u > 0,

(3) if one of P(X), ABIL(X), STJU(X), Q(X) is a complete subspace of X then

(i) P and STJU have a coincidence point and

(ii) Q and ABIL have a coincidence point.

Further if

(4) AB = BA, AI = IA, AL = LA, BI = IB, BL = LB, IL = LI, QL = LQ,QI = IQ, QB = BQ, ST = TS, SJ = JS, SU = US, TJ = JT,TU = UT, JU = UJ, PU = UP, PJ = JP, PT = TP, (1.5) the pairs {P, STJU} and {Q, ABIL} are weakly compatible, then A, B, S, T, I, J, L, U, P and Q have a unique point in X. Here we prove Theorem D under weaker condition using a new property. Moreover complete subspace condition (3) of Theorem D is replaced by closed subspace.

Main Results

Theorem 1 : Let A, B, S, T, I, J, L, U, P and Q be self maps on a Menger space (X, F, t) with t(a, a) ≥ a for all a ∈ [0, 1], satisfying

(1.1) P(X) ⊂ ABIL(X), Q(X) ⊂ STJU(X),

(1.2) {P, STJU} or {Q, ABIL} satisfies the property (S-B),

(1.3) there exists $k \in (0, 1)$ such that

$F_{Px,Qy}(ku) \geq \min \{F_{ABILy,STJUx}(u), F_{Px,STJUx}(u),$

$F_{Qy,ABILy}(u), F_{Qy,STJUx}(u), F_{Px,ABILy}(u)\}$

for all $x, y \in X$ and $u > 0$,

(1.4) if one of P(X), ABIL(X), STJU(X) or Q(X) is a closed subspace of X then

(i) P and STJU have a coincidence point and

(ii) Q and ABIL have a coincidence point.

Further if

(1.5) AB = BA, AI = IA, AL = LA, BI = IB, BL = LB, IL = LI, QL = LQ, QI = IQ, QB = BQ, ST = TS, SJ = JS, SU = US, TJ = JT,TU = UT, JU = UJ, PU = UP, PJ = JP, PT = TP, (1.6) the pairs {P, STJU} and {Q, ABIL} are weakly compatible. Then A, B, S, T, I, J, L, U, P and Q have a unique common point in X.

Proof : Suppose that **{Q, ABIL}** satisfies the property (S-B). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} ABILx_n = z$ for some $z \in X$. Since $Q(X) \subset STJU(X)$, there exists in X a sequence $\{y_n\}$ such that $Qx_n = STJUy_n$.

Hence $\lim_{n\to\infty} STJUy_n = z$. Let us show that $\lim_{n\to\infty} Py_n = z$. Suppose for some $t \in X$, $\lim_{n\to\infty} Py_n = t$, where $t \neq z$. Indeed in view of (1.3), we have

$F_{Py_n,Qx_n}(ku) \geq \min \{F_{ABILx_n ,STJUy_n} (u), F_{Py_n,STJUy_n} (u), F_{Qx_n,ABILx_n} (u), F_{Qx_n,STJUy_n} (u), F_{Py_n,ABILx_n} (u)\}$

Letting $n \to \infty$, we have

$F_{t,z} (ku) \geq \min \{F_{z,z} (u), F_{t,z} (u),$

$F_{z,z} (u), F_{z,z} (u), F_{t,z} (u)\}$

$F_{t,z} (ku) \geq F_{t,z} (u),$

By Lemma 2, we have t = z.

Therefore we deduce that

$\lim_{n\to\infty} Py_n = z.$

Suppose that STJU(X) is closed subset of X. Then z = STJUw for some $w \in X$. Subsequently , we have

$\lim_{n\to\infty} Py_n = \lim_{n\to\infty} Qx_n = \lim_{n\to\infty} ABILx_n = \lim_{n\to\infty} STJUy_n$

$= STJUw,$

By (1.3) , we have

$F_{Pw,Qx_n}(ku) \geq \min \{F_{ABILx_n ,STJUw} (u), F_{Pw,STJUw} (u), F_{Qx_n,ABILx_n} (u), F_{Qx_n,STJUw} (u), F_{Pw,ABILx_n} (u)\}.$

Letting $n \to \infty$, we have

$F_{Pw,z} (ku) \geq \min \{F_{z,z} (u), F_{Pw,z} (u),$

$F_{z,z} (u), F_{z,z} (u), F_{Pw,z} (u)\}$

$F_{Pw,z} (ku) \geq F_{Pw,z} (u)$

Therefore, by Lemma 2, we have Pw = z.

