# Statistical Index Of Relative Performance In Two-Sample Problem

I.C.A Oyeka, N.O. Eriobu

**Abstract:** This paper proposes and develops a statistic here termed the '*relative performance index*' or the 'index of relative performance' for rank-ordering subjects by their relative performance both within and between two sampled populations in a contest for preferential selection as the need arises. The proposed method enables the estimation of the median and other tiles of not only each of the sampled populations but also the common median of the two populations as functions of the relative performance indices. The method unlike some other methods used for the analysis of two sample data is based mostly on the individual subjects rather than on only summary indices or averages. Test statistics also based on the relative performance indices are developed to test desired hypotheses concerning the medians and other tiles of each and the common population. The proposed indices being subject specific rather than merely summary averages easily enables one more clearly and succinctly examine individual subjects relative performance or level of seriousness in a condition in comparison with other subjects from the sampled populations thereby providing subject targeted information to better guide any interventionist actions on a condition of research interest. The method is illustrated with some data and shown to compare favorably with some existing methods.

**Keywords:** Relative Performance, parametric, Ranks, Probabilities, Significance

## Introduction

In statistical analysis of two sample data a lot of attention has often been paid and devoted to measures of central tendency and measures of dispersion for ''these'' data sets, their estimation and hypothesis tests concerning them. If these are the only interest of a researcher, then the researcher can use any of the familiar statistical methods such as the two sample't' test, the sign tests, the median test, the Mann-Whitney U test, the mixed effects model analysis of variance tests by ranks or any other such methods to analyse the data (Gibbons 1973, Oyeka 2009, Oyeka et al 2010). But two sample data sets intrinsically contain much more unexplored information than only a few parameters. Such types of information are the relative relationships between the observations themselves as well as relationships within and between the sampled populations. For example often assessors, decision makers, judges teachers etc. may assess or judge samples drawn from two populations of subjects, objects, entities or conditions and score them both within and between the two samples for preferential selection relative to one another to fill vacant positions when opportunities or resources are limited or scarce. A medical or health researcher or health management official may have data or information by some demographic classifications on subjects or patients on their state of health, medical test results, level of concentration of some contaminants, disease load, injury levels and other such conditions and may wish to relatively rank-order the subjects by the severity of their conditions both within and between the various demographic classifications to guide decisions on the distribution and use of amenities when suppliers are limited. In business, industry and governmental affairs, one may wish to know how various outfits, producers, supplies and distributors of goods and services such as banks, transport operators, ministries, parastatals etc. compare in performance when juxtaposed against one another to guide any interventionist remedial actions by management or supervising body. The problem before the decision makers is how using these observations to rationally select the required number of subjects, objects or outfits from the group of available subjects or options to ensure that meritocracy is upheld in the presence of scarcity. Here although any desired hypothesis may be tested, this may not however be as important as the need to find appropriate ways to systematically rank-order the subjects or available options according to their level of need or performances in a given test or situation to facilitate judicious selection to achieve a desired objective. This is because although hypothesis testing is important and useful it may often not be as important and useful as the need to find ways to rank-order subjects or objects relative to one another for preferential selection both within and between sampled populations of subjects or observations. This paper proposes to develop an index of relative performance that may be of use in rank-ordering subjects, objects or entities according to performance on tests, experiments or conditions for preferential selection both within and between two populations of interest. The proposed method may be used in data analysis even when necessary assumptions for the application of some statistical methods may not be satisfied by the data.

## The Proposed Method

Let $X_{ij}$ be the performance value, observation or score by the j[th] subject in a random sample of size $n_i$ drawn from population i, for j = 1,2….$n_i$; $i$=1,2. The sampled populations should be measurements on at least the ordinal scale, but may or may not be (a) continuous, (b) independent, or (c) numeric and the samples drawn from the populations may or may not be of equal size. To construct a measure, indicator, or an index of relative performance or achievement by subjects, items or entities in tests, experiments, situations or conditions in time or space, we may let

$$U_{ilk} = \begin{cases} 1, if\ x_{il} > x_{ik} \\ 0, if\ x_{il} = x_{ik} \\ -1, if\ x_{il} < x_{ik} \end{cases} \quad (1)$$

for $l$,k = 1,2…$n_i$; $i$=1,2; $l$≠k

Thus $U_{ilk}$ assumes the value 1, 0 or -1 respectively if the performance of or score earned by the $j^{th}$ subject drawn from population '$i$' is higher (better, greater), the same as (equal to) or lower (worse, less) than the score earned by the $k^{th}$ subject drawn from the same population; for $l$,k=1,2…$n_i$, $l$=1,2; $l$≠k.Let

$\pi_{il}^{+} = p(U_{ilk} = 1); \quad \pi_{il}^{0} = p(U_{ilk} = 0); \quad \pi_{il}^{-} = p(U_{ilk} = -1)$ 2

Where

$\pi_{il}^{+} + \pi_{il}^{0} + \pi_{il}^{-} = 1$ 3

Also let

$$W_{il} = \sum_{\substack{k=1 \\ k \neq l}}^{ni} u_{ilk} \qquad 4$$

$W_{il}$ could be defined as the so called index of 'relative performance' by the $l^{th}$ subject from the $i^{th}$ population when compared with other subjects from the same population. It measures the total number of the subjects in population '$i$' whose scores are less (worse, smaller) less than the total number of subjects whose performance or scores are higher (better, greater) than the score by the $l^{th}$ subject from the same population '$i$' for $l$=1,2.........$n_{i}$, $i$=1,2

Now

$E(u_{ilk}) = \pi_{il}^{+} - \pi_{il}^{-}, \; var(u_{ilk}) = \pi_{il}^{+} + \pi_{il}^{-} - (\pi_{il}^{+} - \pi_{il}^{-})^{2}$ 5

Also

$$E(W_{il}) = \sum_{\substack{k=1 \\ k \neq l}}^{ni} E(u_{ilk}) = (n_i - 1)(\pi_{il}^{+} - \pi_{il}^{-})$$

6

And

$$Var(W_{il}) = \sum_{\substack{k=1 \\ k \neq l}}^{ni} Var(u_{ilk}) = (n_i\text{-}1)\left(\pi_{il}^{+} + \pi_{il}^{-} - (\pi_{il}^{+} - \pi_{il}^{-})^{2}\right)$$

7

Now $\pi_{il,}^{+}$ $\pi_{il}^{0}$ and $\pi_{il}^{-}$ are respectively the probabilities that the performance or score by the $l^{th}$ randomly selected subject from population $i$ is higher (better, greater), the same as (equal to) or lower (worse, less) than the score by all other subjects from the same population '$i$';$l$= 1,2... $n_{i;}$ $i$=1,2. Their sample estimates are respectively

$\hat{\pi}_{il}^{+} = \frac{f_{il}^{+}}{n_{i}-1}; \; \hat{\pi}_{il}^{0} = \frac{f_{il}^{0}}{n_{i-1}}; \; \hat{\pi}_{il}^{-} = \frac{f_{il}^{-}}{n_{i-1}}$ 8

Where $f_{il}^{+}, f_{il}^{0}$ and $f_{il}^{-}$ are respectively the total number of subjects in population '$i$', the $l^{th}$ subject score is higher (better, greater), the same as (equal to) or lower (worse, less) than, for $l$ =1,2...$n_i$; $i$= 1,2. Note $f_{il}^{+}, f_{il}^{0}$ and $f_{il}^{-}$ that are respectively the total number of 1s, 0s and -1s in the frequency distribution of the $n_i$-1 values of these numbers in $U_{ilk}$ $l$,k=1,2...$n_i$; $i$ =1,2, $l$≠k. Use of Equation 8 in Equation 6 for the expected value of $W_{il}$ yields the sample estimate of the relative performance index of the $l^{th}$ subject in comparison with all other subjects in population '$i$' as

$W_{il} = (n_i - 1)(\hat{\pi}_{il}^{+} - \hat{\pi}_{il}^{-}) = f_{il}^{+} - f_{il}^{-}$ 9

for $l$=1,2....$n_i$; $i$=1,2.