Since STJUw = z, thus we have Pw = z = STJUw , that is w is coincidence point of P and STJU. This proves (i).

Since **P(X) ⊂ ABIL(X),** Pw = z implies that $z \in ABIL(X)$.

Let $v \in (ABIL)^{-1}z$. Then ABILv = z. By (1.3) , we have

$F_{Py_n,Qv} (ku) \geq \min \{F_{ABILv ,STJUy_n} (u), F_{Py_n,STJUy_n} (u), F_{Qv,ABILv} (u), F_{Qv,STJUy_n} (u), F_{Py_n,ABILv} (u)\}$

Letting $n \to \infty$, we have

$F_{z,Qv}(ku) \geq \min \{F_{z,z} (u), F_{z,z} (u),$

$F_{Qv,z} (u), F_{Qv,z} (u), F_{z,z} (u)\}$

$F_{z,Qv}(ku) \geq F_{z,Qv}(u)$

Then by Lemma 2, we have Qv = z. Since ABILv = z, we have Qv = z ABILv, that is v is coincidence point of Q and ABIL. This proves (ii). The remaining two cases pertain essentially to the previous cases. Indeed if P(X) or Q(X) is closed then by (i), $z \in P(X) \subset ABIL(X)$ or $z \in Q(X) \subset STJU(X)$. Thus (i) and (ii) are completely established. Since the pair {P, STJU} is weakly compatible therefore P and STJU commute at their coincidence point that is P(STJUw)

= (STJU)Pw or Pz = STJUz.

Since the pair {Q, ABIL} is weakly compatible therefore Q and ABIL commute at their coincidence point that is Q(ABILv) = (ABIL)Qv or Qz = ABILz. Now we prove that Pz = z. By (1.3), we have

$F_{Pz,Qx_{2n+1}}(ku)$

$\geq \min\{F_{y_{2n},STJUz}(u), F_{Pz,STJUz}(u), F_{y_{2n+1},y_{2n}}(u),$

$F_{y_{2n+1},STJUz}(u), F_{Pz,y_{2n}}(u)\}.$

Proceeding limit as $n \to \infty$, we have

$F_{Pz,z} (ku)$

$\geq \min\{F_{z,z} (u), F_{Pz,z} (u), F_{z,z} (u),$

$F_{z,z} (u), F_{Pz,z} (u)\}.$

This yields

FPz,z(ku) ≥ FPz,z(u).

Therefore by Lemma 2, we have Pz = z. So Pz = STJUz = z. By (1.3), we have

$FPx_{2n+2},Qz(ku) \geq$

$\min\{FABILz,y_{2n+1}(u),Fy_{2n+2},y_{2n+1}(u),FQz,ABILz(u),$

$FQz,y_{2n+1}(u), Fy_{2n+2},ABILz(u)\}.$

Proceeding limit as n →∞, we have

Fz,Qz(ku)

≥ min{Fz,z (u),Fz,z (u),FQz,z (u),

FQz,z (u), Fz,z (u)}.

This gives

Fz,Qz (ku) ≥ FQz,z(u).

Therefore by Lemma 2, we have Qz = z, so Qz = ABILz = z. By (1.4), and using (1.5), we have

FPz,Q(Lz)(ku)

≥ min{FABIL(Lz),STJz(u),FPz,STJUz(u),

FQ(Lz)z,ABIL(Lz)(u),

FQ(Lz),STJUz(u), FPz,ABIL(Lz)(u)}.

Thus we have

Fz,Lz(ku)

≥ min{FLz,z(u),Fz,z(u),FLz,Lz(u),FLz,z(u), FLz,z(u)}.

Thus

Fz,Lz(ku) ≥ Fz,Lz(u)

Therefore by Lemma 2, we have Lz = z. Since ABILz = z therefore ABIz = z. By (1.3), and using (1.5), we have

FPz,Q(Iz)(ku)

≥ min{FABIL(Iz),STJUz(u),

FPz,STJUz(u),

FQ(Iz),ABIL(Iz)(u),FQ(Iz),STJUz(u), FPz,ABIL(Iz)(u)}.