Similarly the corresponding sample variance of $W_{il}$ is estimated from Equation 7 using Equation 8 as

$Var(W_{il}) = (n_i\text{-}1)(\pi_{il}^{+} + \pi_{il}^{-} - (\pi_{il}^{+} - \pi_{il}^{-})^{2})$ 10

Rank ordering the values of $W_{il}$ of Equation 9 from the largest to the smallest or smallest to the largest and assigning the largest value of $W_{il}$ the rank 1(or $n_i$), the next largest the rank 2 (or $n_i$-1) and so on until the smallest value is assigned the rank $n_i$ (or 1) enables one rank-order subjects drawn from the $i^{th}$ population for preferential selection from the highest performer, achiever or best, say, to the lowest performer, achiever or worst, or vice versa based on the ranks assigned to the values of their relative performance index, $W_{il}$, $l$=1,2...$n_i$, $i$=1,2. All tied values of $W_{il}$ are assigned their mean ranks. Now Note that $W_{il}$ the so called gap in relative performance or the relative performance index by the $l^{th}$ subject from population '$i$' whose estimate is given by Equation 9 with rank $r_{il}$ is the total number of subjects in population '$i$' the performance or score by the $l^{th}$ subject from that population is higher (better, greater) less than the total number of subjects from the population the subjects performance or score is lower (worse, smaller) than, for $l$=1,2...$n_i$; $i$=1,2. If the $l^{th}$ subject from population '$i$' performs better or scores higher than all other subjects from that population and $W_{il}$ are not tied in values, then, $W_{il} = (n_i - 1) = f_{il}^{+}, f_{il}^{-} = 0, \; \hat{\pi}_{il}^{+} = 1, \; \hat{\pi}_{il}^{-} = 0$ and the rank $r_{il} = 1$ (or $n_i$) so that the $l^{th}$ subject from population. $i$ is considered the best(most preferred) in the preferential rank-ordering of all the $n_i$ subject from population $i$ in terms of their performance or scores in the test of interest. If the $l^{th}$ subject from population '$i$' performs or scores higher (better, greater) than as many subjects in that population that subjects own score is lower (worse, less) than, then $W_{il} = 0, f_{il}^{+} = f_{il}^{-}, \hat{\pi}_{il}^{+} = \hat{\pi}_{il}^{-}$ if '$n_i$' is odd or the two middle-most values of $W_{il}$ are 1 and -1 respectively, if $n_i$ is even in this case $r_{il}$ is assigned the middle-most rank and the corresponding $W_{il}$ if '$n_i$' is odd or the average of their two middle-most values which is now also 0 if '$n_i$' is even is the median relative performance index and hence the corresponding value if '$n_i$' is odd or the average value if $n_i$ is even of $X_{il}$ is then the estimated median of the $i^{th}$ population ,$l$=1,2...$n_i$; $i$=1,2. If the $l^{th}$ subject from population '$i$' performs worse or scores lower than all other subject from that population, then, $W_{il} = -(n_i - 1) = f_{il}^{-}$; $f_{il}^{+} = 0$; $\hat{\pi}_{il}^{+} = 0$; $\hat{\pi}_{il}^{-} = 1$ and the rank $r_{il} = n_i(or\ 1)$ so that the $l^{th}$ subject from population $i$ is considered the worst, least (most) preferred in the preferential rank-ordering of all the $n_i$ subjects from population '$i$' in terms of their performance. Thus the larger and positive the value of $W_{il}$ is, the more (less) highly rated and preferred is the $l^{th}$

subject from population '$i$' relative to other subjects from that population in terms of performance or condition; the smaller and negative the value of $W_{il}$, the less (more) the rating of the $l^{th}$ subject from population '$i$' relative to other subjects from that population in terms of performance or condition. As already shown above, if for instance $n_i$ is odd, and the $l^{th}$ subject from population '$i$' performs better (or worse) or scores higher (lower) than as many subjects in that population as the subjects performance or score is worse or lower (better or higher) than, then $W_{il} = 0, f_{il}^{+} = f_{il}^{-}, \hat{\pi}_{il}^{+} = \hat{\pi}_{il}^{-}$, and the rank assigned to that subject $r_{il}$ becomes the median rank of the $n_i$ sample scores or observations from population $i$. In this case the $l^{th}$ subject from population '$i$' is considered better than and more preferred to one-half and worse than and less preferred to another one-half of all the subjects from population '$i$'. Thus in particular if the rank-ordering of the values of $W_{il}$ is from the largest to the smallest, then the larger and positive the value of $W_{il}$ is, the higher, better or greater the performance or score by the $l^{th}$ subject from population '$i$' relative to the performance or scores by other subjects from that population and hence, the more preferable is the $l^{th}$ subject in comparison to other subjects from that population, in terms of performance; the smaller and negative the value of $W_{il}$ is, the poorer, lower, worse and hence less preferred is the $l^{th}$ subject in performance, relative to other subjects from the population. As already noted above, research here may not necessarily be in hypothesis testing but more in rank-ordering subjects by their relative performance in a context or condition for possible preferential selection. One may however still wish to test any desired hypothesis. If in the population of interest, population '$i$', the performance or score by the $l^{th}$ subject is higher, (better or greater) than the scores by one-half of the subjects but lower, (worse or smaller) than the performance or scores by the other one-half of the subjects in the population, then $W_{il}$ would be expected to be zero, that is $W_{il}$= 0 and also assigned the median rank in the absence of ties, for some randomly selected subject drawn from population $i$, $i$=1,2. Thus a null hypothesis that may be of interest would be that a randomly selected subject from population '$i$' performs averagely in the given test or that the subject performs better (worse) than one-half and worse (better) than the other one-half of the subjects in the population, so that $W_{il} = 0$.This hypothesis may however be stated under a more general null hypothesis.

$$H_o: \pi_{il}^{+} - \pi_{il}^{-} \geq \theta_{i0} \quad \text{Versus} \quad H_1: \pi_{il}^{+} - \pi_{il}^{-} < \theta_{i0}, say(-1 \leq \theta_{i0} \leq 1) \qquad 11$$

For $l$=1, 2...$n_i$; $i$=1, 2

The null hypothesis of Equation 11 is tested using the test statistic

$$\chi^2 = \frac{\left(W_{il-(n_i-1).\theta_{i0}}\right)^2}{var\ (W_{il})} = \frac{\left(W_{il-(n_i-1).\theta_{i0}}\right)^2}{(n_i-1)\left(\hat{\pi}_{il}^{+}+\hat{\pi}_{il}^{-}-(\hat{\pi}_{il}^{+}-\hat{\pi}_{il}^{-})^2\right)} \qquad 12$$

which under $H_0$ has approximately the chi-square distribution with 1 degree of freedom for sufficiently Large '$n_i$' ; $l$=1,2...$n_i$; $i$=1,2.

The null hypothesis $H_0$ is rejected at the $\alpha$ level of significance if

$$\chi^2 \geq \chi_{1-\alpha;1}^2 \qquad 13$$

Otherwise $H_0$ is accepted. It is possible in some circumstances, perhaps based on a quota system of preferential selection that any subject from population i must a-priori statistically out-perform or exceed a specified proportion $\theta_{i0}$ of subjects from that population before such a subject can be considered qualified for inclusion (or exclusion) among the subjects preferentially selected from that population, given the condition of interest. Now if the desired statistical significance level is $\alpha$ then we would have from Equations 12 and 13 that for the subject not to qualify for exclusion (or inclusion), the subjects relative performance index, $W_{il} = (n_i - 1)(\hat{\pi}_{il}^{+} - \hat{\pi}_{il}^{-})$, must be such that

$$\frac{\left((\hat{\pi}_{il}^{+}-\hat{\pi}_{il}^{-})-\theta_{i0}\right)^2}{Var\ (\hat{\pi}_{il}^{+}-\hat{\pi}_{il}^{-})} = \frac{(n_i-1)\left((\hat{\pi}_{il}^{+}-\hat{\pi}_{il}^{-})-\theta_{i0}\right)}{\hat{\pi}_{il}^{+}+\hat{\pi}_{il}^{-}-(\hat{\pi}_{il}^{+}-\hat{\pi}_{il}^{-})2} \geq \chi_{1-\alpha;1}^2 \qquad 14$$

Hence for the $l^{th}$ subject from population '$i$' to be qualified for inclusion (or exclusion) the subjects relative performance index or proportion, $\hat{\pi}_{il}^{+} - \hat{\pi}_{il}^{-}$ must be such that the a-priori specified proportion $\theta_{i0}$ must lie, that is, be contained or included within the

$$\text{interval} \ (\hat{\pi}_{il}^{+} - \hat{\pi}_{il}^{-}) \pm \sqrt{\frac{\left(\hat{\pi}_{il}^{+}+\hat{\pi}_{il}^{-} - (\hat{\pi}_{il}^{+}-\hat{\pi}_{il}^{-})^2\right).\chi_{1-\alpha;1}^2}{n_i-1}} \qquad 15$$

for $l$=1,2...$n_i$; $i$=1,2.

Research interest may also be in determining whether any two randomly selected subjects '$l$' and 'g' from population '$i$' perform equally well, that is have equal relative performance indices. To do this we may let

$$W_{ilg} = W_{il} - W_{ig} = \sum_{\substack{k=1\\k\neq l}}^{ni} u_{ilk} - \sum_{\substack{h=1\\h\neq g}}^{ni} u_{igh} \qquad 16$$

Now from Equation 9 we have that.

$$E(W_{ilg}) = E(W_{il} - W_{ig}) = (n_i - 1)\left((\pi_{il}^{+} - \pi_{il}^{-}) - (\pi_{ig}^{+} - \pi_{ig}^{-})\right) \qquad 17$$

Also from Equation 10

$$Var(W_{ilg}) = var(W_{il} - W_{ig}) = var(W_{il}) + var(W_{ig}) - 2cov(W_{il}; W_{ig}) \qquad 18$$

Where

$$Cov(W_{il};W_{ig}) = E(W_{il}.W_{ig}) - E(W_{il}).E(W_{ig}) = \sum_{\substack{k=1\\k\neq l}}^{ni}\sum_{\substack{h=1\\h\neq g}}^{ni} \mathrm{E}(u_{ilk}.u_{igh}) - (n_i-1)^2 (\pi_{il}^+ - \pi_{il}^-)(\pi_{ig}^+ - \pi_{ig}^-)$$

Now $u_{ilk}.u_{igh}$ can only assume the values 1, 0 or -1. It assumes the value 1 if and only if $u_{ilk}$ and $u_{igh}$ both assume the value 1 or the value -1 with probability $\pi_{il}^+.\pi_{ig}^+ + \pi_{il}^-.\pi_{ig}^-$; $u_{ilk}.u_{igh}$ assumes the value 0 if and only if $u_{ilk}$ and $u_{igh}$ both assume the value 0 or $u_{ilk}$ assumes the value 0 no matter the value assumed by $u_{igh,}$ or $u_{igh}$ assumes the value 0 no matter the value assumed by $u_{ilk}$ with probability $\pi_{ilo}.\pi_{igo} + \pi_{ilo}(\pi_{ig}^+ + \pi_{ig}^-) + (\pi_{igo})(\pi_{il}^+ + \pi il-;uilk.uigh$ assumes the value -1 if and only if $uilk$ assumes the value 1 and $u_{igh}$ assumes the value -1 OR $u_{ilk}$ assumes the value -1 and $u_{igh}$ assumes the value 1 with probability $\pi_{il}^+.\pi_{ig}^- + \pi_{il}^-.\pi_{ig.}^+$

Therefore

$$\mathrm{cov}(W_{il}:W_{ig}) = (n-1)^2\left(\left(\pi_{il}^+.\pi_{ig}^+ + \pi_{il}^-.\pi_{ig}^- - \left(\pi_{il}^+.\pi_{ig}^- + \pi_{il}^-.\pi_{ig}^+\right)\right) - \left(\pi_{il}^+ - \pi_{il}^-\right)\left(\pi_{ig}^+ - \pi_{ig}^-\right)\right) = 0 \qquad 19$$

Hence from Equations 18 and 10 we have that

$$\mathrm{var}(W_{il} - W_{ig}) = \mathrm{var}(W_{il}) + \mathrm{var}(W_{ig}) = (n_i - 1)\left(\left(\pi_{il}^+ + \pi_{il}^- - \left(\pi_{il}^+ - \pi_{ij}^-\right)^2\right) + \left(\pi_{ig}^+ + \pi_{ig}^- - \left(\pi_{ig}^+ - \pi_{ig}^-\right)^2\right)\right)$$

The sample estimate of $W_{ilg}$ is from Equation 9

$$W_{ilg} = W_{il} - W_{ig} = (n_i - 1)\left((\hat{\pi}_{il}^+ - \hat{\pi}_{il}^-) - (\hat{\pi}_{ig}^+ - \hat{\pi}_{ig}^-)\right) \qquad 20$$

The estimate of the corresponding sample variance is from Equations 19 and 10

$$\mathrm{var}(W_{il} - W_{ig}) = \mathrm{var}(W_{il}) + \mathrm{var}(W_{ig}) = (n_i - 1)\left(\left(\hat{\pi}_{il}^+ + \hat{\pi}_{il}^- - \left(\hat{\pi}_{il}^+ - \hat{\pi}_{il}^-\right)^2\right) + \left(\hat{\pi}_{ig}^+ + \hat{\pi}_{ig}^- - \left(\hat{\pi}_{ig}^+ - \hat{\pi}_{ig}^-\right)^2\right)\right) \qquad 21$$

for $l, g$=1,2…$n_i$,$i$=1,2, $l$≠g

To test the hypothesis of interest, that is

$$H_o: (\pi_{il}^+ - \pi_{il}^-) - (\pi_{ig}^+ - \pi_{ig}^-) = 0 \text{ versus } H_1: (\pi_{il}^+ - \pi_{il}^-) - (\pi_{ig}^+ - \pi_{ig}^-) \neq 0 \qquad 22$$

We may use the test statistic

$$\chi^2 = \frac{W_{ilg}^2}{var(W_{ilg})} = \frac{(W_{il} - W_{ig})^2}{var(W_{il}) + var(W_{ig})} \qquad 23$$

which under $H_O$ has approximately the chi-square distribution with 1 degree of freedom for sufficiently large '$n_i$'; for $l$,g=1,2…$n_i$; $i = 1,2$. $l$≠g where $W_{ilg}$ and $var(W_{ilg})$ are given by Equations 20 and 21 respectively. The null hypothesis $H_0$ of Equation 22 is rejected at the $\alpha$ level of significance if Equation 13 is satisfied; otherwise $H_0$ is accepted. Furthermore research interest may be to determine whether randomly selected subjects from populations 1 and 2 perform or score equally well in the test, experiment or condition. To do this we may let

$$W_{lg} = W_{1l} - W_{2g} = \sum_{\substack{k=1\\k\neq l}}^{n_1} u_{1lk} - \sum_{\substack{h=1\\h\neq g}}^{n_2} u_{2hg} \qquad 24$$

for $c$=1,2…$n_1$;g=1,2…$n_2$

Now

$$E(W_{lg}) = E(W_{1l} - W_{2g}) = \sum_{\substack{k=1\\k\neq l}}^{n_1} E(u_{1lk}) - \sum_{\substack{h=1\\h\neq j}}^{n_2} E(u_{2gh})$$

OR

$$E(W_{lg}) = (n_1 - 1)(\pi_{1l}^+ - \pi_{1l}^-) - (n_2 - 1)(\pi_{2g}^+ - \pi_{2g}^-) \qquad 25$$

With variance

$$Var(W_{lg}) = var(W_{1l} - W_{2g}) = var(W_{1l}) + var(W_{2g}) - 2cov(W_{1l};W_{2g}) \qquad 26$$

Where

$$\mathrm{cov}(W_{1l};W_{2g}) = E(W_{1l}.W_{2g}) - E(W_{1l}).E(W_{2g}) = E(W_{1l}.W_{2g}) - (n_1 - 1)(n_2 - 1)(\pi_{1l}^+ - \pi_{1l}^-)(\pi_{2g}^+ - \pi_{2g}^-) \qquad 27$$