Thus we have

FIz,z(ku)

≥ min{FIz,z(u), Fz,z(u), FIz,Iz(u), FIz,z(u), FIz,z(u)}.

Therefore by Lemma 2, we have Iz = z. Since ABIz = z therefore ABz =z. Now to prove Bz =z we put x = z, y = Bz in (1.3) and using (1.5), we have

FPz,Q(Bz)(ku)

≥ min{FABIL(Bz),STJUz(u), FPz,STJUz(u),

FQ(Bz),ABIL(Bz)(u),

FQ(Bz),STJUz(u),FPz,ABIL(Bz)(u)}.

Thus we have

Fz,Bz(ku) ≥ min{FBz,z(u), Fz,z(u), FBz,Bz(u),FBz,z(u), FBz,z(u)}.

Therefore by Lemma 2, we have Bz = z. Since ABz = z therefore Az = z. By (1.2) and using (1.5), we have

FP(Uz),Qz(ku)

≥ min{FABILz,STJU(Uz)(u),

FP(Uz),STJU(Uz)(u),

FQz,ABILz(u),

FQz,STJU(Uz)(u),FP(Uz),ABILz(u)}.

Thus we have

FUz,z(ku) ≥ min{FUz,z(u), FUz,Uz(u), Fz,z(u), FUz,z(u), FUz,z(u)}.

Therefore by Lemma 2, we have Uz = z. Since STJUz = z therefore STJz = z. To prove Jz = z put x = Jz, y = z in (1.3) and using (1.5), we have

FP(Jz),Qz(ku)

≥ min{FABILz,STJU(Jz)(u), FP(Jz),STJU(Jz)(u), FQz,ABILz(u),

FQz,STJU(Jz)(u), FP(Jz),ABILz(u)}.

Thus we have

FJz,z (ku) ≥ min{FJz,z(u), FJz,Jz(u), Fz,z(u), FJz,z(u), FJz,z(u)}.

Therefore by Lemma 2, we have Jz = z. Since STJz = z therefore STz = z. To prove Tz = z put x = Tz, y = z in (1.3) and using (1.5), we have

FP(Tz),Qz(ku)

≥ min{FABILz,STJU(Tz)(u),

FP(Tz),STJU(Tz)(u),

FQz,ABILz(u), FQz,STJU(Tz)(u),

FP(Tz),ABILz(u)}.

Thus we have

$F_{Tz,z}(ku)$
$\geq \min\{F_{Tz,z}(u), F_{Tz,Tz}(u), F_{z,z}(u), F_{z,Tz}(u), F_{Tz,z}(u)\}$.

Therefore by Lemma 2, we have Tz = z. Since STz = z therefore Sz = z. By combining the above results we have Az = Bz = Sz = Tz = Iz = Jz = Lz = Uz = Pz = Qz = z. that is z is a common fixed point of A, B, S, T, I, J, L, U, P and Q. For uniqueness of the common fixed point let z_1 ($z_1 \neq z$) be another common fixed point of A, B, S, T, I, J, L , U, P and

Q. Therefore, by (1.3),we have

$F_{z,z_1}(ku) = F_{Pz,Qz_1}(ku)$

$\geq \min\{F_{z_1,z}(u), F_{z,z}(u),$

$F_{z_1,z_1}(u), F_{z_1,z}(u), F_{z,z_1}(u)\}$

$F_{z,z_1}(ku) \geq F_{z,z_1}(u)$

Therefore, by Lemma 2, we have $z = z_1$.

This completes the proof of the theorem.

Theorem 2 : Let A, B, S, T, I, J, L, U, P and Q be self maps on a metric space (X, d) satisfying (1.1) and (1.2) there exists $k \in (0, 1)$ such that d(Px, Qy) ≤ k max{d(ABILy,

STJUx), d(Px, STJUx), d(Qy, ABILy)

(1/2){d(Qy, STJUx) + d(Px, ABILy)}

for all $x, y \in X$.

In addition if condition (1.4) is satisfied then we have (i) and (ii). Further if (1.5) and (1.6) are satisfied then A, B, S, T, I, J, L, U, P and Q have a unique common fixed point.

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