Where $E(W_{1l}.W_{2g}) = E\left(\sum_{\substack{k=1\\k\neq l}}^{n_1} u_{1lk}\right)\left(\sum_{\substack{h=1\\h\neq g}}^{n_2} u_{2gh}\right) =$

$$\sum_{k=1}^{n_1}\sum_{h=1}^{n_2} E(u_{1lk}.u_{2gh})$$

Now $u_{1lk}.u_{2gh}$ can only assumed values 1,0 and -1. It assumes the value 1 if and only if $u_{1lk}$ and $u_{2gh}$ both assume the value 1 or both assume the value -1 with probability $\pi_{1l}^{+}.\pi_{2g}^{+} + \pi_{1l}^{-}.\pi_{2g}^{-}; u_{1lk}.u_{2gh}$ assumes the value 0 if $u_{1lk}$ and $u_{2gh}$ both assume the value 0 or $u_{1lk}$ assumes the value 0 no matter the value assumed by $u_{2gh}$ OR $u_{2gh}$ assumes the value 0 no matter the value assumed by $u_{1lk}$ with probability $\pi_{1l}^{0}.\pi_{2g}^{0} + \pi_{1l}^{0}(\pi_{2g}^{+} + \pi_{2g}^{-}) + \pi_{2g}^{0}(\pi_{1l}^{+} +$ $\pi 1l-; u1lk.u2gh$ assumes the value -1 if and only if $u1lk$ assumes the value 1 and $u_{2gh}$ assumes the value -1 or $u_{1lk}$ assumes the value -1 and $u_{2gh}$ assume the value 1 with probability $\pi_{1l}^{+}.\pi_{2g}^{-} + \pi_{1l}^{-}.\pi_{2g}^{+}$. Hence collecting terms and evaluating Equation 27 we have that $cov(W_{1l}; W_{2g}) = (n_1-1)(n_2-1)\left((\pi_{1l}^{+}.\pi_{2g}^{+} + \pi_{1l}^{-}.\pi_{2g}^{-} - (\pi_{1l}^{+}.\pi_{2g}^{-} + \pi_{1l}^{-}.\pi_{2g}^{+}) - \right.$ $\pi 1l+-\pi 1l-\pi 2g+-\pi 2g-=0.$ Therefore using these results in Equation 26 we have that

$$\operatorname{var}(W_{lg}) = \operatorname{var}(W_{1l}) + \operatorname{var}(W_{2g}) = (n_1-1)\left(\pi_{1l}^{+} + \pi_{il}^{-} - (\pi_{1l}^{+} - \pi_{il}^{-})^2\right) + (n_2-1)\left(\pi_{2g}^{+} + \pi_{2g}^{-} - (\pi_{2g}^{+} - \pi_{2g}^{-})^2\right) \quad 28$$

Using Equation 8 in Equation 25 we have that the sample estimate of $W_{lg}$ is

$W_{lg} = W_{1l} - W_{2g} = (n_1-1)(\hat{\pi}_{1l}^{+} - \hat{\pi}_{1l}^{-}) - (n_2-1)(\hat{\pi}_{2g}^{+} -$ $\pi 2g-$ 29

The corresponding sample estimate of the variance of $W_{1g}$ is from Equations 10 and 28

$$\operatorname{var}(W_{lg}) = \operatorname{var}(W_{1l}) + \operatorname{var}(W_{2g}) = (n_1-1)\left(\hat{\pi}_{1l}^{+} + \hat{\pi}_{1l}^{-} - (\hat{\pi}_{1l}^{+} - \hat{\pi}_{1l}^{-})^2\right) + (n_2-1)\left(\hat{\pi}_{2g}^{+} + \hat{\pi}_{2g}^{-} - (\hat{\pi}_{2g}^{+} - \hat{\pi}_{2g}^{-})^2\right) \quad 30$$

for $l$=1,2...$n_1$; g=1,2...$n_2$.

Note that if any randomly selected subject from population 1 performs or scores as well as a randomly selected subject from population 2 then the two subjects would be equally rated and their relative performance indices would be equal so that the difference between $W_{1l}$ and $W_{2g}$ would be expected to be zero, that is $E(W_{lg}) = E(W_{1l}) - E(W_{2g}) = 0$ for randomly selected subjects from populations 1 and 2. Hence on this bases and consistent with Equation 25 a null hypothesis that may be of interest would be

$$H_O : (n_1-1).\pi_{1l} - (n_2-1).\pi_{2g} = 0 \; verus \; H_1 : (n_1-1).\pi_{1l} - (n_2$$

Where $\pi_{1l} = \pi_{1l}^{+} - \pi_{1l}^{-}$ and $\pi_{2g} = \pi_{2g}^{+} - \pi_{2g}^{-}$ for $l$=1,2...$n_1$; g=1,2...$n_2$

The null hypothesis of Equation 31 may be tested using the test statistic.

$$\chi^2 = \frac{(W_{lg})^2}{var(W_{lg})} = \frac{(w_{1l} - W_{2g})^2}{var(W_{1l}) + var(W_{2g})} = \frac{\left((n_1-1)(\hat{\pi}_{1l}^{+} - \hat{\pi}_{1l}^{-}) - (n_2-1)(\hat{\pi}_{2g}^{+} - \hat{\pi}_{2g}^{-})\right)^2}{(n_1-1)\left(\hat{\pi}_{1l}^{+} + \hat{\pi}_{1l}^{-} - (\hat{\pi}_{1l}^{+} - \hat{\pi}_{1l}^{-})^2\right) + (n_2-1)\left(\hat{\pi}_{2g}^{+} + \hat{\pi}_{2g}^{-} - (\hat{\pi}_{2g}^{+} - \hat{\pi}_{2g}^{-})^2\right)}$$

32

Which under $H_0$ has approximately the chi-square distribution with 1 degree of freedom for sufficiently large $n_1$and$n_2$. Equation 9 may easily be used to estimate differences in relative performance indices expressed as proportions by subjects within each sampled population. Similarly Equation 29 may be used to provide estimates of differences in relative performance indices also expressed as proportions by subjects between the two populations. This would enable a researcher or policy implementer rank-order subjects between two sampled populations in terms of the differences between the estimated proportions of their relative performance indices for preferential selection as needed. Thus from Equation 29 the difference in proportions $P_{12.lg} = p_{1l.2} - p_{2g.1} = P_{1l} - P_{2g}$ between the relative performance indices by the $l^{th}$ subject from population 1 and $g^{th}$ subject from population 2 is estimated as

$$\frac{W_{lg}}{(n_1-1)+(n_2-1)} = \frac{(n_1-1)(\hat{\pi}_{1l}^{+} - \hat{\pi}_{1l}^{-}) - (n_2-1)(\hat{\pi}_{2g}^{+} - \hat{\pi}_{2g}^{-})}{(n_1-1)+(n_2-1)} \quad 33$$

With estimated variance from Equation 30 given as $var(p_{12.lg}) = \frac{var(w_{lg})}{((n_1-1)+(n_2-1))^2} = \frac{var(W_{1l} - W_{2g})}{((n_1-1)+(n_2-1))^2} = \frac{var(W_{1l}) + var(W_{2g})}{((n_1-1)+(n_2-1))^2}$ OR

$$var(p_{12.lg}) = \frac{(n_1-1)\left(\hat{\pi}_{1l}^{+} + \hat{\pi}_{1l}^{-} - (\hat{\pi}_{1l}^{+} - \hat{\pi}_{1l}^{-})^2\right) + (n_2-1)\left(\hat{\pi}_{2g}^{+} + \hat{\pi}_{2g}^{-} - (\hat{\pi}_{2g}^{+} - \hat{\pi}_{2g}^{-})^2\right)}{((n_1-1)+(n_2-1))^2} \quad 34$$

Where

$$P_{1l.2} = \frac{W_{1l}}{(n_1-1)+(n_2-1)} = \frac{(n_1-1)(\hat{\pi}_{1l}^{+}-\hat{\pi}_{1l}^{-})}{(n_1-1)+(n_2-1)};\ P_{2g.1} = \frac{W_{2g}}{(n_1-1)+(n_2-1)} = \frac{(n_2-1)(\hat{\pi}_{2g}^{+}-\hat{\pi}_{2g}^{-})}{(n_1-1)+(n_2-1)}$$

$l = 1, 2 \ldots n_1;\ g = 1,2 \ldots n_2$

To compare subjects from population 1 with subjects from population 2 in terms of their relative performance indices in the condition of interest we may first cross classify values of $P_{1l.2}$ *and* $P_{2g.1}$ and then take the pair-wise differences between these proportions of relative performance indices for $l = 1,2 \ldots n_1;\ g = 1,2 \ldots n_2$. We would then next determine for each subject from population 1 say the largest or highest (smallest or lowest) value of the differences between that subjects estimated relative performance index and those of all subjects from population 2 say. Rank-ordering as usual these estimated differences in proportions of relative performance indices would enable one compare subjects from population 1 with one another in terms of performance relative to, that is when juxtaposed against the performance of all subjects from population 2. A similar approach with respect to subjects from population 2 enables the comparison of subjects from this population with one-another relative to the performance of all the subjects from population 1 in the condition of interest for possible preferential selection as necessary. Note that in considering $P_{1l.2} - P_{2g.1}$ the differences between the proportions of relative performance indices by subjects from populations 1and 2, positive values of these differences assigned higher (lower) ranks indicate that subjects from population 1 perform relatively better than and hence preferred to subjects from population 2. The converse is the case when these differences have negative values assigned lower (higher) ranks and indicate that subjects from population 2 perform better than and hence preferred to subjects from population 1. In practical applications the researcher may use for this analysis only one subject from within each set of subjects tied in their scores or observations and hence have the same relative performance indices W in each of the two samples since these subjects are treated alike in the preferential selection process. However if perhaps again based on a quota system subjects from one of the two populations, population 1 say, is required to statistically out-perform, that is exceed subjects drawn from the other population . Population 2 say by a pre-determined relative performance index $W_o$ say before such a subject may be preferentially included (or excluded) among the subjects preferentially selected from that population at a given α level, then the gap between the relative performance indices of subject '$l$' drawn from population 1 and subject 'g' drawn from population 2 namely $W_{lg} = W_{1l} - W_{2g}$ must be such that the a-prior specified value $W_o$ must lie that is be included within the interval

$$(W_{1l} - W_{2g}) \pm \sqrt{\left(var(W_{1l}) + var(W_{2g})\right)} \cdot \chi^2_{1-\alpha;1} \quad 36$$

where $W_{1l} - W_{2g}$ and $var(W_{1l} - W_{2g})$ are given by Eqns 29 and 30 respectively for $l = 1,2 \ldots n_1;\ g = 1,2 \ldots n_2$ finally research interest may also be in knowing whether on the average subjects in populations 1 and 2 perform or score equally well in the test. In other words whether the median scores of the two populations of subjects from which the samples are drawn are the same. This would in effect mean that one-half of the subjects in each population would be expected to score above as below, that is perform better or higher as worse or lower than the other one-half. In other words if the two samples are drawn from populations with equal medians, then one would expect that one half of the observations in each sample would be above as below that samples median just as one half of each sample would be above as below the common median of the combined sample if the two samples were pooled together into one sample. In such a case the relative performance index of a randomly selected subject from each population with a score that corresponds with the samples median would then be expected to be zero, if $n_i$ is odd, or the two middle most indices would have values of 1 and -1 summing to zero, if $n_i$ is even for $i = 1,2$ and hence assigned the median rank for that sample which would be expected to coincide with the common median of the scores of all the subjects if the two populations were pooled and have a common median. That is this would in effect mean that the relative performance index with the value zero if $n_i$ is odd or with the two middle most ranked values of 1 and -1 which sum to zero if '$n_i'$' is even for $i = 1,2$ in each sample would be expected to correspond to observations with equal values in each of the two samples which would in fact be the common median of the two samples if the two sampled populations have equal medians. The expected implication is that the relative performance index of a randomly selected subject with a performance or score that corresponds with the median score from such a pooled population would also be zero. Notational and specifically if the two populations have equal medians and there are no tied observations in each of the populations so that there are also no ties in the relative performance indices, and the relative performance index $W_{il}$ of the $l^{th}$ subject from population '$i$' is ranked 'r', $l = 1,2 \ldots n_i;\ i = 1,2,$ *then we would expect that* $E(W_{il(n_i+1)/2}) = 0$, if $n_i$ is odd or $E(W_{il(n_i/2)}) = 1$ and $E\left(W_{il\left(\frac{n_i}{2}+1\right)}\right) = -1$ or vice versa, so that the sum of these two middle-most indices is also zero if $n_i$ is even, for

$$l = 1,2 \ldots n_i;\ i = 1.2 \quad 37$$

To evaluate this possibility in terms of relative performance indices of subjects drawn from the two populations, if there are ties in the data ,the same procedure is still followed except that in this situation the ranks assigned to the relative performance indices in each sample are examined to determine those that are the middle-most and hence the median rank. The average value of the observations in each sample corresponding to these ranks and hence to the relative performance indices are then taken. If these averages are equal for the two samples then one may

35

conclude that the two sampled populations probably have equal medians, otherwise one may infer that the two population medians are different. One only needs to examine and compare the two series of relative performance indices from the two sampled populations. If the one with the value 0 if $n_i$ is odd or with the two middle-most ranked values of 1 and -1, if $n_i$ is even correspond with observations of equal values in both samples then one would be able to infer that the two populations have equal medians and hence an estimate of the individual population medians and the possibility of a common population median. However in practical applications, if interest is not necessarily in hypothesis testing but in rank-ordering subjects relative to one another according to their performance or scores in an endeavor, then the researcher may need to use only untied subjects from each sampled population in the analysis. In this case if a number of subjects are tied in values, then only one value of the observations among each tied set would be used in the analysis, so that the 'effective sample size' for this purpose would be only the number of untied subjects in each sample. This is because in the ranking and preferential selection of subjects relative to their performance, subjects with tied values or observations would have equal relative performance indices and hence treated alike that is as one set in the preferential selection process. As already noted above, statistical tests for significance sometimes may not be as important and useful as the need to rank-order subjects, objects or entities by their relative performance in tests, experiments or conditions in time or space by sample subjects within and between populations for possible preferential selection for policy and management purposes when opportunities or resources are limited or scare. The proposed method would easily enable one achieve such an objective by simply examing for each population. The magnitude and direction of the estimated relative performance indices $W_{il}$ and selecting subjects or subsets of subjects from the population with either the highest or the lowest values of these indeces depending on one's interest. If however statistical tests for significance of the percentiles and other tiles are of research interest, then one may use any of the test statistics already provided above as appropriate for these purposes. But to avoid a situation in which the denominators of these equations are zero because the response or score by the $l^{th}$ subject form population '$i$' is greater (or less) than those of all other subjects, objects or items drawn from the population so that $f_{il}^{+} = n_i - 1 \; and \; f_{il}^{-} = 0 \; or \; f_{il}^{+} = 0 \; and \; f_{il}^{-} = -(n_i - 1)$ so that $\hat{\pi}_{il}^{+} = 1 \; and \; \hat{\pi}_{il}^{-} = 0$ or vice versa, yielding a meaningless value of the chi-square test statistic, it is recommended that in such cases a correction factor of $\frac{1}{2(n_i-1)}$ be subtracted from $\hat{\pi}_{il}^{+}$ and added to $\hat{\pi}_{il}^{-}$ or vice-versa depending on which of the two currently has a value of 1(or -1) or a value of 0 for that subject before calculating the variance of $W_{il}, l = 1,2 \ldots n_i; i = 1,2.$ Now note that the above ranking procedure for $W_{il}$ for each population results in essentially the same rank for each observation or score $x_{il}$ and hence each subject drawn from the population as would have been obtained if only subjects' scores had been ranked. Nevertheless the procedure enables the researcher immediately has a birds eye-view in the form of a spread sheet of the overall ranking of subjects relative to one another in performance or response in a test both within and between the sampled populations and also determine by how much a given subject fares better averagely as well as or worse than other subjects in the populations which provides additional useful information. Based on these rankings a policy implementer may decide to introduce any desired interventionist measures for subjects either right of the average (median) or centre, left of the center or both depending on the condition being remediated for the sample populations. The method also enables easy and quick estimation with minimal calculations of the percentiles and other tiles of the distribution of the population of interest using their ranks. Thus the $k^{th}$ percentile of the distribution of the $i^{th}$ population is estimated as the value of the observation $x_{il}$ corresponding to $W_{il}$, the relative performance index of the $l^{th}$subject from population $i$ with rank $r_{il}(kp)$ is

$$r_{il}(kp) = \begin{cases} \frac{kn_i - 1}{p} \; if \; n_i \; is \; odd \\ \frac{\frac{kn_i}{p} + \frac{kn_i + 1}{p}}{2} \; if \; n_i \; is \; even \end{cases}$$

38

For $k$ =1, 2…99, $p$ =100; $l$=1,2…$n_i$, $i = 1,2$.

Other tiles of the distribution of a population of interest may be similarly estimated. This approach easily enables one have a holistic view of the percentiles of the populations of interest in the form of a spread sheet for easy comparisons.

**Illustrative Example**

To illustrate the proposed method we here use the following results in letter grades earned by random samples of 11 Biology and 14 Psychology graduate students who took an introductory course in Bio-Statistics

Biology (B) B,A,C,A,A,B,F,A,C,E,B

Psychology (P) E,F,E,F,E,D,E,F,E,C,D,A,A,B

We now apply Equation 1 separately to the sample results or grades of Biology and Psychology students to obtain values of $u_{ilk}: l, k = 1,2 \ldots \ldots \ldots . n_i, i = 1,2, l \neq k$ which are shown in Tables 1 and 2 enabling us obtain the summary values of $f^{+}, f^{0}, f^{-}, \hat{\pi}^{+}, \hat{\pi}^{0}, \hat{\pi}^{-}, W, r$ and other statistics shown in tables 3 for Biology and 4 for Psychology students respectively. It is seen from Table 3 that Biology students $B_2$, $B_4$, $B_5$ and $B_8$ each with an A grade and each with a relative performance index of $W = f^{+}$ =7 and each ranked 2.5 are the best or most highly performed among Biology students; while the poorest students with a relative performance index of $W = -(n_1 - 1) = -10 = f^{-}, \hat{\pi}^{+} = 0; \hat{\pi}^{-} = 1.0$ ranked 11 is student number $B_7$ with an F grade. Biology student numbers $B_1$, $B_6$ and $B_{11}$ each performed better than one-half $(f_{il}^{+} = 4)$ than one-half $(f_{il}^{-} = 4); \hat{\pi}^{+} = \hat{\pi}^{-}$ of all Biology students and hence achieved a relative performance index of 0.

**Table 1 Values of $U_{ilk}$ (Equation 1) for Biology Students**

| | $B_1$ B | $B_2$ A | $B_3$ C | $B_4$ A | $B_5$ A | $B_6$ B | $B_7$ F | $B_8$ A | $B_9$C | $B_{10}$ E | $B_{11}$ B | $f_i^+$ | $f_i^0$ | $f_i^-$ | $W_{il}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$B_1$ B** | | -1 | 1 | -1 | -1 | 0 | 1 | -1 | 1 | 1 | 0 | 4 | 2 | 4 | 0 |
| **$B_2$ A** | 1 | | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 7 | 3 | 0 | 7 |
| **$B_3$ C** | -1 | -1 | | -1 | -1 | -1 | -1 | -1 | 0 | 1 | -1 | 2 | 1 | 7 | -5 |
| **$B_4$ A** | 1 | 0 | 1 | | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 7 | 3 | 0 | 7 |
| **$B_5$ A** | 1 | 0 | 1 | 0 | | 1 | 1 | 0 | 1 | 1 | 1 | 7 | 3 | 0 | 7 |
| **$B_6$ B** | 0 | -1 | 1 | -1 | -1 | | 1 | -1 | 1 | 1 | 0 | 4 | 2 | 4 | 0 |
| **$B_7$ F** | -1 | -1 | -1 | -1 | -1 | -1 | | -1 | -1 | -1 | -1 | 0 | 0 | 10 | -10 |
| **$B_8$ A** | 1 | 0 | 1 | 0 | 0 | 1 | 1 | | 1 | 1 | 1 | 7 | 3 | 0 | 7 |
| **$B_9$ C** | -1 | -1 | 0 | -1 | -1 | -1 | 1 | -1 | | 1 | -1 | 2 | 1 | 7 | -5 |
| **$B_{10}$ E** | -1 | -1 | -1 | -1 | -1 | -1 | 1 | -1 | -1 | | -1 | 1 | 0 | 9 | -8 |
| **$B_{11}$ B** | 0 | -1 | 1 | -1 | -1 | 0 | 1 | -1 | 1 | 1 | | 4 | 2 | 4 | 0 |

**Table 2: Values of (Equation 1) for Psychology Students**

| | $P_1$ E | $P_2$ F | $P_3$ E | $P_4$ F | $P_5$ E | $P_6$ D | $P_7$ E | $P_8$ F | $P_9$ E | $P_{10}$ C | $P_{11}$ D | $P_{12}$ A | $P_{13}$ A | $P_{14}$ B | $f_i^+$ | $f_i^0$ | $f_i^-$ | $W_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$P_1$ E** | | 1 | 0 | 1 | 0 | -1 | 0 | 1 | 0 | -1 | -1 | -1 | -1 | -1 | 3 | 4 | 6 | -3 |
| **$P_2$ F** | -1 | | -1 | 0 | -1 | -1 | -1 | 0 | -1 | -1 | -1 | -1 | -1 | -1 | 0 | 2 | 11 | -11 |
| **$P_3$ E** | 0 | 1 | | 1 | 0 | -1 | 0 | 1 | 0 | -1 | -1 | -1 | -1 | -1 | 3 | 4 | 6 | -3 |
| **$P_4$ F** | -1 | 0 | -1 | | -1 | -1 | -1 | 0 | -1 | -1 | -1 | -1 | -1 | -1 | 0 | 2 | 11 | -11 |
| **$P_5$ E** | 0 | 1 | 0 | 1 | | -1 | 0 | 1 | 0 | -1 | -1 | -1 | -1 | -1 | 3 | 4 | 6 | -3 |
| **$P_6$ D** | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | 1 | -1 | 0 | -1 | -1 | -1 | 8 | 1 | 4 | 4 |
| **$P_7$ E** | 0 | 1 | 0 | 1 | 0 | -1 | | 1 | 0 | -1 | -1 | -1 | -1 | -1 | 3 | 4 | 6 | -3 |
| **$P_8$ F** | -1 | 0 | -1 | 0 | -1 | -1 | -1 | | -1 | -1 | -1 | -1 | -1 | -1 | 0 | 2 | 11 | -11 |
| **$P_9$ E** | 0 | 1 | 0 | 1 | 0 | -1 | 0 | 1 | | -1 | -1 | -1 | -1 | -1 | 3 | 4 | 6 | -3 |
| **$P_{10}$C** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 1 | -1 | -1 | -1 | 10 | 0 | 3 | 7 |
| **$P_{11}$D** | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | -1 | | -1 | -1 | -1 | 8 | 1 | 4 | 4 |
| **$P_{12}$A** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | 0 | 1 | 12 | 1 | 0 | 12 |
| **$P_{13}$A** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | | 1 | 12 | 1 | 0 | 12 |
| **$P_{14}$B** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | | 11 | 0 | 2 | 9 |

**Table 3: Relative Performance Indices and other Statistics for Biology Students**

| S/N | $f_{ij}^+$ | $f_{ij}^0$ | $f_{ij}^-$ | $n_i - 1$ | $\hat{\pi}_{ij}^+$ | $\hat{\pi}_{ij}^0$ | $\hat{\pi}_{ij}^-$ | $W_{ij}(f_{ij}^+ - f_{ij}^-)$ | $r_{ij}$ | $var(W_{ij})$ | $\chi_{ij}^2$ | $p_{value}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$B_1$ B** | 4 | 2 | 4 | 10 | 0.40 | 0.20 | 0.40 | 0 | 6 | 8.0 | 0 | 0.99 |
| **$B_2$ A** | 7 | 3 | 0 | 10 | 0.70 | 0.30 | 0.00 | 7 | 2.5 | 2.1 | 23.333 | 0.01 |
| **$B_3$ C** | 2 | 1 | 7 | 10 | 0.20 | 0.10 | 0.70 | -5 | 8.5 | 6.5 | 3.846 | 0.99 |
| **$B_4$ A** | 7 | 3 | 0 | 10 | 0.70 | 0.30 | 0.00 | 7 | 2.5 | 2.1 | 23.333 | 0.01 |
| **$B_5$ A** | 7 | 3 | 0 | 10 | 0.70 | 0.03 | 0.00 | 7 | 2.5 | 2.1 | 23.333 | 0.01 |
| **$B_6$ B** | 4 | 2 | 4 | 10 | 0.40 | 0.20 | 0.40 | 0 | 6 | 8.0 | 23.333 | 0.99 |
| **$B_7$ F** | 0 | 0 | 10 | 10 | 0.00 | 0.00 | 1.00 | -10 | 11 | 0.0 | 0.000 | 0.01 |
| **$B_8$ A** | 7 | 3 | 0 | 10 | 0.70 | 0.30 | 0.00 | 7 | 2.5 | 2.1 | undefined | 0.01 |
| **$B_9$ C** | 2 | 1 | 7 | 10 | 0.20 | 0.10 | 0.70 | -5 | 8.5 | 6.5 | 3.846 | 0.99 |
| **$B_{10}$E** | 1 | 0 | 9 | 10 | 0.10 | 0.07 | 0.90 | -8 | 10 | 3.6 | 17.778 | 0.20 |
| **$B_{11}$B** | 4 | 2 | 4 | 10 | 0.40 | 0.20 | 0.40 | 0 | 6 | 8.0 | 0.000 | 0.99 |

**Table 4 Relative Performance Indices and other Statistics for Psychology Students**

| Student score No | $f^+_{2g}$ | $f^0_{2g}$ | $f^-_{2g}$ | $\hat{\pi}^+_{2g}$ | $\hat{\pi}^0_{2g}$ | $\hat{\pi}^-_{2g}$ | $W_{2g}$ | $p_{2g}$ | $r_{2g}$ | $var(W_{2g})$ | $\chi^2_{2g}$ | $p_{value}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$P_1$ E** | 3 | 4 | 6 | 0.231 | 0.308 | 0.461 | -3 | -0.231 | 9 | 8.315 | 1.082 | 0.99 |
| **$P_2$ F** | 0 | 2 | 11 | 0.000 | 0.154 | 0.846 | -11 | -0.846 | 13 | 1.694 | 71.429 | 0.01 |
| **$P_3$ E** | 3 | 4 | 6 | 0.231 | 0.308 | 0.461 | -3 | -0.231 | 9 | 8.315 | 1.082 | 0.99 |
| **$P_4$ F** | 0 | 2 | 11 | 0.000 | 0.154 | 0.846 | -11 | -0.846 | 13 | 1.694 | 71.429 | 0.01 |
| **$P_5$ E** | 3 | 4 | 6 | 0.231 | 0.308 | 0.461 | -3 | -0.231 | 9 | 8.315 | 1.082 | 0.99 |
| **$P_6$ D** | 8 | 1 | 4 | 0.692 | 0.077 | 0.231 | 4 | 0.308 | 5.5 | 10.774 | 1.485 | 0.99 |
| **$P_7$ E** | 3 | 4 | 6 | 0.231 | 0.308 | 0.461 | -3 | -0.231 | 9 | 8.315 | 1.082 | 0.99 |
| **$P_8$ F** | 0 | 2 | 11 | 0.000 | 0.154 | 0.846 | -11 | -0.846 | 13 | 1.694 | 71.429 | 0.01 |
| **$P_9$ E** | 3 | 4 | 6 | 0.231 | 0.308 | 0.461 | -3 | -0.231 | 9 | 8.315 | 1.082 | 0.99 |
| **$P_{10}$ C** | 10 | 0 | 3 | 0.769 | 0.000 | 0.231 | 7 | 0.538 | 4 | 9.237 | 5.305 | 0.99 |
| **$P_{11}$ D** | 8 | 1 | 4 | 0.615 | 0.077 | 0.308 | 4 | 0.308 | 5.5 | 10.774 | 1.485 | 0.99 |
| **$P_{12}$ A** | 12 | 1 | 0 | 0.923 | 0.077 | 0.000 | 12 | 0.923 | 1.5 | 0.924 | 155.84 | 0.01 |
| **$P_{13}$ A** | 12 | 1 | 0 | 0.923 | 0.077 | 0.000 | 12 | 0.923 | 1.5 | 0.924 | 155.84 | 0.01 |
| **$P_{14}$ B** | 11 | 0 | 2 | 0.846 | 0.000 | 0.154 | 9 | 0.692 | 3 | 6.775 | 11.956 | 0.80 |

**Table 5 Relative Performance Indices for Biology and Psychology Students using one observation from each tied set**

| | **$B_1$ B** | **$B_2$ A** | **$B_3$ C** | **$B_7$ F** | **$B_{10}$ E** | $f^+_i$ | $f^0_i$ | $f^-_i$ | $\hat{\pi}^+_i$ | $\hat{\pi}^0_i$ | $\hat{\pi}^-_i$ | $W_i = f^+_{il} - f^-_{il}$ | $p_{il} = \hat{\pi}^+_{il} - \hat{\pi}^-_{il}$ | $r_{il}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$B_1$ B** | | -1 | 1 | 1 | 1 | 3 | 0 | 1 | 0.75 | 0.00 | 0.25 | 2 | 0.50 | 2 |
| **$B_2$ A** | 1 | | 1 | 1 | 1 | 4 | 0 | 0 | 1.00 | 0.00 | 0.00 | 4 | 1.00 | 1 |
| **$B_3$ C** | -1 | -1 | | 1 | 1 | 2 | 0 | 2 | 0.50 | 0.00 | 0.50 | 0 | 0.00 | 3 |
| **$B_7$ F** | -1 | -1 | -1 | | -1 | 0 | 0 | 4 | 0.00 | 0.00 | 1.00 | -4 | -1.00 | 5 |
| **$B_{10}$ E** | -1 | -1 | -1 | 1 | | 1 | 0 | 3 | 0.25 | 0.00 | 0.75 | -2 | -0.50 | 4 |

| | **$P_1$ E** | **$P_2$ F** | **$P_6$ D** | **$P_{10}$ C** | **$P_{12}$ A** | **$P_{14}$ E** | $f^+_i$ | $f^0_i$ | $f^-_i$ | $\hat{\pi}^+_i$ | $\hat{\pi}^0_i$ | $\hat{\pi}^-_i$ | $W_i = f^+_{il} - f^-_{il}$ | $p_{il} = \hat{\pi}^+_{il} - \hat{\pi}^-_{il}$ | $r_{il}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$P_1$ E** | | 1 | -1 | -1 | -1 | -1 | 1 | 0 | 4 | 0.20 | 0.00 | 0.80 | -3 | -0.60 | 5 |
| **$P_2$ F** | -1 | | -1 | -1 | -1 | -1 | 0 | 0 | 5 | 0.00 | 0.00 | 1.00 | -5 | -1.00 | 6 |
| **$P_6$ D** | 1 | 1 | | -1 | -1 | -1 | 2 | 0 | 3 | 0.40 | 0.00 | 0.60 | -1 | -0.20 | 4 |
| **$P_{10}$ C** | 1 | 1 | 1 | | -1 | 1 | 3 | 0 | 2 | 0.60 | 0.00 | 0.40 | 1 | 0.20 | 3 |
| **$P_{12}$ A** | 1 | 1 | 1 | 1 | | 1 | 5 | 0 | 0 | 1.00 | 0.00 | 0.00 | 5 | 1.00 | 1 |
| **$P_{14}$ E** | 1 | 1 | 1 | 1 | -1 | | 4 | 0 | 1 | 0.80 | 0.00 | 0.20 | 3 | 0.60 | 2 |

earning a median rank of 6 and grade of B which is seen to be the median grade for Biology Students. Similarly it is seen from Table 4 that Psychology students Numbers $P_{12}$ and $P_{13}$ each with an A grade are the most highly performed among psychology students. Each of these students therefore has a relative performance index of 12 ($f^+_{2l}$ = 12; $f^-_{2l}$ = 0) ranked 1.5 each. Also the most poorly performed psychology students ($P_2$, $P_4$ and $P_8$) each has a relative performance index of -11 ranked lowest as 13 with an F grade. Five psychology students are tied at the median grade E with a relative performance index of -2 rank 8.5, the median rank for psychology students. Thus it is seen from these analyses that Biology and Psychology graduate students on the average differ in performance in Bio-Statistic. Biology students have a median grade of B while Psychology Students earned a median grade of E only. Thus on the average psychology graduate students would seem to need more intensive help in the Bio-Statistics course. Table 5 shows re-analysis of the illustrative example using only one observation from within each set of tied observations in each of the two samples. This is possible because as already noted above all subjects within each set of tied observations are treated alike in the relative rank-ordering of subjects by performance in a preferential selection process. The results of table 5 are clearly consistent with the specifications and results expected to be obtained with the proposed method in the absence of tied observations in the sampled populations. Thus it is seen from table 5 that for biology students now with $n_1 = 5$ which is odd, the student who performed more than all other student in the sample is student number $B_2$ with an A grade , so that the students relative performance index is W=$n_1 - 1 = f_i^+ = 4$ ranked 1. The worst student $B_7$ with an F grade has a relative performance index of $W = -(n_1 - 1) = f^- = -4$ ranked 5; the last with $\hat{\pi}^+ = 0\ and\ \hat{\pi}^- = 1.0.$ Student $B_3$ with a C grade scored higher than two and worse than two other students and earned a relative performance index of $W = 0\ (\hat{\pi}^+ = \hat{\pi}^-)$ ranked 3 which is the median rank, showing that the median score by Biology students would be a C grade in the absence of ties. Similarly the results for Psychology students now reduced to an effective size of $n_2 = 6$ which is even, show that the best student in terms of relative performance index is student $P_2$ with $W = n_2 - 1 = 5\ (\hat{\pi}^+ = 1, \hat{\pi}^- = 0.0)$ ranked 1, the first. The worst is student $P_2$ with an F grade and hence $W = -(n_2 - 1) = -5\ (\hat{\pi}^+ = 0.0, \hat{\pi}^- = 1.0)$ ranked 6 the last. Student $P_{10}$ with a C grade and W=1 ranked 3 and student $P_6$ with D grade and with W= -1 ranked 4 are students with the two middle most W values and hence assigned the ranks 3 and 4 respectively in the rank-ordering of these values. Therefore the median score of psychology students would be about a $C^-\ or\ CD$ grade ranked 3.5 in the absence of ties and the number of observations 'n' is even. In the preferential ranking of the students these conclusions would normally be applied in the selection of students within each group of students tied in scores in each of the two programs. Other comparisms can be made between Biology and psychology student by taking the difference between their relative performances indices expressed as proportions and cross-classifying them by grades if desired. It would be instructive to compare the results obtained using the proposed method with what would have been obtained if the data of the illustrative example had been analyzed using the median test or the Mann-Whitney u-test. Because there are many tied observations, we would use the modified median test (Oyeka etal 2010) to reanalyze the data. Using the modified median test with the illustrative data we have that $f^+ = 104, f^0 = 21, f^- = 29, m = 11, n = 14$, so that $\hat{\pi}^+ = \frac{f^+}{mn} = \frac{104}{154} = 0.675$; $\hat{\pi}^0 = \frac{f^0}{mn} = \frac{21}{154} = 0.136$; $\hat{\pi}^- = \frac{f^-}{mn} = \frac{29}{154} = 0.188$; and $w = f^+ - f^-$=104-29=75 and $var(W) = mn\left(\hat{\pi}^+ + \hat{\pi}^- - (\hat{\pi}^+ - \hat{\pi}^-)^2\right) = 154\left(0.675 + 0.188 - (0.675 - 0.188)^2\right) = 96.404$. hence the corresponding chi-square test statistics for the equality of the two population medians is $\chi^2 = \frac{W^2}{var(W)} = \frac{75^2}{96.404} = \frac{5625}{96.404} = 58.348(P_{value=0.0000})$ which with 1 degree of freedom is highly statistically significant, indicating that Biology and Psychology students differ in their performance in the course.Similarly, applying the Mann-whitney U test we have that the sum of the ranks assigned To Biology when the two samples are combined and ranked is R=105.5; so that $U = mn + m\left(\frac{m+1}{2}\right) - R = 154 + 11\left(\frac{12}{2}\right) - 105.5 = 114.5$. $E(U) = \frac{mn}{2} = \frac{154}{2} = 77$; $var(U) = \frac{mn(m+n+1)}{12} = \frac{154(26)}{12} = \frac{4004}{12} = 333.667$ so that $Se(U) = \sqrt{333.667}$ =18.267. Hence the corresponding Z-test statistic is $Z = \frac{U-E(U)}{Se(U)} = \frac{114.5-77}{18.267} = \frac{37.5}{18.267} = 2.053(P_{value} = 0.0202)$ which is also statistically significant again indicating differences between the two population medians. Thus both the modified median test and the Mann-whitney U test like the proposed method have shown the existence of significant differences between the performance of Biology and Psychology graduate students in the Bio-Statistics course. However these two methods unlike the proposed method are unable to be used to readily provide sample estimates of the median grades of the two populations of students. This feature is an added advantage of the proposed method over some of the existing statistical methods currently used for the analysis of two sample data.

## Summary and Conclusion

We have in this paper proposed and presented a statistical method for the analysis of two sample data and for comparing the performance or scores of subjects both within and between the sampled populations. The statistic here termed 'index of relative performance' or 'relative performance index' is developed, that would enable the comparison of the performance or scores of subjects relative to one another in a test for preferential selection both within and between sampled populations for management purposes as the need arises. It is shown that in the absence of ties, the value of the observation whose relative performance index is zero if the sample size is odd or the average of the observations whose two middle-most ranked relative performance indices have values of 1 and -1 respectively if 'n' is even in each sample provides an estimate of that samples population median and hence an estimate of the common population median if the two samples had been pooled together into one combined sample. It is also shown that these sample medians which always correspond to the middle-most ranked relative performance indices if 'n' is odd or the average of the two observations with the two middle-most ranked relative performance indices if 'n' is even whether or not there are ties in the samples are the same values that would have been obtained if the observations had simply been rank-ordered for the purpose of identifying the tiles of the data distribution. But the proposed method enables a more in depth analysis of the data and provides more useful information for policy purposes.Test statistics are developed for testing the equality of the relative performance indices and the associated medians and the other tiles both within and between the populations. The proposed indices that are individual subject specific rather than merely summary averages easily enables one more clearly examine individual subject relative performance or level of seriousness in a condition in comparison with other subjects from sampled populations thereby providing subject targeted information to better guide any interventionist actions on a condition of research interest. The proposed method which is illustrated with some sample data is shown to compare favorably with the modified median and Mann-whitney U tests. However, if the intension of the research and analysis is to systematically rank-order subjects in a contest for preferential selection based on the performance or score, then the proposed method is more useful and hence may be preferable to the median, Mann-whitney or other similar tests that are based on only summary indices or averages rather than on individual subjects and so are unable to achieve the set objective.

## References

[1]. Gibbons, J.D (1971): Non parametric statistical inference Mc Graw hill, New York.

[2]. Oyeka,(2009); An introduction to Applied Statistical Methods (5th editions), Nobern Avocation pub.coy, Enugu.

[3]. Oyeka C.A, Ebuh G.U, Nwankwo C.C, Obiora-ilouno H, Ibeakuzie P.O, Utazi .C (2010). A Statistical comparison of Test Score; A Non-parametric approach.j. math.sci 21(1); 77-87

[4]. Hollander, M. and Wolfe,D.A.(1973); Non parametric statistics. New York; J.Wiley

[5]. Siegel, and Casterllan, N.J.Jr(1988); Non parametric statistics for Behavioural Sciences.2nd ed.Mc Graw hill. New York